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STAT 110 Slides

موقع المكتب

Main Reference

Elementary Statistics

A Step by Step Approach

By

Allan Bluman

Chapter 4

Probability

and

Counting Rules

Objectives

- Demonstrate knowledge of basic probability concepts.
- Calculate and interpret the probability of an event using classical probability or empirical probability.
- Calculate and interpret the probability of compound events using the addition rules.
- Calculate and interpret the probability of compound events using the multiplication rules.
- Find the total number of outcomes in a sequence of events using the fundamental counting rule.
- Find the number of ways that r objects can be selected from n objects using the permutation rule.
- Find the number of ways that r objects can be selected from n objects without regard to order using the combination rule.

Introduction

Probability as a general concept can be defined as the chance of an event occurring.

- Probability are used in games of chance, insurance, investments, weather forecasting and in various other areas.
- Also, probability is the basis of inferential statistics. For example; predictions are based on probability and hypotheses are tested by using probability.
- Rules such as the fundamental counting rule, permutation rule and combination rule allow us to count the number of ways in which events can occur.

Basic Concepts

- A *probability experiment* is a chance process that leads to well-defined results called outcomes.
- An *outcome* is the result of a single trial of a probability experiment.
- A *sample space* is the set of all possible outcomes of a probability experiment.
- An *event* consists of a set of outcomes of a probability experiment.
- An event with one outcome is called a simple event and with more than one outcome is called compound event.

Experiment	**Outcome**	**Number of outcomes in the sample space**	**Sample Space**	**Event**	**Event Type**
Toss a coin	T	2^1 = 2	S={H, T}	E={H}	Simple
Toss two coins	HT	2^2 = 2 × 2 = 4	S={HH, HT, TH, TT}	E={HT, TH}	Compound
Roll a die	4	6^1 = 6	S={1, 2, 3, 4, 5, 6}	E={1, 3, 5}	Compound

EXAMPLE

Find the sample space for the gender of children if a family has three children and give an example for a simple event and another one for a compound event. Use B for boy and G for girl.

There are two genders and three children, so there are $\mathbf{2^3 = 8}$ possibilities as shown here;

BBB BBG BGB GBB BGG GBG GGG GGB

So, the sample space is

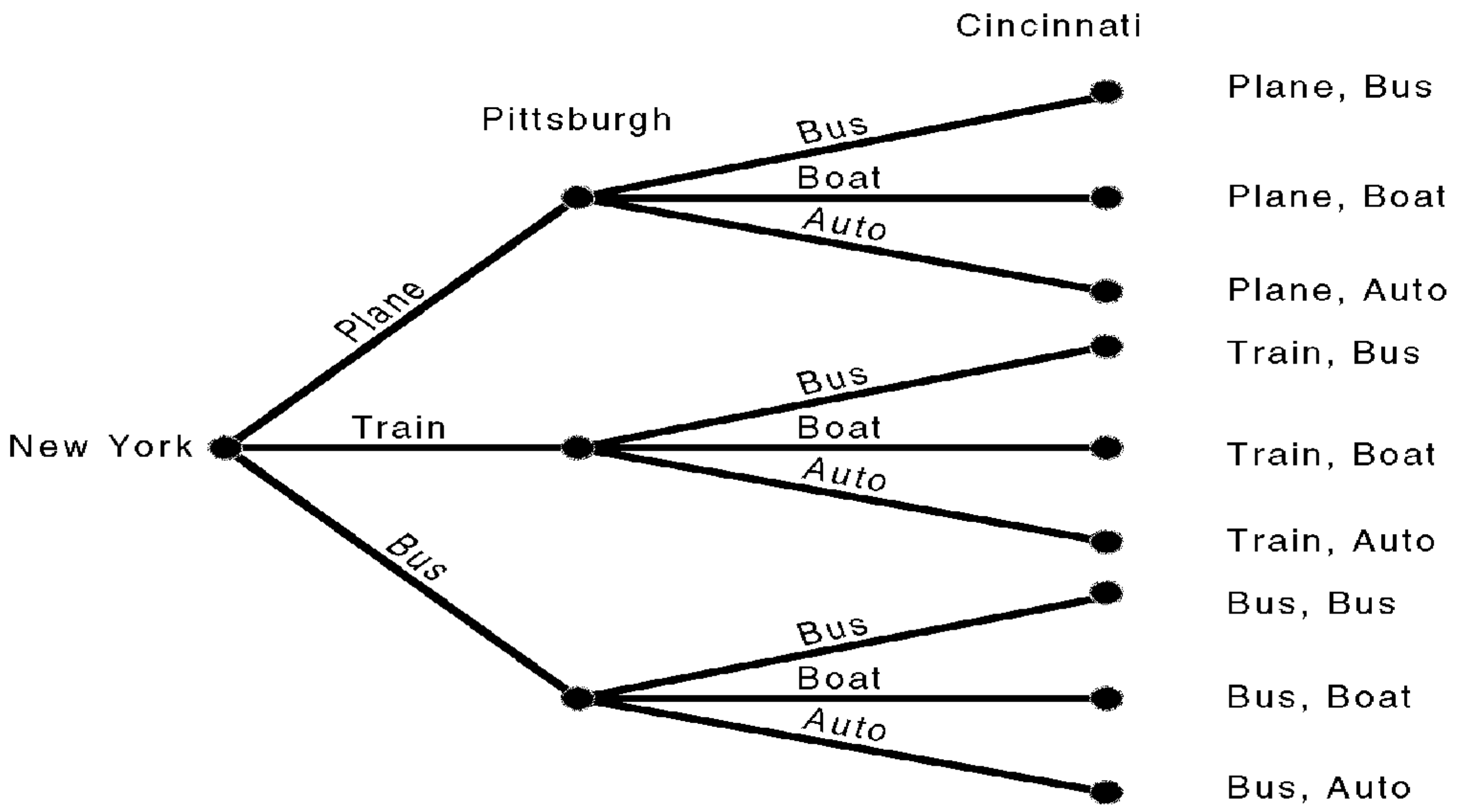
S ={BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG}

