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Estimation of Lomax Parameters Based on Generalized Probability Weighted Moment

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Abstract: Probability weighted moments method, introduced and recommend earlier as an alternate method to the classical moments, for fitting statistical distributions to data; is thought to be less affected by sampling variability and be more efficient at producing robust parameter estimates in case of small samples.

In this paper, the generalized probability weighted moments method is applied for estimating the parameters of Lomax distribution.

1. Introduction

Greenwood *et al*.[1] introduced a new method for estimating the parameters of distribution, which is called the method of probability weighted moments (PWM) which can be applied to the probability distribution functions which have inverse form, such as Weibull, generalized Pareto, Log logistic.

Landweher *et al*.[2,3] compared the performance of the estimations, that have been obtained by probability weighted moments, and the maximum likelihood method, and the method of moments for the parameters of Gumbel distribution and found that the method of probability weighted moments gives better results compared to the others.

Hosking *et al*.[4] noted that the new definition of probability weighted moments described by Greenwood *et al*.[1] can be used as the traditional Moments. Moreover, Song and Ding[5] stated the probability weighted distributions of the distributions can be applied to the distributions that can not be expressed in inverse form, and they have applied this method to Pearson type III, Inverse gamma, and Log normal distributions.

Rasmussen[6] introduced a new class of probability weighted moments as a generalization method which is called the generalized probability weighted moments (GPWM).

Rasmussen[6] applied the generalized probability weighted moments to the generalized Pareto distribution and noted that this method gave the best estimation compared to other methods. Also, he compared the performance of estimation that have been obtained by probability weighted moments and by the method of moments and found that the method of generalized probability weighted moments gives the least squares error of the method of moments.

Moreover, Ashkar and Mahdi[7] applied the generalized probability weighted moments method to log-logistic distribution. They compared between the performance of estimation that have been obtained through generalized probability weighted moments and that of the maximum likelihood method. They noted that the generalized probability weighted moments gives better estimation than the maximum likelihood method, especially for small samples. Recently, Allam[8] applied the method of partial probability weighted moments to estimate the parameters of generalized exponential distribution and Elharoun[9] used the generalized probability weighted moments to estimate the parameters of generalized exponential and Pearson type III distributions.

The Lomax distribution is known as Pareto distribution of the second kind or Pearson Type VI distribution. It has been used in the analysis of income data, and business failure data. It may describe the lifetime of a decreasing failure rate component as a heavy tailed alternative to the exponential distribution. Zagan[10] deals with the properties of the Lomax distribution with three parameter. The relation between the Lomax distribution and some other distributions are discussed in the literature. These distributions are Weibull distribution, Compound Weibull distribution which is called the three parameter Burr type XII distribution, exponential distribution, Rayleigh distribution, beta type II, beta type I distribution, uniform distribution, generalized uniform distribution, and extreme value distribution.

The objective of this paper is to study the estimation problem for the parameters of Lomax distribution using the GPWM method.

This paper is organized as follows: In section 2, we discuss, the definition of sample estimators of probability weighted moments. The GPWMs are discussed in section 3. In section 4, we discuss the GPWMs estimation of Lomax distribution. In section 5, the approximate asymptotic variance and covariance matrix of GPWMs for Lomax are investigated.

2. The Sample Estimators of Probability Weighted Moments

Landwehr *et al.*[3] introduced unbiased estimators of α_s and β_r, when s and r are nonnegative integer, which are based on the ordered sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ from the distribution F and are given by:

$$\hat{\alpha}_s = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} \binom{n-i}{s} \Big/ \binom{n-1}{s}, \tag{2.1}$$

and

$$\hat{\beta}_r = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} \binom{i-1}{r} \Big/ \binom{n-1}{r} \tag{2.2}$$

Landwehr *et al.*[4] introduced a biased estimator of α_s, which is constructed in accordance with the concept of plotting position and is given by:

$$W_s = \frac{1}{n}\sum_{i=1}^{n} \left[\frac{n-i+h}{n}\right]^s x_{(i)}, \tag{2.3}$$

where $h = 0.35$. They mentioned that it is not necessary to specify that s is an integer, s is assumed only to be nonnegative. Hosking et al.[2] introduced an estimator of β_r which is given by:

$$b_r = \frac{1}{n}\sum_{i=1}^{n} p_{i,n}^{r} x_{(i)} \tag{2.4}$$

and estimator of α_s is given by:

$$c_s = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}\left(1 - p_{i,n}\right) \tag{2.5}$$

where $P_{i,n}$ is a plotting position that is, a distribution free estimate of $F\left(x_{(i)}\right)$. They mentioned that reasonable choices of $P_{i,n}$, such as:

$$P_{i,n} = \begin{cases} \dfrac{(i-a)}{n}, & 0 < a < 1 \\ P_{i,n} = \dfrac{(i-a)}{(n+1-2a)} & , \dfrac{1}{2} < a < \dfrac{1}{2}, \end{cases}$$

yield estimators b_r that are asymptotically equivalent to $\hat{\beta}_r$, $\hat{\alpha}_s$ and, therefore, consistent estimators of β_r and α_s.

