CHARACTERIZATION FROM EXPONENTIATED GAMMA DISTRIBUTION BASED ON RECORD VALUES

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ABSTRACT

In this paper, we study the lower record values from an exponentiated gamma distribution and derive explicit expressions for the single, product, triple and quadruple moments. We also, establish recurrence relations for the single, product, triple and quadruple moments and moment generating function.

Key words

Lower record values; Exponentiated gamma distribution; Recurrence relations; Single moments; Product moments; Triple moments; Quadruple moments; Moment generating function.

1. Introduction

Record values are important in many real-life situations involving data relating to weather, sports, economics, and life-tests. The statistical study of record values have been pursued in different directions by several authors; for example, see, Arnold *et al.*(1992, 1998) and Ahsanullah (1995). For the Rayleigh and Weibull distributions by Balakrishnan and chan (1993). Also, Sultan *et al.* (2002) derived moments from generalized power function based on record values. Balakrishnan and Ahsanullah (1994) and Al-Zaid and Ahsanullah (2003) have established some recurrence relations for single and product moments of record values from Lomax and Gumbel distributions respectively. Pawlas and Szynal (1999) dealt with Pareto, generalized Pareto and Burr distributions. Also, general recurrence relations based on upper record values was established by Mohie El-Din *et al.* (2000).

Now, let $\{X_n, n \geq 1\}$ be an infinite sequence of i.i.d. random variables from an absolutely continuous distribution function F, and probability density function (p.d.f.) f. Let $X_{i:j}$ denote the i^{th} order statistic of the random sample $X_1, X_2, ..., X_j$, and $F_{i:j}$ be its cumulative distribution function (c.d.f.). Let $T_k = \min\{X_1, X_2, ..., X_k\}, k \geq 1.$ We say that X_j is a lower record value of this sequence if $T_j < T_{j-1}, j \geq 2.$ By definition, X_1 is a record value. Let $L(n) = \min\{j : j > L(n-1), X_j < X_{L(n-1)}\}, n \geq 2$ with $L(1) = 1.$ Then $X_{L(n)}, n \geq 1,$ denotes the sequence of lower record values. From the above definition, the sequence of record statistics can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations.

Consider the exponentiated gamma (*EG*) distribution with p.d.f. and c.d.f., respectively,

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$$f(x)=\theta x e^{-x}[1-e^{-x}(x+1)]^{\theta-1},\ \ x>0,\theta>0, \tag{1.1}$$

and

$$F(x)=[1-e^{-x}(x+1)]^{\theta},\ x>0,\theta>0, \tag{1.2}$$

for details about this distribution, see Shawky and Bakoban (2006).

In this paper, we consider the lower record values from an exponentiated gamma distribution. In section 2, we derive explicit expressions for the single, product, triple and quadruple moments. We also, establish recurrence relations for the single, product, triple and quadruple moments in section 3. Finally we derive the single, product, triple and quadruple moment generating function (MGF) and recurrence relations for the single one in section 4.

2. Moments of Lower Record Values

Let $X_{L(1)}, X_{L(2)},\ldots, X_{L(n)}$ be the first n lower record values from the *EG* distribution given in (1.1). Then the single, double, triple and quadruple moments of lower record values are given as follows.

2.1 Single moments

The p.d.f. of the n^{th} lower record value $X_{L(n)}$ is given by (Ahsanullah (1995))

$$f_n(x)=\frac{1}{\Gamma(n)}[-\log F(x)]^{n-1} f(x),\qquad x>0,\ n=1,2,\ldots, \tag{2.1}$$

where $f(.)$ and $F(.)$ are given, respectively, by (1.1) and (1.2).

The single moments of the n^{th} lower record value, $E(X^{a}_{L(n)})$, denoted by $\mu_n^{(a)}, n=1,2,\ldots$ and $a=0, 1, 2, \ldots$, is given by

$$\mu_n^{(a)}=\frac{1}{\Gamma(n)}\int_0^\infty x^a[-\log F(x)]^{n-1} f(x)dx. \tag{2.2}$$

The exact explicit expression for the single moments of the n^{th} lower record value $X_{L(n)}$ from *EG* distribution is given by the following theorem.

Theorem 1

For n =1, 2, …, $a\geq 0$ and θ is a real value, then

$$\mu_n^{(a)}=\frac{\theta^n}{\Gamma(n)}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{n-1+i+k}(-1)^k a_i(n-1)\binom{\theta-1}{k}\binom{n-1+i+k}{j}\frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}}, \tag{2.3}$$

where $a_i(n-1)$ is the coefficient of $e^{-(n-1+i)x}(x+1)^{n-1+i}$ in the expansion of $[\sum_{i=1}^{\infty}\frac{e^{-ix}(x+1)^i}{i}]^{n-1}$ (see, the Appendix).

Proof

From (2.2) and (1.2) we get

$$\mu_n^{(a)}=\frac{\theta^{n-1}}{\Gamma(n)}\int_0^\infty x^a\{-\log[1-e^{-x}(x+1)]\}^{n-1} f(x)dx. \tag{2.4}$$

Using the logarithmic expansion we get

$$\mu_n^{(a)}=\frac{\theta^{n-1}}{\Gamma(n)}\int_0^\infty x^a[\sum_{i=1}^{\infty}\frac{e^{-ix}(x+1)^i}{i}]^{n-1} f(x)dx$$

$$\mu_n^{(a)} = \frac{\theta^{n-1}}{\Gamma(n)} \sum_{i=0}^{\infty} a_i(n-1) \int_0^{\infty} x^a e^{-(n-1+i)x} (x+1)^{n-1+i} f(x) dx,$$

where $a_i(n-1)$ is defined in the Appendix, from (1.1) we have

$$\mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} a_i(n-1) \int_0^{\infty} x^{a+1} e^{-(n+i)x} (x+1)^{n-1+i} [1-e^{-x}(x+1)]^{\theta-1} dx,$$

From the Binomial theorem we find

$$\mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k a_i(n-1) \binom{\theta-1}{k} \int_0^{\infty} x^{a+1} e^{-(n+i+k)x} (x+1)^{n-1+i+k} dx,$$

Again, from the Binomial theorem we get

$$\mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n-1+i+k} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \int_0^{\infty} x^{a+j+1} e^{-(n+i+k)x} dx,$$

Since the integration is a complete gamma function, then, the theorem is proved.
If θ is a positive integer number, then the relation (2.3) takes the form

$$\mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\theta-1} \sum_{j=0}^{n-1+i+k} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}}. \qquad (2.5)$$

The single moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.5) by setting $\theta = 1$.

2.2 Double moments

The joint p.d.f. of $X_{L(m)}$ and $X_{L(n)}, 1 \le m < n$ is given by (Ahsanullah (1995))

$$f_{m,n}(x,y) = \frac{1}{\Gamma(m)\Gamma(n-m)} [-\log F(x)]^{m-1} \frac{f(x)}{F(x)} [-\log F(y) + \log F(x)]^{n-m-1} f(y), \quad x > y > 0, \qquad (2.6)$$

where $f(.)$ and $F(.)$ are given, respectively, by (1.1) and (1.2).

The double moments of the lower record values, $E(X_{L(m)}^a X_{L(n)}^b)$, denoted by $\mu_{m,n}^{(a,b)}, m,n = 1,2,\dots,m<n$ and a,b =0, 1, 2, …, is given by

$$\mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^{\infty} \int_y^{\infty} x^a y^b [-\log F(x)]^{m-1} [-\log F(y) + \log F(x)]^{n-m-1} \frac{f(x)}{F(x)} f(y)\, dx dy. \qquad (2.7)$$

The exact explicit expression for the double moments of lower record values from an *EG* distribution is given by the following theorem.

Theorem 2

For *m, n* =1, 2, …, $m < n$, *a*, $b \ge 0$ and θ is a real value, then

$$\mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{t=0}^{\infty} \sum_{j=0}^{n-s+i+t-2a+j+1} \sum_{p=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_1=0}^{i_1+s+k} (-1)^{n-m+s+k-1} a_i(n-s-2)$$
$$\times a_{i_1}(s) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \binom{\theta-1}{k} \binom{i_1+s+k}{j_1} \frac{\Gamma(a+j+2)}{(n-s+i+t-1)^{a+j+2-p}}$$
$$\times \frac{\Gamma(b+p+j_1+2)}{p!(n+t+i+i_1+k)^{b+p+j_1+2}}, \qquad (2.8)$$

where $a_i(n-s-2)$ and $a_{i_1}(s)$ are defined as the Appendix.