- Simple event as E ={BBB}

- Compound event as E ={BBG, BGB, GBB}

Basic Concepts

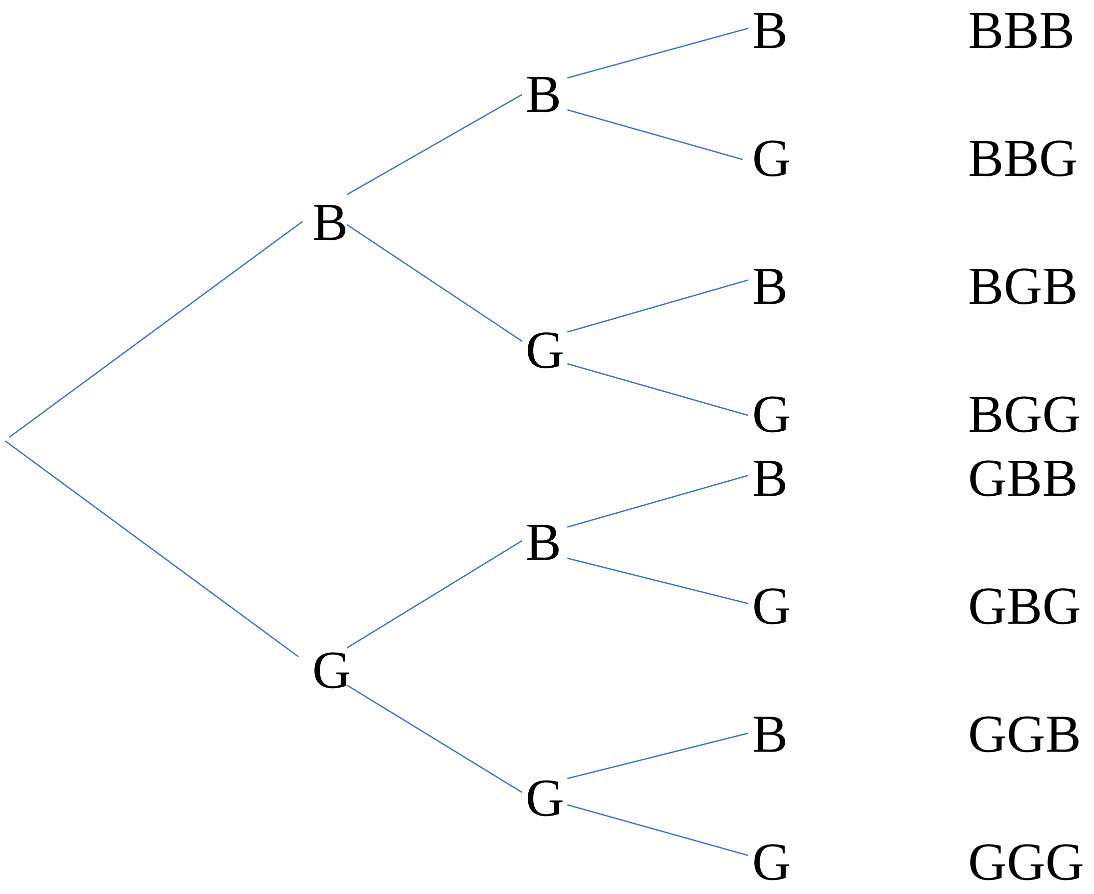
A *tree diagram* is a device used to list all possibilities of a sequence of events in a systematic way.



Basic Concepts

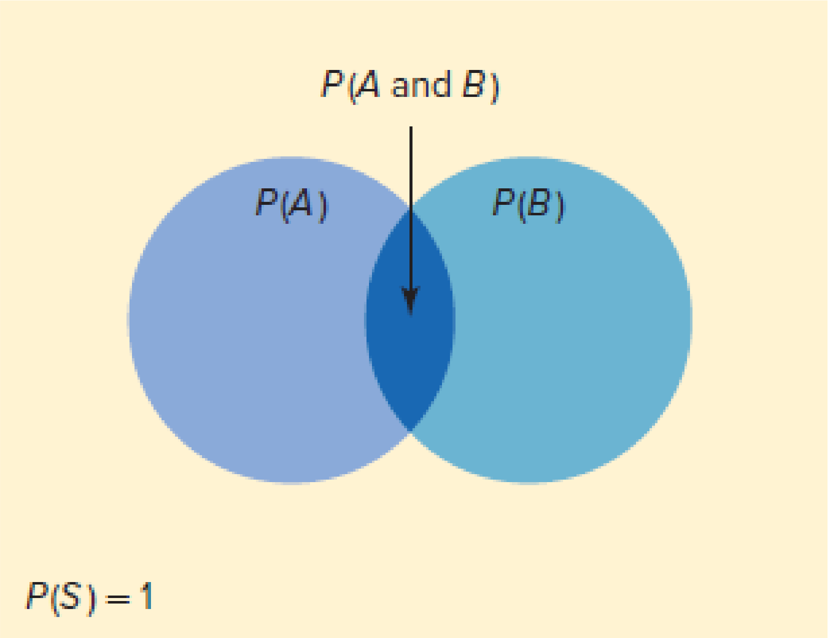
EXAMPLE

Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.



Basic Concepts

- *Equally likely events* are events that have the same probability of occurring.
- *Venn diagrams* are used to represent probabilities pictorially.



Classical Probability

Classical probability uses sample spaces to determine the numerical probability that an event will happen. It assumes that all outcomes in the sample space are equally likely to occur. The probability of an event E can be defined as

$$P(E)=\frac{n(E)}{n(S)}=\frac{Number\ of\ outcomes\ in\ E}{Number\ of\ outocmes\ in\ the\ sample\ space}$$

EXAMPLE

Find the probability of two girls in a family that has three children.