3. The Generalized Probability Weighted Moments Method

Song and Ding[5] and Ding *et al.*[11] applied the method of probability weighted moments (PWM) for estimating the unknown parameters of several distributions in expressible in inverse form. among them they considered the Pearson type three distribution.

The method of generalized probability weighted moments was introduced by Rasmussen[6] as an extension of the original method of probability weighted moments. It used to estimate the parameters for several distributions such as generalized Pareto, Log-logistic and Weibull distribution.

Rasmussen[6] proposed the GPWM as a generalization for PWM method. All applications of the PWM method have been considered small nonnegative integer on the exponents. While the GPWM are not restricted to small nonnegative integer on the exponent. The GPWM of order $p = 1$ and $v = 0$, is given by $M_{1,u,0}$, therefore:

$$M_{1,u,0} = E\left(X\left(F\left(X\right)\right)^u\right)$$

$$= \int_0^\infty xF\left(x\right)^u f\left(x\right)dx, \tag{3.1}$$

where, X is continuous variable whose distribution is being estimated and F is the cumulative distribution function of X, and u has neither to be small, nor nonnegative integer. The idea behind the PWM and GPWM methods is to obtain parameter estimators by equating PWM ($M_{1,u,0}$) to the sample estimators of PWM and solving the resulting system of equations for the distribution parameters.

For example, if the distribution has a two parameters, the tradition PWM method, involves consideration of $u = 0$ and $u = 1$ in equation (3.1). On the other hand, the GPWM method considers $u = u_1$ and $u = u_2$ where u_1 and u_2 has either to be small, or a non negative integer. The sample estimates in this case is given by:

$$\hat{M}_{1,u_1,u_2} = \sum_{i=1}^{n} x_{(1)} p_i^{u_1} \left(1-p_i\right)^{u_2}, \tag{3.2}$$

where, $p_i = \dfrac{i-0.35}{n}$.

The following are the steps for GPWM procedure for estimating the unknown parameters of the distribution expressible in inverse form:

Step (1): Obtaining the inverse distribution function $x(F)$ for the given distribution function F of the distribution, if possible.

Step(2):Calculating the theoretical GPWM from one of the two formulas $M_{1,u,0}$ or $M_{1,0,v}$ whichever is possible, where u and v take values $u_1, u_2, \ldots$ or $v_1, v_2, \ldots$ depending on the number of the unknown parameters. General considering a distribution has three parameters at most, let $u = u_1, u_2, u_3$ then the formula $M_{1,u,0}$ takes the following forms

$$M_{1,u_1,0} = E\left(X\left(F(X)\right)^{u_1}\right) = \int_0^1 x(F)\, F^{u_1}\, dF \tag{3.3}$$

$$M_{1,u_2,0} = E\left(X\left(F(X)\right)^{u_2}\right) = \int_0^1 x(F)\, F^{u_2}\, dF \tag{3.4}$$

and $M_{1,u_3,0} = E\left(X\left(F(X)\right)^{u_3}\right) = \int_0^1 x(F)\, F^{u_3}\, dF$ (3.5)

Step (3)*:* obtaining the parameters of interest in terms of one of the two theoretical formulas $M_{1,u,0}$ or $M_{1,0,v}$ any of which is possible.

Step(4): Replacing the theoretical GPWM by one of their sample estimators $\hat{M}_{1,u,0}$ or $\hat{M}_{1,0,v}$.