Proof

Relation (2.7) can be written as

$$\mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty y^b I(y) f(y)dy, \tag{2.9}$$

Where

$$I(y) = \int_y^\infty x^a[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}dx \tag{2.10}$$

$$= \sum_{s=0}^{n-m-1}(-1)^{n-m-1}\binom{n-m-1}{s}[\log F(y)]^s\int_y^\infty x^a[-\log F(x)]^{n-s-2}\frac{f(x)}{F(x)}dx$$

$$= \sum_{s=0}^{n-m-1}(-1)^{n-m-1}\theta^{n-s-2}\binom{n-m-1}{s}[\log F(y)]^s\int_y^\infty x^a\frac{f(x)}{F(x)}\{-\log[1-e^{-x}(x+1)]\}^{n-s-2}dx$$

$$= \sum_{s=0}^{n-m-1}\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{n-s+i+t-2}(-1)^{n-m-1}\theta^{n-s-1}a_i(n-s-2)\binom{n-m-1}{s}\binom{n-s+i+t-2}{j}$$

$$\times[\log F(y)]^s\int_y^\infty x^{a+j+1}e^{-(n-s+i+t-1)x}dx.$$

The last integration is an incomplete gamma function which can be defined as

$$\frac{\lambda^r}{\Gamma(r)}\int_y^\infty x^{r-1}e^{-\lambda x}dx = \sum_{j=0}^{r-1}\frac{e^{-\lambda y}(\lambda y)^j}{j!},$$

r is an integer number, then

$$I(y) = \sum_{s=0}^{n-m-1}\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{n-s+i+t-2}\sum_{p=0}^{a+j+1}(-1)^{n-m-1}\theta^{n-s-1}a_i(n-s-2)\binom{n-m-1}{s}$$

$$\times\binom{n-s+i+t-2}{j}[\log F(y)]^s\frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}}y^p e^{-(n-s+i+t-1)y}. \tag{2.11}$$

Substituting $I(y)$ in (2.9), we get

$$\mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{n-s+i+t-2}\sum_{p=0}^{a+j+1}(-1)^{n-m-1}\theta^{n-s-1}a_i(n-s-2)$$

$$\times\binom{n-m-1}{s}\binom{n-s+i+t-2}{j}\frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}}$$

$$\times\int_0^\infty y^{b+p}[\log F(y)]^s e^{-(n-s+i+t-1)y}f(y)dy$$

$$= \frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{i=0}^{\infty}\sum_{t=0}^{\infty}\sum_{j=0}^{n-s+i+t-2}\sum_{p=0}^{a+j+1}(-1)^{n-m-1}a_i(n-s-2)$$

$$\times\binom{n-m-1}{s}\binom{n-s+i+t-2}{j}\frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}}$$

$$\times\int_0^\infty y^{b+p+1}[-\sum_{i=1}^{\infty}\frac{e^{-iy}(y+1)^i}{i}]^s e^{-(n-s+i+t)y}[1-e^{-y}(y+1)]^{\theta-1}dy$$

$$\mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{t=0}^{\infty} \sum_{j=0}^{n-s+i+t-2} \sum_{p=0}^{a+j+1} \sum_{i_1=0}^{\infty} (-1)^{n-m-1+s} a_i(n-s-2) a_{i_1}(s)$$

$$\times \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}}$$

$$\times \int_0^\infty y^{b+p+1} (y+1)^{i_1+s} e^{-(n+i+i_1+t)y} [1-e^{-y}(y+1)]^{\theta-1} dy,$$

where $a_{i_1}(s)$ is defined as in the Appendix.

$$\mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{t=0}^{\infty} \sum_{j=0}^{n-s+i+t-2} \sum_{p=0}^{a+j+1} \sum_{i_1=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{n-m+s+k-1} a_i(n-s-2)$$

$$\times a_{i_1}(s) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \binom{\theta-1}{k} \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}}$$

$$\times \int_0^\infty y^{b+p+1} (y+1)^{i_1+s+k} e^{-(n+i+i_1+t+k)y} dy$$

$$= \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{t=0}^{\infty} \sum_{j=0}^{n-s+i+t-2} \sum_{p=0}^{a+j+1} \sum_{i_1=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_1=0}^{i_1+s+k} (-1)^{n-m+s+k-1}$$

$$\times a_i(n-s-2) a_{i_1}(s) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \binom{\theta-1}{k} \binom{i_1+s+k}{j_1}$$

$$\times \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+2-p}} \int_0^\infty y^{b+p+j_1+1} e^{-(n+i+i_1+t+k)y} dy.$$

Hence the theorem is proved.

If θ is a positive integer number, then the relation (2.8) becomes

$$\mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{t=0}^{\infty} \sum_{j=0}^{n-s+i+t-2} \sum_{p=0}^{a+j+1} \sum_{i_1=0}^{\infty} \sum_{k=0}^{\theta-1} \sum_{j_1=0}^{i_1+s+k} (-1)^{n-m+s+k-1}$$

$$\times a_i(n-s-2) a_{i_1}(s) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \binom{\theta-1}{k} \binom{i_1+s+k}{j_1}$$

$$\times \frac{\Gamma(a+j+2)}{(n-s+i+t-1)^{a+j+2-p}} \frac{\Gamma(b+p+j_1+2)}{p!(n+t+i+i_1+k)^{b+p+j_1+2}}. \quad (2.12)$$

The double moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.12) by setting $\theta = 1$.

2.3 Triple moments

The joint p.d.f. of $X_{L(m)}, X_{L(n)}$ and $X_{L(l)}, 1 \le m < n < l$ is given by (Ahsanullah (1995))

$$f_{m,n,l}(x,y,z) = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} [-\log F(x)]^{m-1} [-\log F(y) + \log F(x)]^{n-m-1}$$

$$\times [-\log F(z) + \log F(y)]^{l-n-1} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} f(z), \; x > y > z > 0, \quad (2.13)$$

where $f(.)$ and $F(.)$ are given, respectively, by (1.1) and (1.2).

The triple moments of the lower record values, $E(X_{L(m)}^a X_{L(n)}^b X_{L(l)}^c)$, denoted by $\mu_{m,n,l}^{(a,b,c)}, m,n,l = 1,2,...,m < n < l$ and $a,b,c = 0,\ 1,\ 2,\ \ldots$, is given by

$$\mu_{m,n,l}^{(a,b,c)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\int_0^\infty\int_z^\infty\int_y^\infty x^a y^b z^c[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}f(z)dxdydz. \quad (2.14)$$

The exact explicit expression for the triple moments of lower record values from *EG* distribution is given by the following theorem.

Theorem 3

For *m, n, l* =1, 2, …, $m<n<l$, *a*, *b*, $c\geq 0$ and θ is a real value, then

$$\mu_{m,n,l}^{(a,b,c)}=\frac{\theta^l}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2a+j_1+1}\sum_{p_1=0}^{l-n-1}\sum_{s_2=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}$$
$$\times\sum_{p_2=0}^{b+p_1+j_2+1}\sum_{k=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{i_3+s_2+k}(-1)^{l-m+s_1+s_2+k-2}a_{i_1}(n-s_1-2)\,a_{i_2}(s_1+l-n-1-s_2)\,a_{i_3}(s_2)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}\binom{\theta-1}{k}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}$$
$$\times\binom{i_3+s_2+k}{j_3}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}$$
$$\times\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(c+p_2+j_3+2)}{(t_1+t_2+i_1+i_2+i_3+l+k)^{c+p_2+j_3+2}}, \quad (2.15)$$

where $a_{i_1}(n-s_1-2), a_{i_2}(s_1+l-n-1-s_2)$ and $a_{i_3}(s_2)$ are defined in the Appendix.

Proof

Relation (2.14) can be written as

$$\mu_{m,n,l}^{(a,b,c)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\int_0^\infty\int_z^\infty y^b z^c I(y)[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(y)}{F(y)}f(z)dydz, \quad (2.16)$$

where $I(y)$ is defined in (2.10).

By integrating $I(y)$ as it was shown in theorem 2, then by substituting (2.11) into (2.16) we get

$$\mu_{m,n,l}^{(a,b,c)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2a+j_1+1}\sum_{p_1=0}(-1)^{n-m-1}\theta^{n-s_1-1}a_{i_1}(n-s_1-2)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}\int_0^\infty z^c J(z)f(z)dz, \quad (2.17)$$

where

$$J(z)=\int_z^\infty y^{b+p_1}[\log F(y)]^{s_1}e^{-(n-s_1+i_1+t_1-1)y}[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(y)}{F(y)}dy. \quad (2.18)$$

We can find $J(z)$ in (2.18) in the similar way that was shown in theorem 2, then by substituting the result into (2.17) we get

$$\mu_{m,n,l}^{(a,b,c)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2}\sum_{p_1=0}^{a+j_1+1}\sum_{s_2=0}^{l-n-1}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}\sum_{p_2=0}^{b+p_1+j_2+1}\theta^{l-s_2-1}$$
$$\times(-1)^{l-m+s_1-2}a_{i_1}(n-s_1-2)\,a_{i_2}(s_1+l-n-1-s_2)\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}$$
$$\times\binom{l-n-1}{s_2}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}$$
$$\times\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\int_0^\infty z^{c+p_2}e^{-(l-s_2+i_1+i_2+t_1+t_2-1)z}[\log F(z)]^{s_2}f(z)dz,$$

In the same way of theorem 2, we can get (2.15). Hence the theorem is proved.
If θ is a positive integer number, then the relation (2.15) becomes

$$\mu_{m,n,l}^{(a,b,c)}=\frac{\theta^l}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2}\sum_{p_1=0}^{a+j_1+1}\sum_{s_2=0}^{l-n-1}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}$$
$$\times\sum_{p_2=0}^{b+p_1+j_2+1}\sum_{k=0}^{\theta-1}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{i_3+s_2+k}(-1)^{l-m+s_1+s_2+k-2}a_{i_1}(n-s_1-2)\,a_{i_2}(s_1+l-n-1-s_2)\,a_{i_3}(s_2)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}\binom{\theta-1}{k}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}$$
$$\times\binom{i_3+s_2+k}{j_3}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}$$
$$\times\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(c+p_2+j_3+2)}{(t_1+t_2+i_1+i_2+i_3+l+k)^{c+p_2+j_3+2}},\qquad(2.19)$$

The triple moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.19) by setting $\theta=1$.