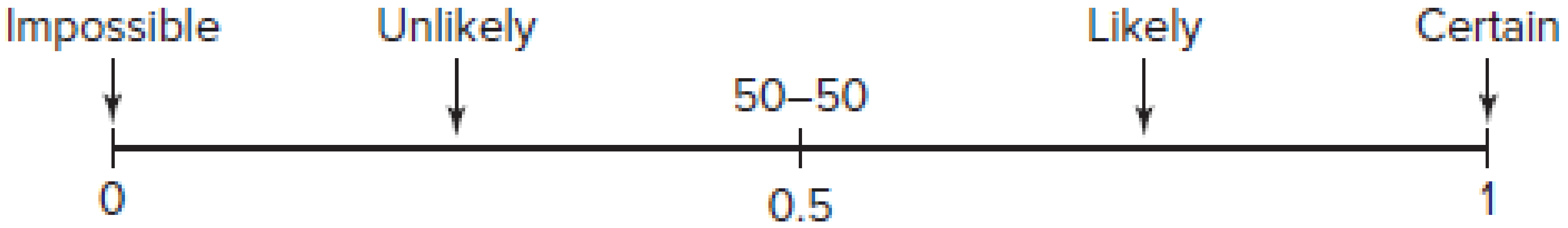
The sample space is S ={BBB, BBG, BGB, GBB, GGG, GGB, GBG, BGG}

The event of two girls is E ={GGB, GBG, BGG}

Hence, the probability of two girls is $\boldsymbol{P(E)=\frac{n(E)}{n(S)}=\frac{3}{8}=0.375}$

Probability Rules

1. The probability of an event E is a number (either a fraction or decimal) between and including 0 and 1. Thus, $0 \le P(E) \le 1$.
2. If an event E cannot occur (i.e., the event contains no members in the sample space), the probability is zero.
3. If an event E is certain, then the probability of E is one.
4. The sum of the probabilities of the outcomes in the sample space is one.



EXAMPLE

When a single die is rolled, the sample space is S={1, 2, 3, 4, 5, 6}.

Find the probability of getting an even number.

Since the event of even outcomes E={2, 4, 6}, then $\boldsymbol{P(E) = \frac{3}{6} = 0.5}$

Find the probability of getting a 9.

It is impossible to get a 9, E={ }=Φ. Hence, $\boldsymbol{P(9) = \frac{0}{6} = 0}$

What is the probability of getting a number less than 7?

Since all outcomes in the sample space are less than 7, E={1, 2, 3, 4, 5, 6}, then

$$\boldsymbol{P(< 7) = \frac{6}{6} = 1}$$

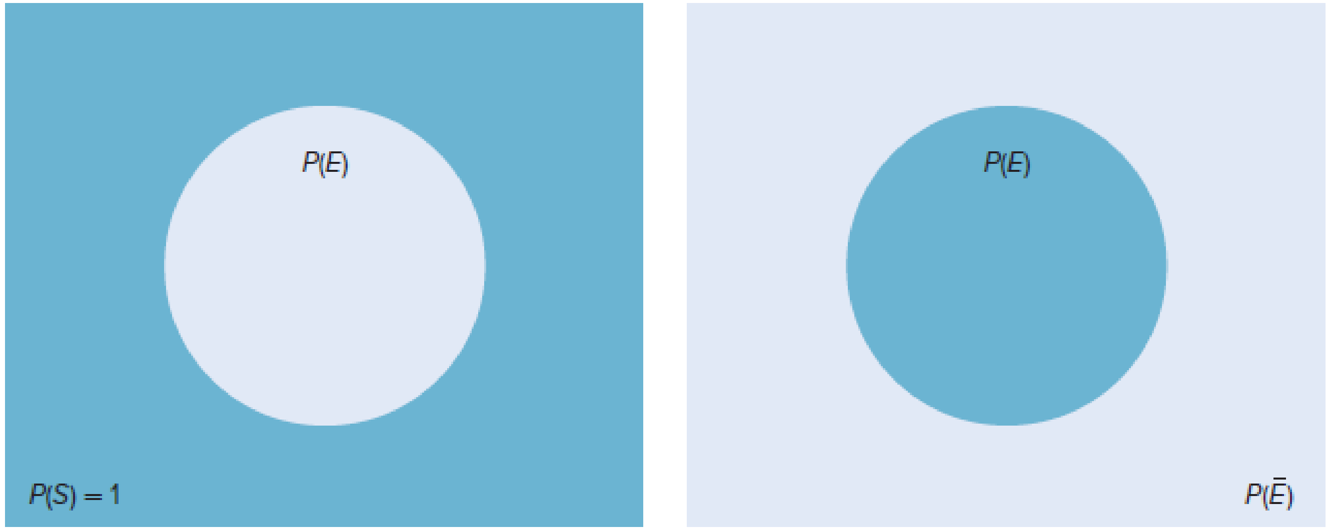
Complementary Events

The *complement of an event* ***E*** is the set of outcomes in the sample space that are not included in the outcomes of event ***E***. The complement of ***E*** is denoted by $\bar{E}$ (read "E bar").

Rule for Complementary Events, $P(E) + P(\bar{E}) = 1$

Thus, $P(\bar{E}) = 1 - P(E)$ or $P(E) = 1 - P(\bar{E})$.

Complementary events are mutually exclusive.



(a) Simple probability (b) $P(\bar{E}) = 1 - P(E)$

Complementary Events

EXAMPLES

Find the complement of each event.

a. Rolling a die and getting a 4.

Getting 1,2,3,5 or 6

b. Selecting a letter of the alphabet and getting a vowel.