The following are the steps for GPWM procedure for estimating the unknown parameters of the distribution can not be expressed in inverse form:

Step(1): calculating the theoretical GPWM from one of the two formulas $M_{1,u,0}$ or $M_{1,0,v}$ which one is possible.

$$M_{1,u,0} = E\left(X\left(F(X)\right)^{u}\right) = \int_{0}^{1} x(F)\, F^{u}\; dF\,.$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{x} f(t)dt \right)^{u} f(x)dx \tag{3.6}$$

or $M_{1,0,v} = E\left(X\left(1-F(X)\right)^{v}\right) = \int_{0}^{1} x(1-F)^{v}\, F\; dF\,.$

$$= \int_{-\infty}^{\infty} x \left(1 - \int_{-\infty}^{x} f(t)dt \right)^{v} f(x)dx \tag{3.7}$$

each of the two values u and v taken values $u_1, u_2, ...$ or $v_1, v_2, ...$ depending on the number of the unknown distributions parameters. Generally considering a distribution has three parameters. Let $u = u_1, u_2, u_3$ then the formula $M_{1,u,0}$ takes the following forms, $M_{1,u_1,0} = E\left(X\left(F(X)\right)^{u_1}\right) = \int_{0}^{1} x(F)\, F^{u_1} dF\,,$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{x} f(t)dt \right)^{u_1} f(x)dx \tag{3.8}$$

$$M_{1,u_2,0} = E\left(X\left(F(X)\right)^{u_2}\right) = \int_{0}^{1} x(F)\, F^{u_2} dF\,,$$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{x} f(t)dt \right)^{u_2} f(x)dx\,. \tag{3.9}$$

and $M_{1,u_3,0} = E\left(X\left(F(X)\right)^{u_3}\right) = \int_{0}^{1} x(F)\, F^{u_3} dF\,,$

$$= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{x} f(t)dt \right)^{u_3} f(x)dx \tag{3.10}$$

Step(2): obtaining the parameter of interest in terms of one of the two theoretical formulas $M_{1,u,0}$ and $M_{1,0,v}$ whichever is possible.

Step(3): Replace the theoretical GPWM, $M_{1,u_1,0}$, $M_{1,u_2,0}$ and $M_{1,u_3,0}$ by their sample estimators, $\hat{M}_{1,u_1,0}$ $\hat{M}_{1,u_2,0}$ and $\hat{M}_{1,u_3,0}$. The formula $\hat{M}_{1,u,v}$ is based on the order complete sample $x_1 < x_2 ... < x_n$ of size n. It is given by;

$$\hat{M}_{1,u,v} = \sum_{i=1}^{n} x_{(i)} p_i^u (1 - p_i)^v \tag{3.11}$$

where $x_{(i)}$ is the i th observation in the ordered sample .

and $p_i = \dfrac{i - 0.35}{n}$.

From equation (3.11), the sample estimators of $\hat{M}_{1,u_2,0}$ are given by:

$$\hat{M}_{u_1} = \frac{1}{n} \sum_{i=1}^{n} x_i \left(\frac{i - 0.35}{n} \right)^{u_1} . \tag{3.12}$$

$$\hat{M}_{u_2} = \frac{1}{n} \sum_{i=1}^{n} x_i \left(\frac{i - 0.35}{n} \right)^{u_2} \tag{3.13}$$

and $$\hat{M}_{u_3} = \frac{1}{n} \sum_{i=1}^{n} x_i \left(\frac{i - 0.35}{n} \right)^{u_3} \tag{3.14}$$

4. GPWM Estimators for Lomax Distribution

The GPWM estimators for parameters of the Lomax distribution will be obtained. As mentioned before, two sets of GPWM of the form $M_{1,0,v}$. For the Lomax distribution the GPWM of the form $M_{1,u,0}$ will be used because of it has a simpler analytical structure than $M_{1,0,v}$.

The procedure of GPWM estimation for the Lomax distribution is summarized in the following steps.

Step (1): The inverse cumulative distribution function of the Lomax distribution is given by:

$$x\left(F\right)=\lambda\left((1-F)^{\frac{-1}{\alpha}}-1\right),\ \alpha,\lambda,x>0 \tag{4.1}$$

where the pdf and CDF of Lomax distribution are respectively,

$$f(x)=\frac{\alpha}{\lambda}(1+\frac{x}{\lambda})^{-(\alpha+1)},\ x>0,\lambda,\alpha>0 \text{ and } F(x)=1-(1+\frac{x}{\lambda})^{-\alpha}.$$

Step (2): Obtain the theoretical probability weighted moments, for the Lomax distribution, that is:

$$M_{1,u,0}=\int_0^1 x F^u \ dF, \tag{4.2}$$

u takes the values $u_1, u_2, ...$ according to the number of parameters.