2.4 Quadruple moments

The joint p.d.f. of $X_{L(m)},X_{L(n)},X_{L(l)}$ and $X_{L(v)}, 1\le m<n<l<v$ is given by (Ahsanullah (1995))

$$f_{m,n,l,v}(x,y,z,w)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w),$$
$$x>y>z>w>0,\qquad(2.20)$$

where $f(.)$ and $F(.)$ are given, respectively, by (1.1) and (1.2).

The quadruple moments of the lower record values, $E(X_{L(m)}^{a}X_{L(n)}^{b}X_{L(l)}^{c}X_{L(v)}^{d})$, denoted by $\mu_{m,n,l,v}^{(a,b,c,d)}, m,n,l,v=1,2,\ldots,m<n<l<v$ and $a,b,c,d=0,\ 1,\ 2,\ \ldots$, is given by

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\int_0^\infty\int_z^\infty\int_y^\infty x^a y^b z^c w^d[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w)dxdydzdw. \quad (2.21)$$

The exact explicit expression for the quadruple moments of lower record values from *EG* distribution is given by the following theorem.

Theorem 4

For *m, n, l, v* =1, 2, …, $m<n<l<v$, $a, b, c, d\geq 0$ and θ is a real value, then

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{\theta^v}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2}\sum_{p_1=0}^{a+j_1+1}\sum_{s_2=0}^{l-n-1}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}\sum_{p_2=0}^{b+p_1+j_2+1}$$
$$\times\sum_{s_3=0}^{v-l-1}\sum_{i_3=0}^{\infty}\sum_{t_3=0}^{\infty}\sum_{j_3=0}^{v-l-1+s_2-s_3+i_3+t_3}\sum_{p_3=0}^{c+p_2+j_3+1}\sum_{k=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{j_4=0}^{i_4+s_3+k}(-1)^{v-m+s_1+s_2+s_3+k-3}a_{i_1}(n-s_1-2)$$
$$\times a_{i_2}(s_1+l-n-1-s_2)\,a_{i_3}(v-l-1+s_2-s_3)\,a_{i_4}(s_3)\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}$$
$$\times\binom{\theta-1}{k}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}\binom{v-l-1}{s_3}\binom{v-l-1-s_3+s_2+i_3+t_3}{j_3}\binom{i_4+s_3+k}{j_4}$$
$$\times\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(c+p_2+j_3+2)}{p_3!(v-s_3+t_1+t_2+t_3+i_1+i_2+i_3-1)^{c+p_2+j_3+2-p_3}}$$
$$\times\frac{\Gamma(d+p_3+j_4+2)}{(k+v+t_1+t_2+t_3+i_1+i_2+i_3+i_4)^{d+p_3+j_4+2}}, \quad (2.22)$$

where $a_{i_1}(n-s_1-2), a_{i_2}(s_1+l-n-1-s_2), a_{i_3}(v-l-1+s_2-s_3)$ and $a_{i_4}(s_3)$ are defined in the Appendix.

Proof

Relation (2.21) can be written as

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\int_0^\infty\int_w^\infty\int_z^\infty y^b z^c w^d I(y)[-\log F(z)+\log F(y)]^{l-n-1}$$
$$\times[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w)dydzdw, \quad (2.23)$$

where $I(y)$ is defined in (2.10).

By integrating $I(y)$ as it was shown in theorem 2, then by substituting (2.11) into (2.23) we get

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2}\sum_{p_1=0}^{a+j_1+1}\theta^{n-s_1-1}(-1)^{n-m-1}a_{i_1}(n-s_1-2)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}\int_0^\infty\int_w^\infty z^c w^d J(z)$$
$$\times[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(z)}{F(z)}f(w)dzdw, \quad (2.24)$$

where $J(z)$ is defined in (2.18). By integrating $J(z)$ as it was shown in theorem 3, then by substituting the result into (2.24) we get

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2a+j_1+1}\sum_{p_1=0}^{l-n-1}\sum_{s_2=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}$$
$$\times\sum_{p_2=0}^{b+p_1+j_2+1}(-1)^{l-m+s_1-2}\theta^{l-s_2-1}a_{i_1}(n-s_1-2)a_{i_2}(s_1+l-n-1-s_2)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}$$
$$\times\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}$$
$$\times\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}\int_0^\infty w^d K(w)f(w)dw, \tag{2.25}$$

where

$$K(w)=\int_w^\infty z^{c+p_2}e^{-(l-s_2+i_1+i_2+t_1+t_2-1)z}[\log F(z)]^{s_2}[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(z)}{F(z)}dz, \tag{2.26}$$

We can find $K(w)$ in (2.26) in the similar way that was shown in theorem 2, then by substituting the result into (2.25) we get

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2a+j_1+1}\sum_{p_1=0}^{l-n-1}\sum_{s_2=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}\sum_{p_2=0}^{b+p_1+j_2+1}\sum_{s_3=0}^{v-l-1}$$
$$\times\sum_{i_3=0}^{\infty}\sum_{t_3=0}^{\infty}\sum_{j_3=0}^{v-l-1+s_2-s_3+i_3+t_3}\sum_{p_3=0}^{c+p_2+j_3+1}\theta^{v-s_3-1}(-1)^{v-m+s_1+s_2-3}a_{i_1}(n-s_1-2)a_{i_2}(s_1+l-n-1-s_2)a_{i_3}(v-l-1+s_2-s_3)$$
$$\times\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}\binom{v-l-1}{s_3}$$
$$\times\binom{v-l-1-s_3+s_2+i_3+t_3}{j_3}\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(c+p_2+j_3+2)}{p_3!(v-s_3+t_1+t_2+t_3+i_1+i_2+i_3-1)^{c+p_2+j_3+2-p_3}}$$
$$\times\int_0^\infty w^{d+p_3}[\log F(w)]^{s_3}e^{-(v-s_3+t_1+t_2+t_3+i_1+i_2+i_3-1)w}f(w)\,dw.$$

In the same way of theorem 2, we can get (2.22). Hence the theorem is proved.
If θ is a positive integer number, then the relation (2.22) becomes

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{\theta^v}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{n-m-1}\sum_{i_1=0}^{\infty}\sum_{t_1=0}^{\infty}\sum_{j_1=0}^{n-s_1+i_1+t_1-2a+j_1+1}\sum_{p_1=0}^{l-n-1}\sum_{s_2=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{t_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_1-s_2+i_2+t_2}\sum_{p_2=0}^{b+p_1+j_2+1}\sum_{s_3=0}^{v-l-1}$$
$$\times\sum_{i_3=0}^{\infty}\sum_{t_3=0}^{\infty}\sum_{j_3=0}^{v-l-1+s_2-s_3+i_3+t_3}\sum_{p_3=0}^{c+p_2+j_3+1}\sum_{k=0}^{\theta-1}\sum_{i_4=0}^{\infty}\sum_{j_4=0}^{i_4+s_3+k}(-1)^{v-m+s_1+s_2+s_3+k-3}a_{i_1}(n-s_1-2)a_{i_2}(s_1+l-n-1-s_2)$$
$$\times a_{i_3}(v-l-1+s_2-s_3)a_{i_4}(s_3)\binom{n-m-1}{s_1}\binom{n-s_1+i_1+t_1-2}{j_1}\binom{l-n-1}{s_2}$$
$$\times\binom{\theta-1}{k}\binom{l-n-1+s_1-s_2+i_2+t_2}{j_2}\binom{v-l-1}{s_3}\binom{v-l-1-s_3+s_2+i_3+t_3}{j_3}\binom{i_4+s_3+k}{j_4}$$
$$\times\frac{\Gamma(a+j_1+2)}{p_1!(n-s_1+i_1+t_1-1)^{a+j_1+2-p_1}}\frac{\Gamma(b+p_1+j_2+2)}{p_2!(t_1+t_2+i_1+i_2+l-s_2-1)^{b+p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(c+p_2+j_3+2)}{p_3!(v-s_3+t_1+t_2+t_3+i_1+i_2+i_3-1)^{c+p_2+j_3+2-p_3}}$$
$$\times\frac{\Gamma(d+p_3+j_4+2)}{(k+v+t_1+t_2+t_3+i_1+i_2+i_3+i_4)^{d+p_3+j_4+2}}, \tag{2.27}$$

The quadruple moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.27) by setting $\theta = 1$.