Getting a consonant

c. Selecting a month and getting a month that begins with a J.

Getting February, March, April, May, August, September, October, November or December

d. Selecting a day of the week and getting a weekday.

Getting a weekend (Friday or Saturday)

Complementary Events

EXAMPLES

If the probability that a person lives in an industrialized country is $\frac{1}{5}$, find the probability that a person does not live in an industrialized country.

$$P(\textit{not living in an industrialized country}) = 1 - \frac{1}{5} = \frac{4}{5} = 0.8$$

In a study, it was found that 24% of people who were victims of a violent crime were ages 20 to 24. If a person is selected at random, find the probability that the person is younger than 20 or older than 24.

$$P(\textit{not aged } 20 \textit{ to } 24) = 1 - 0.24 = 0.76 = 76\%$$

Empirical Probability

Empirical probability relies on actual experience to determine the likelihood of outcomes. It doesn't assume that all outcomes in the sample space are equally likely to occur.

Given a frequency distribution, the probability of an event being in a class is:

$$P(E) = \frac{frequency\ for\ the\ class}{sample\ size} = \frac{f}{n}$$

EXAMPLE

A researcher asked 50 people who plan to travel over the holiday how they will get to their destination. The results can be categorized in a frequency distribution as shown.

Find the probability that the person will travel by train or bus over the holiday.

$$P(E) = \frac{3}{50} = 0.06$$

Method	Frequency
Drive	41
Fly	6
Train or bus	3
	50

Empirical Probability

EXAMPLE

In a sample of 50 people, 21 had type O blood. 22 had type A, 5 had type B blood and 2 had type AB blood. Set up a frequency distribution and find the following probabilities:

Blood Type	Frequency
A	22
B	5
O	21
AB	2
Total	50

a. a person has type O blood.

$$P(O) = \frac{21}{50} = 0.42$$

b. a person has type A or type B blood.

$$P(A \text{ } or \text{ } B) = \frac{22+5}{50} = 0.54$$

c. a person has neither type A nor type O blood.

$$P(neither \text{ } A \text{ } nor \text{ } O) = \frac{5+2}{50} = 1 - \frac{22+21}{50} = 0.14$$

d. a person does not have type AB blood.

$$P(not \text{ } AB) = 1 - 2/50 = 0.96$$

Mutually Exclusive Events

Two events are *mutually exclusive* if they cannot occur at the same time (i.e., they have no outcomes in common).

EXAMPLES

- Rolling a die and getting an even or an odd number.
- Randomly selecting a student and getting a female student or a male student.

In the case of mutually exclusive and not-mutually exclusive events, the probability of two or more events can be determined by the *addition rules*.

Mutually Exclusive Events

EXAMPLE

When a single die rolled, which of the following events is mutually exclusive and which is not?

a. Getting a 3 and getting an odd number.

The events are not mutually exclusive, since the first event is a 3 and then second event is 1, 3 or 5. Hence, 3 is contained in both events.

b. Getting an odd number and getting a number less than 4.

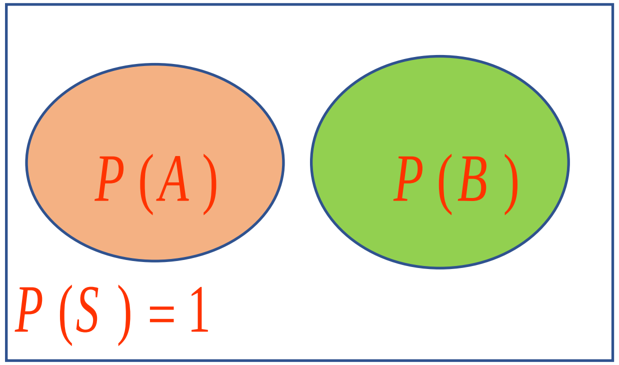
The events are not mutually exclusive, since the first event can be 1, 3 or 5 and the second event is 1, 2 or 3. Hence, 1 and 3 are contained in both events.

c. Getting a number greater than 4 and getting a number less than 4.

The events are mutually exclusive, since the first event is 5 or 6 and the second event is 1, 2 or 3.

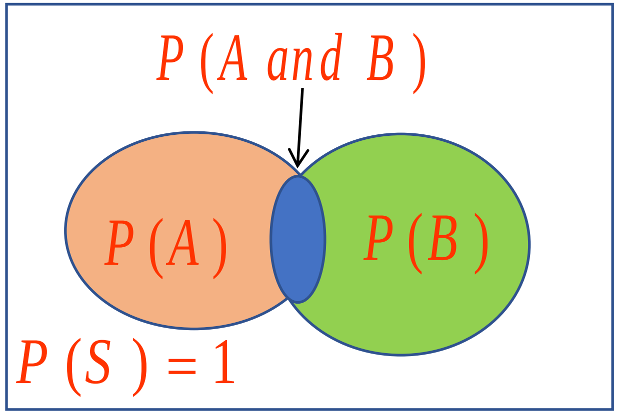
Addition Rules

- When two events A and B are *mutually exclusive*, the probability that A or B will occur is



$$P(A\ or\ B) = P(A) + P(B)$$

- When two events A and B are *not-mutually exclusive*, the probability that A or B will occur is



$$P(A\ or\ B) = P(A) + P(B) - P(A\ \&\ B)$$

Addition Rules

EXAMPLES

A box contains doughnuts, 3 glazed, 4 jelly and 5 chocolate. If a person selects a doughnut at random, find the probability that it is either a glazed or a chocolate doughnut.