Substituting the form of the inverse distribution function of the Lomax distribution into equation (4.2) yield:

$$M_{1,u,0}=\int_0^1 \lambda\left((1-F)^{\frac{-1}{\alpha}}-1\right) F^u \ dF \tag{4.3}$$

$$M_{1,u,0}=\lambda\left[\beta(u+1,1-\frac{1}{\alpha})-\frac{1}{u+1}\right],$$

where $\beta(n,m)$ is the beta function.

Since the Lomax distribution has a two parameters; then only the first two GPWM, $M_{1,u_1,0}$ and $M_{1,u_2,0}$, are needed for parameter estimation. They are given by:

$$M_{1,u_1,0}=\lambda\left[\beta(u_1+1,1-\frac{1}{\alpha})-\frac{1}{u_1+1}\right], \tag{4.4}$$

and $$M_{1,u_2,0}=\lambda\left[\beta(u_2+1,1-\frac{1}{\alpha})-\frac{1}{u_2+1}\right] \tag{4.5}$$

Step (3): Obtain the parameters α , λ in terms of $M_{1,u_1,0}$ and $M_{1,u_2,0}$ using equation (4.4)and (4.5).

From equation (4.4) then;

$$\lambda = \left[\frac{\left[\beta(u_1 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_1 + 1} \right]}{M_{1,u_1,0}} \right]^{-1} \qquad (4.6)$$

Thus from equation (4.5)

$$\lambda = \left[\frac{\left[\beta(u_2 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_2 + 1} \right]}{M_{1,u_2,0}} \right]^{-1} \qquad (4.7)$$

Substituting equation (4.6) into equation (4.5) gives:

$$M_{1,u_2,0} = \left[\frac{\left[\beta(u_1 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_1 + 1} \right]}{M_{1,u_1,0}} \right]^{-1} \left[\beta(u_2 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_2 + 1} \right],$$

$$M_{1,u_2,0} \left[\beta(u_1 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_1 + 1} \right] = M_{1,u_1,0} \left[\beta(u_2 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_2 + 1} \right]$$

$$M_{1,u_2,0} \left[\beta(u_1 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_1 + 1} \right] - M_{1,u_1,0} \left[\beta(u_2 + 1, 1 - \frac{1}{\alpha}) - \frac{1}{u_2 + 1} \right] = 0$$

Step (4): Replace the theoretical GPWM, $M_{1,u_1,0}$ and $M_{1,u_2,0}$ by their sample estimators, $\hat{M}_{1,u_1,0}$ and $\hat{M}_{1,u_2,0}$. The formula $\hat{M}_{1,u,v}$ introduced by Hosking (1986) will be used here. The formula $\hat{M}_{1,u,v}$ is based on the order complete sample $x_1 < x_2 ... < x_n$ of size n. It is given by;

$$\hat{M}_{1,u,v} = \sum_{i=1}^{n} x_{(i)} p_i^u (1 - p_i)^v \qquad (4.8)$$

where $x_{(i)}$ is the i th observation in the ordered sample;

and $$p_i = \frac{i - 0.35}{n}.$$

From equation (3.9), the sample estimators of $\hat{M}_{1,u_2,0}$ are given by:

$$\hat{M}_{u_1} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_1}.$$

and $$\hat{M}_{u_2} = \frac{1}{n}\sum_{i=1}^{n} x_i \left(\frac{i-0.35}{n}\right)^{u_2}.$$

Therefore the estimation of GPWM for the shape parameter α, say $\hat{\alpha}$ will be,

$$\hat{M}_{1,u_2,0}\left[\beta(u_1+1,1-\frac{1}{\alpha}) - \frac{1}{u_1+1}\right] - \hat{M}_{1,u_1,0}\left[\beta(u_2+1,1-\frac{1}{\alpha}) - \frac{1}{u_2+1}\right] = 0 \tag{4.9}$$

Equation (4.9) will be solved for one unknown which is the shape parameter estimate $\hat{\alpha}$ from estimation of GPWM. The solution of this equation requires iterative techniques. Once $\hat{\alpha}$ is found, determination of the scale parameter estimated from GPWM, $\hat{\lambda}$ and the estimation GPWM for λ, say $\hat{\lambda}$ will be;

$$\hat{\lambda} = \left[\frac{\left[\beta(u_1+1,1-\frac{1}{\alpha}) - \frac{1}{u_1+1}\right]}{\hat{M}_{1,u_1,0}}\right]^{-1}.$$