3. Recurrence Relations Between Moments

For the exponentiated gamma distribution, it easily observed that

$$F(x) = \frac{x^{-1}}{\theta}[e^{x} - (x+1)]f(x) \tag{3.1}$$

By using this relation, we establish below some recurrence relations satisfied by the single, product, triple and quadruple moments of record values.

3.1 Recurrence relations between single moments

Theorem 5

For $n = 1,2,...,$ $a = 0,1,2,...,$ and θ is a real value, then

$$\mu_{n}^{(a)} - (1 + \frac{a}{2\theta})\mu_{n+1}^{(a)} = \frac{a}{\theta}\sum_{i=3}^{\infty} \frac{1}{i!}\mu_{n+1}^{(a+i-2)}. \tag{3.2}$$

Proof

Let us replay equation (2.2)

$$\mu_{n}^{(a)} = \frac{1}{\Gamma(n)}\int_{0}^{\infty} x^{a}[-\log F(x)]^{n-1} f(x)dx.$$

Upon integrating by parts, we obtain

$$\mu_{n}^{(a)} = \frac{1}{\Gamma(n+1)}\int_{0}^{\infty} [-\log F(x)]^{n}\{ax^{a-1}F(x) + x^{a} f(x)\}dx$$

$$\mu_{n}^{(a)} - \mu_{n+1}^{(a)} = \frac{a}{\Gamma(n+1)}\int_{0}^{\infty} x^{a-1}[-\log F(x)]^{n} F(x)dx$$

$$= \frac{a}{\theta\Gamma(n+1)}\int_{0}^{\infty} x^{a-2}[e^{x} - (x+1)][-\log F(x)]^{n} f(x)dx \qquad \text{(using (3.1))}$$

$$= \frac{a}{\theta\Gamma(n+1)}\{\int_{0}^{\infty} x^{a-2}e^{x}[-\log F(x)]^{n} f(x)dx - \int_{0}^{\infty} x^{a-2}(x+1)[-\log F(x)]^{n} f(x)dx$$

$$= \frac{a}{\theta\Gamma(n+1)}\{\sum_{i=0}^{\infty}\frac{1}{i!}\int_{0}^{\infty} x^{a+i-2}[-\log F(x)]^{n} f(x)dx$$

$$- \int_{0}^{\infty} x^{a-1}[-\log F(x)]^{n} f(x)dx - \int_{0}^{\infty} x^{a-2}[-\log F(x)]^{n} f(x)dx\}$$

$$= \frac{a}{\theta}\{\sum_{i=0}^{\infty}\frac{1}{i!}\mu_{n+1}^{(a+i-2)} - \mu_{n+1}^{(a-1)} - \mu_{n+1}^{(a-2)}\}$$

After some simplification, we get (3.2). Hence the theorem is proved.

3.2 Recurrence relations between double moments

Theorem 6

For $m < n, m,n = 1,2,...$ and $a,b = 0,1,2,...,$ and θ is a real value, then

$$\mu_{m,n}^{(a,b)} - \mu_{m+1,n}^{(a,b)} = \frac{a}{\theta}\sum_{i=2}^{\infty} \frac{1}{i!}\mu_{m+1,n+1}^{(a+i-2,b)}. \tag{3.3}$$

Proof

Let us consider (2.9), then by integrating $I(y)$, we get

$$I(y)=\frac{n-m-1}{m}\int_y^\infty x^a[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-2}\frac{f(x)}{F(x)}dx$$
$$+\frac{a}{m}\int_y^\infty x^{a-1}[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-1}dx$$

From (3.1) we get

$$I(y)=\frac{n-m-1}{m}\int_y^\infty x^a[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-2}\frac{f(x)}{F(x)}dx$$
$$+\frac{a}{m\theta}\int_y^\infty x^{a-1}[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}x^{-1}[e^x-(x+1)]dx \qquad (3.4)$$

By substituting $I(y)$ in (2.9), we get

$$\mu_{m,n}^{(a,b)}=\frac{1}{\Gamma(m+1)\Gamma(n-m-1)}\int_0^\infty\int_y^\infty x^a y^b[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-2}$$
$$\times\frac{f(x)}{F(x)}f(y)dxdy+\frac{a}{\theta\Gamma(m+1)\Gamma(n-m)}\int_0^\infty\int_y^\infty x^{a-2}y^b[e^x-(x+1)]$$
$$\times[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}f(y)dxdy.$$

$$\mu_{m,n}^{(a,b)}-\mu_{m+1,n}^{(a,b)}=\frac{a}{\theta\Gamma(m+1)\Gamma(n-m)}\{\sum_{i=0}^\infty\frac{1}{i!}\int_0^\infty\int_y^\infty x^{a+i-2}y^b[-\log F(x)]^m$$
$$\times[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}f(y)dxdy$$
$$-\int_0^\infty\int_y^\infty x^{a-2}y^b(x+1)[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}f(y)dxdy.$$

In the same manner of theorem 5, we can get

$$\mu_{m,n}^{(a,b)}-\mu_{m+1,n}^{(a,b)}=\frac{a}{\theta}\{\sum_{i=0}^\infty\frac{1}{i!}\mu_{m+1,n+1}^{(a+i-2,b)}-\mu_{m+1,n+1}^{(a-1,b)}-\mu_{m+1,n+1}^{(a-2,b)}\}.$$

After some simplification, we get (3.3). Hence the theorem is proved.

3.3 Recurrence relations between triple moments

Theorem 7

For $m<n<l, m,n,l=1,2,...$ and $a,b,c=0,1,2,...,$ and θ is a real value, then

$$\mu_{m,n,l}^{(a,b,c)}-\mu_{m+1,n,l}^{(a,b,c)}=\frac{a}{\theta}\sum_{i=2}^\infty\frac{1}{i!}\mu_{m+1,n+1,l+1}^{(a+i-2,b,c)}. \qquad (3.5)$$

Proof

Let us consider (2.16), then by integrating $I(y)$ in (2.10) by parts as it was shown in theorem 6 and substituting the result (3.4) in (2.16), we get

$$\mu_{m,n,l}^{(a,b,c)}=\frac{1}{\Gamma(m+1)\Gamma(n-m-1)\Gamma(l-n)}\int_0^\infty\int_z^\infty\int_y^\infty x^a y^b z^c[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-2}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}f(z)dxdydz$$
$$+\frac{a}{\theta\Gamma(m+1)\Gamma(n-m)}\int_0^\infty\int_z^\infty\int_y^\infty x^{a-2}y^b z^c[e^x-(x+1)][-\log F(x)]^m$$
$$\times[-\log F(y)+\log F(x)]^{n-m-1}[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}f(z)\,dxdydz.$$

In the same manner of theorem 6, we can get

$$\mu_{m,n,l}^{(a,b,c)}-\mu_{m+1,n,l}^{(a,b,c)}=\frac{a}{\theta}\{\sum_{i=0}^{\infty}\frac{1}{i!}\mu_{m+1,n+1,l+1}^{(a+i-2,b,c)}-\mu_{m+1,n+1,l+1}^{(a-1,b,c)}-\mu_{m+1,n+1,l+1}^{(a-2,b,c)}\}.$$

After some simplification, we get (3.5). Hence the theorem is proved.

3.4 Recurrence relations between quadruple moments

Theorem 8

For $m<n<l<v, m,n,l,v=1,2,...$ and $a,b,c,d=0,1,2,...,$ and θ is a real value, then

$$\mu_{m,n,l,v}^{(a,b,c,d)}-\mu_{m+1,n,l,v}^{(a,b,c,d)}=\frac{a}{\theta}\sum_{i=2}^{\infty}\frac{1}{i!}\mu_{m+1,n+1,l+1,v+1}^{(a+i-2,b,c,d)}. \quad (3.6)$$

Proof

Let us consider (2.23), then by integrating $I(y)$ in (2.10) by parts as it was shown in theorem 6 and substituting the result (3.4) in (2.23), we get

$$\mu_{m,n,l,v}^{(a,b,c,d)}=\frac{1}{\Gamma(m+1)\Gamma(n-m-1)\Gamma(l-n)\Gamma(v-l)}\int_0^\infty\int_w^\infty\int_z^\infty\int_y^\infty x^a y^b z^c w^d[-\log F(x)]^m$$
$$\times[-\log F(y)+\log F(x)]^{n-m-2}[-\log F(z)+\log F(y)]^{l-n-1}$$
$$\times[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w)\,dxdydzdw$$
$$+\frac{a}{\theta\Gamma(m+1)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\int_0^\infty\int_w^\infty\int_z^\infty\int_y^\infty x^{a-2}y^b z^c w^d[e^x-(x+1)]$$
$$\times[-\log F(x)]^m[-\log F(y)+\log F(x)]^{n-m-1}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}[-\log F(w)+\log F(z)]^{v-l-1}$$
$$\times\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w)\,dxdydzdw.$$

In the same manner of theorem 6, we can get

$$\mu_{m,n,l,v}^{(a,b,c,d)}-\mu_{m+1,n,l,v}^{(a,b,c,d)}=\frac{a}{\theta}\{\sum_{i=0}^{\infty}\frac{1}{i!}\mu_{m+1,n+1,l+1,v+1}^{(a+i-2,b,c,d)}-\mu_{m+1,n+1,l+1,v+1}^{(a-1,b,c,d)}-\mu_{m+1,n+1,l+1,v+1}^{(a-2,b,c,d)}\}.$$

After some simplification, we get (3.6). Hence the theorem is proved.