The total number of doughnuts in the box is 12 and the events are mutually exclusive, so

$$\boldsymbol{P(G\ or\ C) = P(G) + P(C) = \frac{3}{12} + \frac{5}{12} = \frac{8}{12} = 0.667}$$

A day of the week is selected at random. Find the probability that it is a weekend day (Friday or Saturday)

The total number of days in a week is 7 (5 weekday and 2 weekend) and the events are mutually exclusive, so

$$\boldsymbol{P(F\ or\ S) = P(F) + P(S) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7} = 0.286}$$

EXAMPLE

In a hospital unit there are 9 nurses and 7 physicians; 7 nurses and 3 physicians are females. If a staff is selected, find the probability that the subject is a nurse or a male.

The events are not mutually exclusive and the sample space is

Staff	Female	Male	Total
Nurses	7	2	9
Physicians	3	4	7
Total	10	6	16

$$\boldsymbol{P(N \text{ or } M) = P(N) + P(M) - P(N \,\&\, M) = \frac{9}{16} + \frac{6}{16} - \frac{2}{16} = \frac{13}{16} = 0.8125}$$

Independent and Dependent Events

- Two events A and B are *independent* if the fact that A occurs does not affect the probability of B occurring.
- When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be *dependent*.
- The *multiplication rules* can be used to find the probability of two or more events that occur in sequence.
- When two events are independent, the probability of both occurring is:

$$P(A \text{ and } B) = P(A) * P(B)$$

- When two events are dependent, the probability of both occurring is:

$$P(A \text{ and } B) = P(A) * P(B|A)$$

Independent and Dependent Events

EXAMPLE

An urn contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected and its color noted. Then it is replaced. A second ball is selected and its color noted. Find the probability of each of these:

a. selecting 2 blue balls

$$\boldsymbol{P(B\ and\ B) = P(B) * P(B) = \frac{2}{10} * \frac{2}{10} = \frac{4}{100} = 0.04}$$

b. selecting 1 blue ball and then 1 white ball

$$\boldsymbol{P(B\ and\ W) = P(B) * P(W) = \frac{2}{10} * \frac{5}{10} = \frac{10}{100} = 0.1}$$

c. selecting 1 red ball and then 1 blue ball

$$\boldsymbol{P(R\ and\ B) = P(R) * P(B) = \frac{3}{10} * \frac{2}{10} = \frac{6}{100} = 0.06}$$

Independent and Dependent Events

EXAMPLE

An urn contains 3 red balls, 2 blue balls and 5 white balls. A ball is selected and its color noted (**without replacement**) , then a second ball is selected and its color noted. Find the probability of each of these:

a. selecting 2 blue balls

$$P(B \text{ and } B) = P(B) * P(B|B) = \frac{2}{10} * \frac{1}{9} = \frac{2}{90} = 0.022$$

b. selecting 1 blue ball and then 1 white ball

$$P(B \text{ and } W) = P(B) * P(W|B) = \frac{2}{10} * \frac{5}{9} = \frac{10}{90} = 0.111$$

c. selecting 1 red ball and then 1 blue ball

$$P(R \text{ and } B) = P(R) * P(B|R) = \frac{3}{10} * \frac{2}{9} = \frac{6}{90} = 0.067$$

Independent and Dependent Events

EXAMPLE

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

Let C denote red-green color blindness. Then

$$\boldsymbol{P(C \text{ and } C \text{ and } C) = P(C) * P(C) * P(C) = \frac{9}{100} * \frac{9}{100} * \frac{9}{100}} = 0.000729$$

Hence, the rounded probability is 0.0007

EXAMPLE

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that only the first two of them will have this type of red-green color blindness.

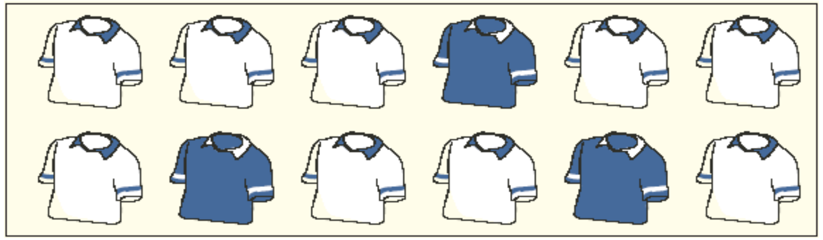
Let C denote red-green color blindness. Then

$$\boldsymbol{P(C \text{ and } C \text{ and } NC) = P(C) * P(C) * P(NC) = \frac{9}{100} * \frac{9}{100} * \frac{91}{100} = 0.007371}$$

Hence, the rounded probability is 0.007

EXAMPLE

A golfer has 12 golf shirts in his closet. Suppose 9 of these shirts are white and the others blue. He gets dressed in the dark, so he just grabs a shirt and puts it on. He plays golf two days in a row and does not do laundry.



Let W be the symbol for selecting a white shirt and

B be the symbol for selecting a blue shirt.

What is the likelihood of selecting both white shirts?

$$P(W \text{ and } W) = P(W)P(W|W) = (9/12)(8/11) = 0.545$$

What is the likelihood of selecting a white then a blue shirt?

$$P(W \text{ and } B) = P(W)P(B|W) = (9/12)(3/11) = 0.205$$

Counting Rule

The multiplication rule can be used to determine the total number of outcomes in a sequence of events.

<u>Fundamental counting rule</u>

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3 and so forth, the total number of possibilities of the sequence will be:

$$k_1 * k_2 * k_3 * \cdots * k_n$$

Note: “And” in this case means to multiply.

Counting Rule

EXAMPLE

A paint manufacturer whishes to manufacture several different paints. The categories include

Color	Red, blue, white, black, green, brown, yellow
Type	Latex, oil
Texture	Flat, semi gloss, high gloss
Use	Outdoor, indoor

How many different kinds of paint can be made if a person can select one color, one type, one texture and one use?

Since there are 7 color choices, 2 type choices, 3 texture choices and 2 use choices, then the total number of possible different paints is

$$\mathbf{7} * \mathbf{2} * \mathbf{3} * \mathbf{2} = 84$$

EXAMPLE

The digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are to be used in a four-digit ID card. How many different cards are possible if

- repetitions are permitted?