5. Variances Covariance Matrix for the Estimators

The asymptotic theory usually provides a good approximation to the parameters of distribution for samples of size $n \geq 50$ and also for small samples of size $n = 10, 20$. Also, it provides some preliminary insight and motivation for the subsequent computer simulation experiment. Asymptotic variance of GPWM avoids the use of extensive computer simulation because it based on analytical expression. The asymptotic variance of λ and α are approximated by using the asymptotic variance covariance of the GPWM estimators $\hat{M}_{1,u_1,0}$ and $\hat{M}_{1,u_2,0}$, where $\lambda * = \phi(\hat{M}_{1,u_1,0} , \hat{M}_{1,u_2,0})$,

and $\alpha * = \varphi(\hat{M}_{1,u_1,0} , \hat{M}_{1,u_2,0})$

Hosking[12] mentioned there are two sets of the asymptotic variance. The first, if X is random variable with cumulative distribution function F and PWM is $M_{1,u,0}$ has the sample PWM $\hat{M}_{1,u,0}$, then the covariance matrix is;

$$A = \frac{1}{n}\left(A_{u_1,u_2}\right) \tag{5.1}$$

where

$$A_{u_1,u_2} = I_{u_1,u_2} + I_{u_2,u_1} \tag{5.2}$$

and I_{u_1,u_2} is the variance of PWM which is

$$I_{u_1,u_2} = \int_0^{\infty}\int_0^{y} \{F(x)\}^{u_1+1} \{F(y)^{u_2}\}\{1-F(y)\} \quad dx \quad dy \tag{5.3}$$

and $I_{u_2,u_1} = \int_0^{\infty}\int_0^{y} \{F(x)\}^{u_2+1} \{F(y)^{u_1}\}\{1-F(y)\} \quad dx \quad dy$ (5.4)

The second if X is random variable with cumulative distribution function F and PWM is $M_{1,0,v}$ has the sample PWM $\hat{M}_{1,0,v}$, then the covariance matrix is;

$$A = \frac{1}{n}\left(A_{v_1,v_2}\right) \tag{5.5}$$

where $A_{v_1,v_2} = I_{v_1,v_2} + I_{v_2,v_1}$ (5.6)

and I_{v_1,v_2} is the variance of PWM which is

$$I_{v_1 v_2} = \iint_{x<y} \left(1-F(x)\right)^{v_1}\left(1-F(y)\right)^{v_2+1} F(x) \; dx \; dy \tag{5.7}$$

and $I_{v_2 v_1} = \iint_{x<y} \left(1-F(x)\right)^{v_2}\left(1-F(y)\right)^{v_1+1} F(x) \; dx \; dy$ (5.8)

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تقدير معالم توزيع لوماكس اعتمادا على طريقة العزوم الاحتمالية المرجحة العامه

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المستخلص: في عام ١٩٧٩م قدمت طريقة لتقدير معالم التوزيعات وأطلق عليها اسم طريقة العزوم الاحتمالية المرجحة، وأثبت فيها إمكانية إيجاد صيغ صريحة وواضحة للتعبير عن معالم التوزيعات، التي لا يمكن التعبير عنها إلا من خلال دالة التوزيع التجميعية المعكوسة، وكذلك يمكن تطبيق هذه الطريقة للتوزيعات التي يمكن إيجاد الشكل المعكوس لها مثل Weibull, generalized Pareto, Log logistic.

أما في عام 2001 فقد قدم فصل جديد من العزوم الاحتمالية المرجحة أطلق عليه العزوم الاحتمالية العامة المرجحة، حيث تبين من خلال تعريف العزوم الاحتمالية المرجحة، أنه لا يشترط أن يكون الأس الترتيبي للعزوم الاحتمالية أعداداً صحيحة موجبة، بل يمكن أن يكون الأس لإعداد غير صحيحة وأقل من الواحد.

وعند تطبيق العزوم الاحتمالية العامة المرجحة لتوزيع باريتو العام، لوحظ أن هذه الطريقة أعطت أفضل المقدرات، مقارنة بالطرق الأخرى، حيث أنه عند مقارنة أداء المقدرات التي تم الحصول عليها بطريقة العزوم الاحتمالية المرجحة، وطريقة العزوم، وجد أن طريقة العزوم الاحتمالية العامة المرجحة قد أعطت أقل متوسط مربعات الخطأ من طريقة العزوم.

إن الهدف من هذه الدراسة هو تقدير معالم توزيع لوماكس، باستخدام طريقة العزوم الاحتمالية العامة المرجحة .