4. Moment Generating Function

Let $X_{L(1)}, X_{L(2)},..,X_{L(n)}$ be the first n lower record values from the *EG* distribution given in (1.1). Then moment generating function (MGF)for the single, double, triple and quadruple moments of lower record values are given as follows.

4.1 Moment generating function for single moments

The MGF of the lower record value $X_{L(n)}$ denoted by $M_n(t)$ is given (see Mohie El-Din *et al*. (2000)) by

$$M_n(t) = E(e^{tX_{L(n)}}) = \int_0^\infty e^{tx} f_n(x)dx, \tag{4.1}$$

where $f_n(x)$ is defined in (2.1).

The exact explicit expression for the MGF for single moments of lower record values from *EG* distribution is given by the following theorem.

Theorem 9

For n =1, 2, …, $a \geq 0$ and θ is a real value, then

$$M_n^{(a)}(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n-1+i+k} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n-t+i+k)^{a+j+2}}. \tag{4.2}$$

where $a_i(n-1)$ is defined as the Appendix.

Proof

From (2.1) and (4.1), we get

$$M_n(t) = \frac{1}{\Gamma(n)} \int_0^\infty e^{tx} [-\log F(x)]^{n-1} f(x)dx. \tag{4.3}$$

By using (1.2), we get

$$M_n(t) = \frac{\theta^{n-1}}{\Gamma(n)} \int_0^\infty e^{tx} \{-\log[1-e^{-x}(x+1)]\}^{n-1} f(x)dx.$$

From the logarithmic expansion, we get

$$M_n(t) = \frac{\theta^{n-1}}{\Gamma(n)} \int_0^\infty e^{tx} [\sum_{i=1}^{\infty} \frac{e^{-ix}(x+1)^i}{i}]^{n-1} f(x)dx$$

$$= \frac{\theta^{n-1}}{\Gamma(n)} \sum_{i=0}^{\infty} a_i(n-1) \int_0^\infty e^{-(n-t-1+i)x} (x+1)^{n-1+i} f(x)dx,$$

where $a_i(n-1)$ is defined in the Appendix, from (1.1), we get

$$M_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} a_i(n-1) \int_0^\infty x e^{-(n-t+i)x} (x+1)^{n-1+i} [1-e^{-x}(x+1)]^{\theta-1} dx,$$

From the Binomial theorem, we get

$$M_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k a_i(n-1) \binom{\theta-1}{k} \int_0^\infty x e^{-(n-t+i+k)x} (x+1)^{n-1+i+k} dx,$$

Again from the Binomial theorem, we get

$$M_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n-1+i+k} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \int_0^\infty x^{j+1} e^{-(n-t+i+k)x} dx,$$

Since the integration is a complete gamma function, we get

$$M_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{n-1+i+k} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \frac{\Gamma(j+2)}{(n-t+i+k)^{j+2}}. \tag{4.4}$$

By differentiating both sides of (4.4) w.r.t. t, a times, we can easily obtain (4.2).

Note that by putting t=0 in (4.2), we get (2.3).

If θ is a positive integer number, then the relation (4.2) becomes

$$M_n^{(a)}(t)=\frac{\theta^n}{\Gamma(n)}\sum_{i=0}^{\infty}\sum_{k=0}^{\theta-1}\sum_{j=0}^{n-1+i+k}(-1)^k\, a_i(n-1)\binom{\theta-1}{k}\binom{n-1+i+k}{j}\frac{\Gamma(a+j+2)}{(n-t+i+k)^{a+j+2}}. \tag{4.5}$$

The MGF for single moments of lower record value from gamma distribution $G(2,1)$ can be obtained from (4.5) by setting $\theta=1$.

4.2 Recurrence relations for single MGF

Theorem 10

For n =1, 2, …, $a\geq 0$ and θ is a real value, then

$$M_n^{(a)}(t)+\frac{t-\theta}{\theta}M_{n+1}^{(a)}(t)+\frac{a}{\theta}M_{n+1}^{(a-1)}(t)=\frac{t}{\theta}E_{n+1}(X^{a-1}(e^{(t+1)X}-e^{tX}))+\frac{a}{\theta}.E_{n+1}(X^{a-2}(e^{(t+1)X}-e^{tX})). \tag{4.6}$$

Proof

Let us consider equation (4.3), upon integrating by parts, we obtain

$$M_n(t)=\frac{1}{\Gamma(n+1)}\int_0^\infty[-\log F(x)]^n\{te^{tx}F(x)+e^{tx}f(x)\}dx$$

$$M_n(t)-M_{n+1}(t)=\frac{t}{\Gamma(n+1)}\int_0^\infty e^{tx}[-\log F(x)]^n F(x)dx$$

$$=\frac{t}{\theta\Gamma(n+1)}\int_0^\infty e^{tx}x^{-1}[e^x-(x+1)][-\log F(x)]^n f(x)dx \qquad \text{(using (3.1))}$$

$$=\frac{t}{\theta\Gamma(n+1)}\{\int_0^\infty x^{-1}e^{(t+1)x}[-\log F(x)]^n f(x)dx-\int_0^\infty x^{-1}e^{tx}(x+1)[-\log F(x)]^n f(x)dx$$

$$=\frac{t}{\theta}E_{n+1}(\frac{e^{(t+1)X}}{X})-\frac{t}{\theta}[M_{n+1}(t)+E_{n+1}(\frac{e^{tX}}{X})].$$

By rearranging the last equation, we obtain

$$M_n(t)+\frac{t-\theta}{\theta}M_{n+1}(t)=\frac{t}{\theta}E_{n+1}(\frac{e^{(t+1)X}}{X}-\frac{e^{tX}}{X}). \tag{4.7}$$

By differentiating both sides of (4.7) w.r.t. t, a times, we can easily obtain (4.6).
Note that by putting t=0 in (4.6), we get (3.2).

4.3 Moment generating function for double moments

The joint MGF of $X_{L(m)}$ and $X_{L(n)}$, $E(e^{t_1X_{L(m)}+t_2X_{L(n)}})$, denoted by $M_{m,n}(t_1,t_2)$ is given by

$$M_{m,n}(t_1,t_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^x e^{t_1x+t_2y}f_{m,n}(x,y)dydx, \tag{4.8}$$

where $f_{m,n}(x,y)$ is defined in (2.6).

The exact explicit expression for the MGF for double moments of lower record values from *EG* distribution is given by the following theorem.

Theorem 11

For *m, n* =1, 2, …, $m < n$, and θ is a real value, then

$$M_{m,n}(t_1,t_2)=\frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{i_1+s+k}\sum_{p=j_1+2}^{\infty}\sum_{v=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{n-s+i_2+v-2}(-1)^{n-m-s+k-1}$$
$$\times a_{i_1}(s)a_{i_2}(n-s-2)\binom{n-m-1}{s}\binom{\theta-1}{k}\binom{i_1+s+k}{j_1}\binom{n-s+i_2+v-2}{j_2}$$
$$\times\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}\frac{\Gamma(p+j_2+2)}{(n+v+i_1+i_2-t_1-t_2+k)^{p+j_2+2}}, \tag{4.9}$$

where $a_{i_1}(s)$ and $a_{i_2}(n-s-2)$ is defined as the Appendix.