Since there are 4 spaces to fill and 10 choices for each space, then the number of possible different cards is

$$\mathbf{10} * \mathbf{10} * \mathbf{10} * \mathbf{10} = \mathbf{10^4} = \mathbf{10000}$$

- repetitions are not permitted?

Since there are 4 spaces to fill and 10 choices for first space, 9 choices for the second space, 8 choices for the third space and 7 choices for fourth space, then the number of possible different cards is

$$\mathbf{10} * \mathbf{9} * \mathbf{8} * \mathbf{7} = 5040$$

Permutations

The arrangement of n objects in a specific order using r objects at a time is called a *permutation of n objects taking r objects at a time*.

It is written as ${}_nP_r$, and the formula is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

where

$$ n! = n \times (n-1) \times (n-2) \times \cdots \times 1 $$
$$ 4! = 4 \times 3 \times 2 \times 1 $$
$$ 0! = 1 $$

Permutations

EXAMPLE

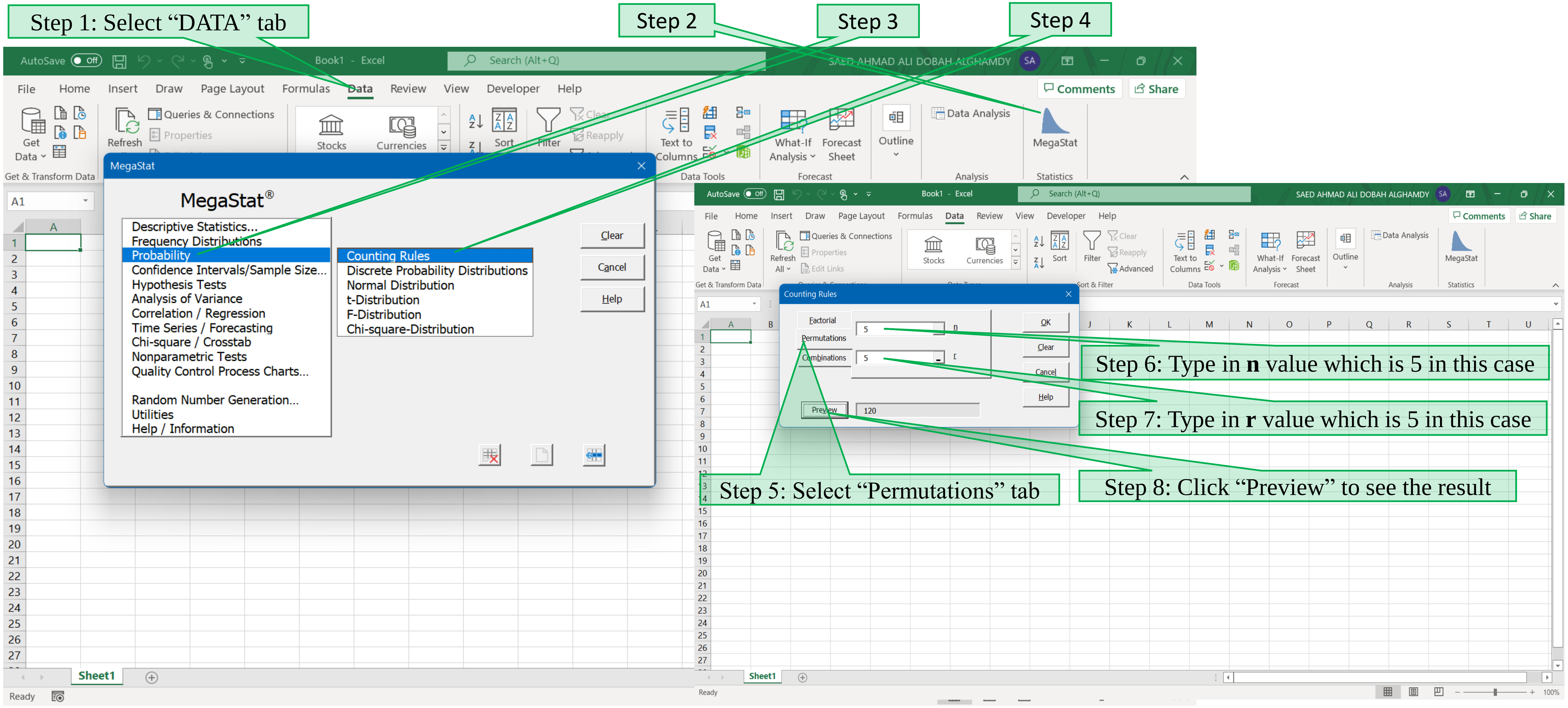
Suppose a business owner has a choice of five locations in which to establish his business. He decide to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can he rank the five locations?

Since there are 5 choices for the first location, 4 choices for the second location, 3 choices for the third location, 2 choices for the fourth location and 1 choice for the last location, then the number of ways is

$$_5P_5 = \frac{5!}{(5-5)!} = 5! = 5 * 4 * 3 * 2 * 1 = 120$$

Permutations

EXAMPLE Ranking five locations



Permutations

EXAMPLE

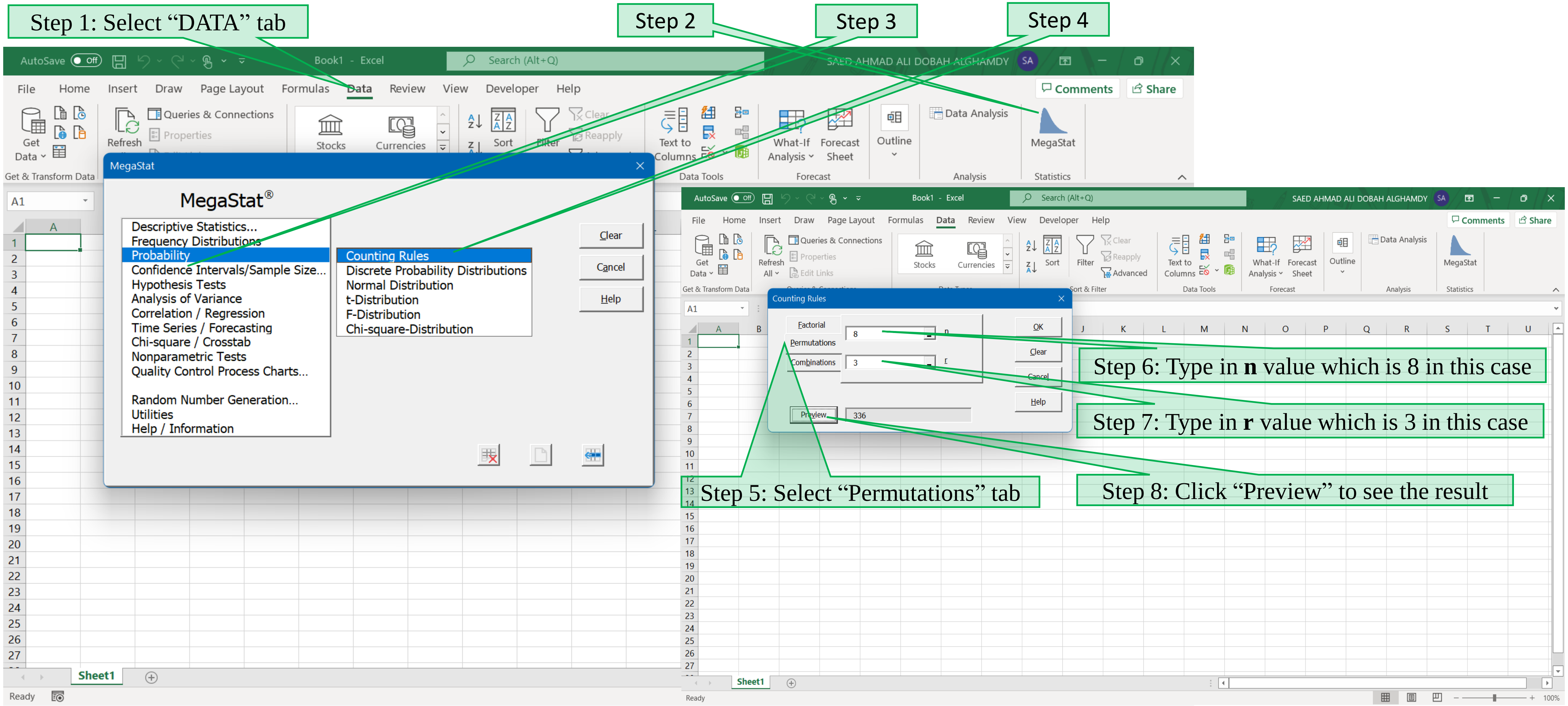
A television news director wishes to use three news stories on an evening show. One story will be the lead story, one will be the second story and the last will be a closing story. If the director has a total of eight stories to choose from, how many possible ways can the program be set up?

Since the order is important, then the number of ways to set up the program is

$$ {}_8P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8*7*6*5!}{5!} = 8*7*6 = 336 $$

Permutations

EXAMPLE Selecting 3 from 8 in order



Combinations

A selection of distinct objects without regard to order is called a *combination*.

The number of combinations of r objects selected from n objects is denoted ${}_nC_r$ and is given by the formula

$$ {}_nC_r = \frac{n!}{(n-r)! * r!} $$

EXAMPLE

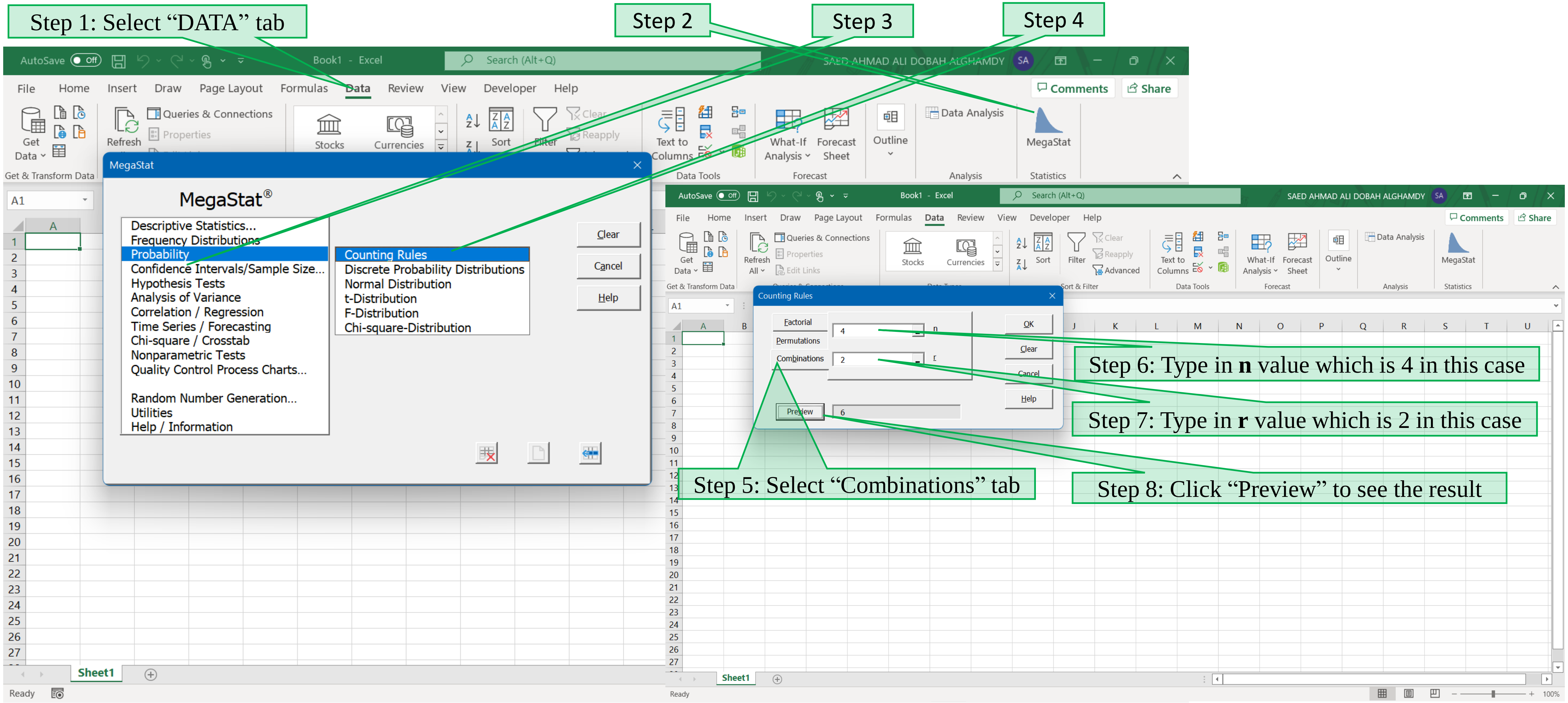
How many combination of 4 objects are there, taken 2 at a time?