Proof

From (2.6) and (4.8), we get

$$M_{m,n}(t_1,t_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty\int_0^x e^{t_1x}e^{t_2y}[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}f(y)\,dydx.$$
$$=\frac{1}{\Gamma(m)\Gamma(n-m)}\int_0^\infty e^{t_1x}\{-\log F(x)\}^{m-1}Q_1(x)\frac{f(x)}{F(x)}dx, \tag{4.10}$$

where

$$Q_1(x)=\int_0^x e^{t_2y}[-\log F(y)+\log F(x)]^{n-m-1}f(y)dy$$
$$=\sum_{s=0}^{n-m-1}\binom{n-m-1}{s}[\log F(x)]^{n-m-1-s}\int_0^x e^{t_2y}[-\log F(y)]^s f(y)dy$$
$$=\sum_{s=0}^{n-m-1}\theta^{s+1}\binom{n-m-1}{s}[\log F(x)]^{n-m-1-s}\int_0^x y\,e^{-(1-t_2)y}\{-\log[1-e^{-y}(y+1)\}^s[1-e^{-y}(y+1)]^{\theta-1}dy$$
$$=\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}(-1)^k\theta^{s+1}\binom{n-m-1}{s}\binom{\theta-1}{k}[\log F(x)]^{n-m-1-s}\int_0^x y\,e^{-(1+k-t_2)y}(y+1)^k\{\sum_{i=1}^{\infty}\frac{e^{-iy}(y+1)^i}{i}\}^s dy$$
$$=\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}(-1)^k\theta^{s+1}a_{i_1}(s)\binom{n-m-1}{s}\binom{\theta-1}{k}[\log F(x)]^{n-m-1-s}\int_0^x ye^{-(1+k+i_1-t_2+s)y}(y+1)^{k+i_1+s}dy$$
$$=\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{k+i_1+s}\sum_{p=j_1+2}^{\infty}(-1)^k\theta^{s+1}a_{i_1}(s)\binom{n-m-1}{s}\binom{\theta-1}{k}$$
$$\times\binom{k+i_1+s}{j_1}[\log F(x)]^{n-m-1-s}e^{-(1+k+i_1-t_2+s)x}\frac{\Gamma(j_1+2)x^p}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}. \tag{4.11}$$

Substituting (4.11) into (4.10), we have

$$M_{m,n}(t_1,t_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{k+i_1+s}\sum_{p=j_1+2}^{\infty}(-1)^{n-m-1+k-s}\theta^{s+1}a_{i_1}(s)\binom{n-m-1}{s}$$
$$\times\binom{\theta-1}{k}\binom{k+i_1+s}{j_1}\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}$$
$$\times\int_0^\infty x^p e^{-(1+k+i_1-t_1-t_2+s)x}\{-\log F(x)\}^{n-s-2}\frac{f(x)}{F(x)}dx$$

$$M_{m,n}(t_1,t_2)=\frac{1}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{k+i_1+s}\sum_{p=j_1+2}^{\infty}(-1)^{n-m-1+k-s}\theta^{s+2}a_{i_1}(s)\binom{n-m-1}{s}$$
$$\times\binom{\theta-1}{k}\binom{k+i_1+s}{j_1}\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}$$
$$\times\int_0^\infty x^{p+1}e^{-(2+k+i_1-t_1-t_2+s)x}\{-\log[1-e^{-x}(x+1)]^{\theta}\}^{n-s-2}$$
$$\times[1-e^{-x}(x+1)]^{-1}dx$$
$$=\frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{k+i_1+s}\sum_{p=j_1+2}^{\infty}\sum_{v=0}^{\infty}(-1)^{n-m-1+k-s}a_{i_1}(s)\binom{n-m-1}{s}$$
$$\times\binom{\theta-1}{k}\binom{k+i_1+s}{j_1}\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}$$
$$\times\int_0^\infty x^{p+1}e^{-(2+k+i_1-t_1-t_2+s+v)x}(x+1)^{v}\{\sum_{i=1}^{\infty}\frac{e^{-ix}(x+1)^{i}}{i}\}^{n-s-2}dx$$
$$=\frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{k+i_1+s}\sum_{p=j_1+2}^{\infty}\sum_{v=0}^{\infty}\sum_{i_2=0}^{\infty}(-1)^{n-m-1+k-s}a_{i_1}(s)a_{i_2}(n-s-2)$$
$$\times\binom{n-m-1}{s}\binom{\theta-1}{k}\binom{k+i_1+s}{j_1}\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}$$
$$\times\int_0^\infty x^{p+1}e^{-(k+i_1+i_2-t_1-t_2+n+v)x}(x+1)^{v+i_2+n-s-2}dx$$
$$=\frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\infty}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{i_1+s+k}\sum_{p=j_1+2}^{\infty}\sum_{v=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{n-s+i_2+v-2}(-1)^{n-m-s+k-1}$$
$$\times a_{i_1}(s)a_{i_2}(n-s-2)\binom{n-m-1}{s}\binom{\theta-1}{k}\binom{i_1+s+k}{j_1}\binom{n-s+i_2+v-2}{j_2}$$
$$\times\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}\int_0^\infty x^{p+j_2+1}e^{-(k-t_1-t_2+i_1+i_2+v+n)x}dx$$

Since the integration is a complete gamma function, we get (4.9). Hence the theorem is proved.

If θ is a positive integer number, then the relation (4.9) becomes

$$M_{m,n}(t_1,t_2)=\frac{\theta^n}{\Gamma(m)\Gamma(n-m)}\sum_{s=0}^{n-m-1}\sum_{k=0}^{\theta-1}\sum_{i_1=0}^{\infty}\sum_{j_1=0}^{i_1+s+k}\sum_{p=j_1+2}^{\infty}\sum_{v=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{n-s+i_2+v-2}(-1)^{n-m-s+k-1}$$
$$\times a_{i_1}(s)a_{i_2}(n-s-2)\binom{n-m-1}{s}\binom{\theta-1}{k}\binom{i_1+s+k}{j_1}\binom{n-s+i_2+v-2}{j_2}$$
$$\times\frac{\Gamma(j_1+2)}{p!(1+k+i_1-t_2+s)^{j_1+2-p}}\frac{\Gamma(p+j_2+2)}{(n+v+i_1+i_2-t_1-t_2+k)^{p+j_2+2}},\qquad(4.12)$$

The MGF for double moments of lower record values from gamma distribution $G(2,1)$ can be obtained from (4.12) by setting $\theta=1$.

4.4 Moment generating function for triple moments

The MGF for triple moments of lower record values, $E(e^{t_1 X_{L(m)}+t_2 X_{L(n)}+t_3 X_{L(l)}})$, denoted by $M_{m,n,l}(t_1,t_2,t_3)$ is given by

$$M_{m,n,l}(t_1,t_2,t_3)=\int_0^\infty\int_0^x\int_0^y e^{t_1x+t_2y+t_3z} f_{m,n,l}(x,y,z)\,dzdydx, \tag{4.13}$$

where $f_{m,n,l}(x,y,z)$ is defined in (2.13).

The exact explicit expression for the MGF for triple moments of lower record values from *EG* distribution is given by the following theorem:

Theorem 12

For $m, n, l = 1, 2, \ldots, m<n<l$, and θ is a real value, then

$$\begin{aligned} M_{m,n,l}(t_1,t_2,t_3)=&\frac{\theta^l}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{n-m-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_2-s_1+i_2+v_1}\\ &\times\sum_{p_2=p_1+j_2+2}^{\infty}\sum_{v_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{n-s_2+i_3+v_2-2}(-1)^{l-m-s_1-s_2+k-2}a_{i_1}(s_1)a_{i_2}(l-n-1-s_1+s_2)a_{i_3}(n-s_2-2)\\ &\times\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{n-m-1}{s_2}\binom{l-n-1-s_1+s_2+i_2+v_1}{j_2}\\ &\times\binom{n-s_2+i_3+v_2-2}{j_3}\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}\\ &\times\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+l-n+s_2+1-t_2-t_3)^{p_1+j_2+2-p_2}}\\ &\times\frac{\Gamma(p_2+j_3+2)}{(v_1+v_2+i_1+i_2+i_3+l+k-t_1-t_2-t_3)^{p_2+j_3+2}}, \end{aligned}\tag{4.14}$$

where $a_{i_1}(s_1), a_{i_2}(l-n-1-s_1+s_2)$ and $a_{i_3}(n-s_2-2)$ are defined as in appendix.

Proof

From (2.13) and (4.13), we get

$$\begin{aligned} M_{m,n,l}(t_1,t_2,t_3)=&\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\int_0^\infty\int_0^x\int_0^y e^{t_1x}e^{t_2y}e^{t_3z}[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}\\ &\times[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}f(z)\,dzdydx.\\ =&\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\int_0^\infty\int_0^x e^{t_1x}e^{t_2y}Q_2(y)[-\log F(x)]^{m-1}\\ &\times[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}dydx, \end{aligned}\tag{4.15}$$

where

$$Q_2(y)=\int_0^y e^{t_3z}[-\log F(z)+\log F(y)]^{l-n-1}f(z)dz \tag{4.16}$$

We can find $Q_2(y)$ in (4.16) in the similar way that was shown in theorem 11 to find $Q_1(x)$, then we get

$$Q_2(y)=\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}(-1)^k\theta^{s_1+1}a_{i_1}(s_1)\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}$$

$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}[\log F(y)]^{l-n-1-s_1}y^{p_1}e^{-(s_1+i_1+k+1-t_3)y} \tag{4.17}$$

Substituting (4.17) into (4.15), we get

$$M_{m,n,l}(t_1,t_2,t_3)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}(-1)^k\theta^{s_1+1}a_{i_1}(s_1)\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}$$

$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}\int_0^{\infty}\int_0^{x}e^{t_1x}e^{t_2y}[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}$$

$$\times[\log F(y)]^{l-n-1-s_1}y^{p_1}e^{-(s_1+i_1+k+1-t_3)y}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}dydx$$

$$=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}(-1)^k\theta^{s_1+1}a_{i_1}(s_1)\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}$$

$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}\int_0^{\infty}e^{t_1x}Q_3(x)[-\log F(x)]^{m-1}\frac{f(x)}{F(x)}dx \tag{4.18}$$

where

$$Q_3(x)=\int_0^{x}[-\log F(y)+\log F(x)]^{n-m-1}[\log F(y)]^{l-n-1-s_1}y^{p_1}e^{-(s_1+i_1+k+1-t_2-t_3)y}\frac{f(y)}{F(y)}dy. \tag{4.19}$$

In similar way that was shown in theorem 2 to find $I(y)$, we can find $Q_3(x)$, then we get

$$Q_3(x)=\sum_{s_2=0}^{n-m-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_2-s_1+i_2+v_1}\sum_{p_2=p_1+j_2+2}^{\infty}(-1)^{l-n-1-s_1}a_{i_2}(l-n-1-s_1+s_2)\theta^{l-n-s_1+s_2}$$

$$\times\binom{n-m-1}{s_2}\binom{l-n-1-s_1+s_2+i_2+v_1}{j_2}\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+l-n+s_2+1-t_2-t_3)^{p_1+j_2+2-p_2}}$$

$$\times[\log F(x)]^{n-m-1-s_2}x^{p_2}e^{-(v_1+i_1+i_2+l-n+s_2+k+1-t_2-t_3)x} \tag{4.20}$$

Substituting (4.20) into (4.18), we get

$$M_{m,n,l}(t_1,t_2,t_3)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{n-m-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_2-s_1+i_2+v_1}\sum_{p_2=p_1+j_2+2}^{\infty}(-1)^{l-m-s_1-s_1+k-2}$$

$$\times\theta^{l-n+1+s_2}a_{i_1}(s_1)a_{i_2}(l-n-1-s_1+s_2)\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{n-m-1}{s_2}$$

$$\times\binom{l-n-1-s_1+s_2+i_2+v_1}{j_2}\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}$$

$$\times\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+l-n+s_2+1-t_2-t_3)^{p_1+j_2+2-p_2}}$$

$$\times\int_0^{\infty}[-\log F(x)]^{n-s_2-2}x^{p_2}\frac{f(x)}{F(x)}e^{-(v_1+i_1+i_2+l-n+s_2+k+1-t_1-t_2-t_3)x}dx \tag{4.21}$$

In similar way that was shown in theorem 2, we can get (4.14). Hence the theorem is proved.

If θ is a positive integer number, then the relation (4.14) becomes

$$M_{m,n,l}(t_1,t_2,t_3)=\frac{\theta^l}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)}\sum_{s_1=0}^{l-n-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\theta-1}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{n-m-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{l-n-1+s_2-s_1+i_2+v_1}\sum_{p_2=p_1+j_2+2}^{\infty}$$
$$\times\sum_{v_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{n-s_2+i_3+v_2-2}(-1)^{l-m-s_1-s_2+k-2}\,a_{i_1}(i_1,s_1)\,a_{i_2}(i_2,l-n-1-s_1+s_2)$$
$$\times\, a_{i_3}(i_3,n-s_2-2)\binom{l-n-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{n-m-1}{s_2}$$
$$\times\binom{l-n-1-s_1+s_2+i_2+v_1}{j_2}\binom{n-s_2+i_3+v_2-2}{j_3}\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{j_1+2-p_1}}$$
$$\times\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+l-n+s_2+1-t_2-t_3)^{p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(p_2+j_3+2)}{(v_1+v_2+i_1+i_2+i_3+l+k-t_1-t_2-t_3)^{p_2+j_3+2}},\qquad(4.22)$$

The MGF for triple moments of lower record values from gamma distribution $G(2,1)$ can be obtained from (4.22) by setting $\theta=1$.

4.5 Moment generating function for quadruple moments

The MGF for quadruple moments of lower record values, $E(e^{t_1X_{L(m)}+t_2X_{L(n)}+t_3X_{L(l)}+t_4X_{L(v)}})$, denoted by $M_{m,n,l,v}(t_1,t_2,t_3,t_4)$ is given by

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\int_0^{\infty}\int_0^{x}\int_0^{y}\int_0^{z} e^{t_1x+t_2y+t_3z+t_4w}\,f_{m,n,l,v}(x,y,z,w)\,dwdzdydx.\qquad(4.23)$$

where $f_{m,n,l,v}(x,y,z,w)$ is defined in (2.20).

The exact explicit expression for the MGF for quadruple moments of lower record values from *EG* distribution is given by the following theorem.

Theorem 13

For *m, n, l, v* =1, 2, …, *m* < *n* < *l*< *v*, and θ is a real value, then

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{\theta^{v}}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{l-n-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{v_1+s_2+i_2+v-l-1-s_1}$$
$$\times\sum_{p_2=p_1+j_2+2}^{\infty}\sum_{s_3=0}^{n-m-1}\sum_{v_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{l-n-1-s_2+s_3+i_3+v_2}\sum_{p_3=p_2+j_3+2}^{\infty}\sum_{v_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{j_4=0}^{n-s_3+i_4+v_3-2}(-1)^{v-m-s_1-s_2-s_3+k-3}$$
$$\times a_{i_1}(s_1),a_{i_2}(s_2+v-l-1-s_1),a_{i_3}(s_3+l-n-1-s_2),a_{i_4}(n-s_3-2)$$
$$\times\binom{v-l-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{l-n-1}{s_2}\binom{v-l-1-s_1+s_2+i_2+v_1}{j_2}$$
$$\times\binom{n-m-1}{s_3}\binom{n-s_3+i_4+v_3-2}{j_4}\binom{l-n-1-s_2+s_3+i_3+v_2}{j_3}$$
$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}$$
$$\times\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)^{p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(p_2+j_3+2)}{p_3!(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_2-t_3-t_4+1)^{p_2+j_3+2-p_3}}$$
$$\times\frac{\Gamma(p_3+j_4+2)}{(k+v+v_1+v_2+v_3+i_1+i_2+i_3+i_4-t_1-t_2-t_3-t_4)^{p_3+j_4+2}},\quad(4.24)$$

where $a_{i_1}(s_1),a_{i_2}(s_2+v-l-1-s_1),a_{i_3}(s_3+l-n-1-s_2)$ and $a_{i_4}(n-s_3-2)$ are defined as in appendix.

Proof

From (2.20) and (4.23), we get

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\int_0^{\infty}\int_0^{x}\int_0^{y}\int_0^{z}e^{t_1x+t_2y+t_3z+t_4w}[-\log F(x)]^{m-1}$$
$$\times[-\log F(y)+\log F(x)]^{n-m-1}[-\log F(z)+\log F(y)]^{l-n-1}$$
$$\times[-\log F(w)+\log F(z)]^{v-l-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}f(w)dwdzdydx.$$
$$=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\int_0^{\infty}\int_0^{x}\int_0^{y}e^{t_1x+t_2y+t_3z}Q_4(z)$$
$$\times[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}$$
$$\times[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}\frac{f(z)}{F(z)}dzdydx,\quad(4.25)$$

where

$$Q_4(z)=\int_0^{z}e^{t_4w}[-\log F(w)+\log F(z)]^{v-l-1}f(w)dw\quad(4.26)$$

We can find $Q_4(z)$ in (4.26) in the similar way that was shown in theorem 11 to find $Q_1(x)$, then we get

$$Q_4(z)=\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}(-1)^{k}\,a_{i_1}(s_1)\theta^{s_1+1}\binom{v-l-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}$$
$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}[\log F(z)]^{v-l-1-s_1}z^{p_1}e^{-(s_1+i_1+k-t_4+1)z}\quad(4.27)$$

Substituting (4.27) into (4.25), we get

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}(-1)^k\theta^{s_1+1}a_{i_1}(s_1)$$
$$\times\binom{v-l-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}\times\int_0^{\infty}\int_0^{x}e^{t_1x+t_2y}$$
$$\times Q_5(y)[-\log F(x)]^{m-1}[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(x)}{F(x)}\frac{f(y)}{F(y)}dydx \qquad (4.28)$$

where

$$Q_5(y)=\int_0^{y}[\log F(z)]^{v-l-1-s_1}z^{p_1}e^{-(s_1+i_1+k-t_3-t_4+1)z}[-\log F(z)+\log F(y)]^{l-n-1}\frac{f(z)}{F(z)}dz \qquad (4.29)$$

In similar way that was shown in theorem 2 to find $I(y)$, we can find $Q_5(y)$, then we get

$$Q_5(y)=\sum_{s_2=0}^{l-n-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{v_1+s_2+i_2+v-l-1-s_1}\sum_{p_2=p_1+j_2+2}^{\infty}(-1)^{v-l-1-s_1}a_{i_2}(s_2+v-l-1-s_1)\theta^{v-l+s_2-s_1}\binom{l-n-1}{s_2}$$
$$\times\binom{v-l-1-s_1+s_2+i_2+v_1}{j_2}\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)^{p_1+j_2+2-p_2}}$$
$$\times[\log F(y)]^{l-n-1-s_2}y^{p_2}e^{-(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)y} \qquad (4.30)$$

Substituting (4.30) into (4.28), we get

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{l-n-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{v_1+s_2+i_2+v-l-1-s_1}$$
$$\times\sum_{p_2=p_1+j_2+2}^{\infty}(-1)^{v-l-s_1+k-1}\theta^{s_2+v-l+1}a_{i_1}(s_1)a_{i_2}(s_2+v-l-1-s_1)\binom{v-l-1}{s_1}$$
$$\times\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{l-n-1}{s_2}\binom{v-l-1-s_1+s_2+i_2+v_1}{j_2}\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}$$
$$\times\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)^{p_1+j_2+2-p_2}}$$
$$\times\int_0^{\infty}e^{t_1x}Q_6(x)[-\log F(x)]^{m-1}\frac{f(x)}{F(x)}dx \qquad (4.31)$$

where

$$Q_6(x)=\int_0^{x}y^{p_2}[-\log F(y)+\log F(x)]^{n-m-1}\frac{f(y)}{F(y)}[\log F(y)]^{l-n-1-s_2}e^{-(v_1+k+i_1+i_2+v-l+s_2-t_2-t_3-t_4+1)y}dy \qquad (4.32)$$

In similar way that was shown in theorem 2 to find $I(y)$, we can find $Q_6(x)$, then we get

$$Q_6(x)=\sum_{s_3=0}^{n-m-1}\sum_{v_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{l-n-1-s_2+s_3+i_3+v_2}\sum_{p_3=p_2+j_3+2}^{\infty}(-1)^{l-n-1-s_2}a_{i_3}(s_3+l-n-1-s_2)\theta^{s_3+l-n-s_2}$$
$$\times\binom{n-m-1}{s_3}\binom{l-n-1-s_2+s_3+i_3+v_2}{j_3}e^{-(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_2-t_3-t_4+1)x}$$
$$\times\frac{\Gamma(p_2+j_3+2)}{p_3!(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_2-t_3-t_4+1)^{p_2+j_3+2-p_3}}$$
$$\times x^{p_3}[\log F(x)]^{n-m-1-s_3}. \qquad (4.33)$$

Substituting (4.33) into (4.31), we get

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{l-n-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{v_1+s_2+i_2+v-l-1-s_1}\sum_{p_2=p_1+j_2+2}^{\infty}\sum_{s_3=0}^{n-m-1}\sum_{v_2=0}^{\infty}$$
$$\times\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{l-n-1-s_2+s_3+i_3+v_2}\sum_{p_3=p_2+j_3+2}^{\infty}\theta^{v-n+s_3+1}(-1)^{v-m-s_1-s_2-s_3+k-3}a_{i_1}(s_1)\,a_{i_2}(s_2+v-l-1-s_1)\,a_{i_3}(s_3+l-n-1-s_2)$$
$$\times\binom{v-l-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{l-n-1}{s_2}\binom{v-l-1-s_1+s_2+i_2+v_1}{j_2}\binom{n-m-1}{s_3}\binom{l-n-1-s_2+s_3+i_3+v_2}{j_3}$$
$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)^{p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(p_2+j_3+2)}{p_3!(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_2-t_3-t_4+1)^{p_2+j_3+2-p_3}}$$
$$\times\int_0^{\infty}x^{p_3}e^{-(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_1-t_2-t_3-t_4+1)x}\,[-\log F(x)]^{n-s_3-2}\,\frac{f(x)}{F(x)}dx \qquad (4.34)$$

In similar way that was shown in theorem 2, we can get (4.24). Hence the theorem is proved.

If θ is a positive integer number, then the relation (4.34) becomes

$$M_{m,n,l,v}(t_1,t_2,t_3,t_4)=\frac{\theta^{v}}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)}\sum_{s_1=0}^{v-l-1}\sum_{i_1=0}^{\infty}\sum_{k=0}^{\theta-1}\sum_{j_1=0}^{i_1+s_1+k}\sum_{p_1=j_1+2}^{\infty}\sum_{s_2=0}^{l-n-1}\sum_{v_1=0}^{\infty}\sum_{i_2=0}^{\infty}\sum_{j_2=0}^{v_1+s_2+i_2+v-l-1-s_1}$$
$$\times\sum_{p_2=p_1+j_2+2}^{\infty}\sum_{s_3=0}^{n-m-1}\sum_{v_2=0}^{\infty}\sum_{i_3=0}^{\infty}\sum_{j_3=0}^{l-n-1-s_2+s_3+i_3+v_2}\sum_{p_3=p_2+j_3+2}^{\infty}\sum_{v_3=0}^{\infty}\sum_{i_4=0}^{\infty}\sum_{j_4=0}^{n-s_3+i_4+v_3-2}(-1)^{v-m-s_1-s_2-s_3+k-3}$$
$$\times a_{i_1}(s_1),a_{i_2}(s_2+v-l-1-s_1),a_{i_3}(s_3+l-n-1-s_2),a_{i_4}(n-s_3-2)$$
$$\times\binom{v-l-1}{s_1}\binom{\theta-1}{k}\binom{i_1+s_1+k}{j_1}\binom{l-n-1}{s_2}\binom{v-l-1-s_1+s_2+i_2+v_1}{j_2}$$
$$\times\binom{n-m-1}{s_3}\binom{n-s_3+i_4+v_3-2}{j_4}\binom{l-n-1-s_2+s_3+i_3+v_2}{j_3}$$
$$\times\frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k-t_4+1)^{j_1+2-p_1}}\frac{\Gamma(p_1+j_2+2)}{p_2!(v_1+k+i_1+i_2+v-l+s_2-t_3-t_4+1)^{p_1+j_2+2-p_2}}$$
$$\times\frac{\Gamma(p_2+j_3+2)}{p_3!(k+s_3-n+v+v_1+v_2+i_1+i_2+i_3-t_2-t_3-t_4+1)^{p_2+j_3+2-p_3}}$$
$$\times\frac{\Gamma(p_3+j_4+2)}{(k+v+v_1+v_2+v_3+i_1+i_2+i_3+i_4-t_1-t_2-t_3-t_4)^{p_3+j_4+2}}. \qquad (4.35)$$

The MGF for quadruple moments of lower record values from gamma distribution $G(2,1)$ can be obtained from (4.35) by setting $\theta=1$.

Appendix

Let us consider

$$\log(1-x)=-\sum_{i=1}^{\infty}\frac{x^i}{i},\ |x|<1,\ \text{then}$$

$$[-\log(1-x)]^m=[\sum_{i=1}^{\infty}\frac{x^i}{i}]^m$$
$$=[x+\frac{x^2}{2}+\frac{x^3}{3}+...+\frac{x^i}{i}+...]^m$$
$$=\sum_{s=0}^{\infty}a_s(m)x^{m+s},$$

where $a_s(m)$ is the coefficient of x^{m+s} in the expansion of $[\sum_{i=1}^{\infty}\frac{x^i}{i}]^m$. (see, Balakrishnan and Cohen (1991)).

The coefficient $a_s(m)$ can be generated very easily as follows. First of all, we see that

$$a_0(1)=1, a_1(1)=\frac{1}{2}, a_2(1)=\frac{1}{3}, \ldots, a_s(1)=\frac{1}{s+1}, \ldots. \qquad \text{(A.1)}$$

Next, let us consider $a_s(m)$ for $m \geq 2,$ given by

$$\begin{aligned} a_s(m) &= \text{coefficient of } x^s \text{ in } [\sum_{i=1}^{\infty}\frac{x^i}{i}]^m \\ &= \sum_{i=1}^{\infty} \{\text{coefficient of } x^i \text{ in } [\sum_{i=1}^{\infty}\frac{x^i}{i}]\}\times\{\text{coefficient of } x^{s-i} \text{ in } [\sum_{i=1}^{\infty}\frac{x^i}{i}]^{m-1}\} \\ &= \sum_{i=1}^{\infty}\frac{1}{i}a_{s-i}(m-1). \end{aligned} \qquad \text{(A.2)}$$

Thus, by starting with the value of $a_s(1)$ given in (A.1), we can compute the coefficients $a_s(m)$ for any value of m by repeated application of the recurrence relation in (A.2). After computing the coefficients $a_s(m)$ by this way, one may either directly compute the required moments or MGF's.

Note that, if $m=0,$ then the summation $[\sum_{i=1}^{\infty}(.)]^m$ is equal the unity, so $a_s(0)=1, s=0$ and $a_s(0)=0, s>0.$

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