Since this is a combination problem, then

$$ {}_4C_2 = \frac{4!}{(4-2)! * 2!} = \frac{4 * 3 * 2!}{2! * 2!} = \frac{4 * 3}{2 * 1} = 6 $$

Combinations

EXAMPLE Selecting 2 from 4 regardless of order



Combinations

EXAMPLE

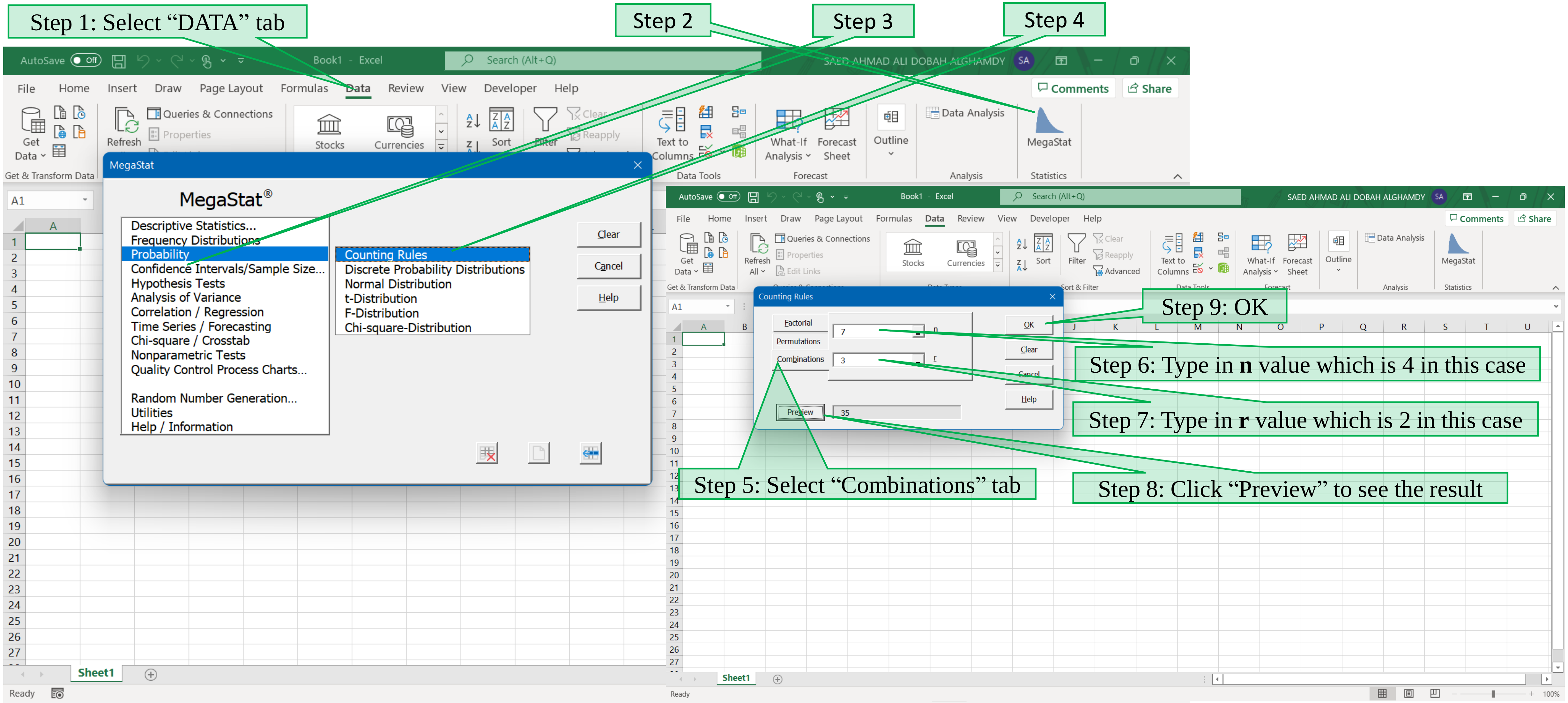
In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

Here, one must select 3 women from 7 women and selects 2 men from 5 men. Then, using the fundamental counting rule, we can find the total number of different possibilities.

$$ {}_7C_3 * {}_5C_2 = \frac{7!}{(7-3)! * 3!} * \frac{5!}{(5-2)! * 2!} = 350 $$

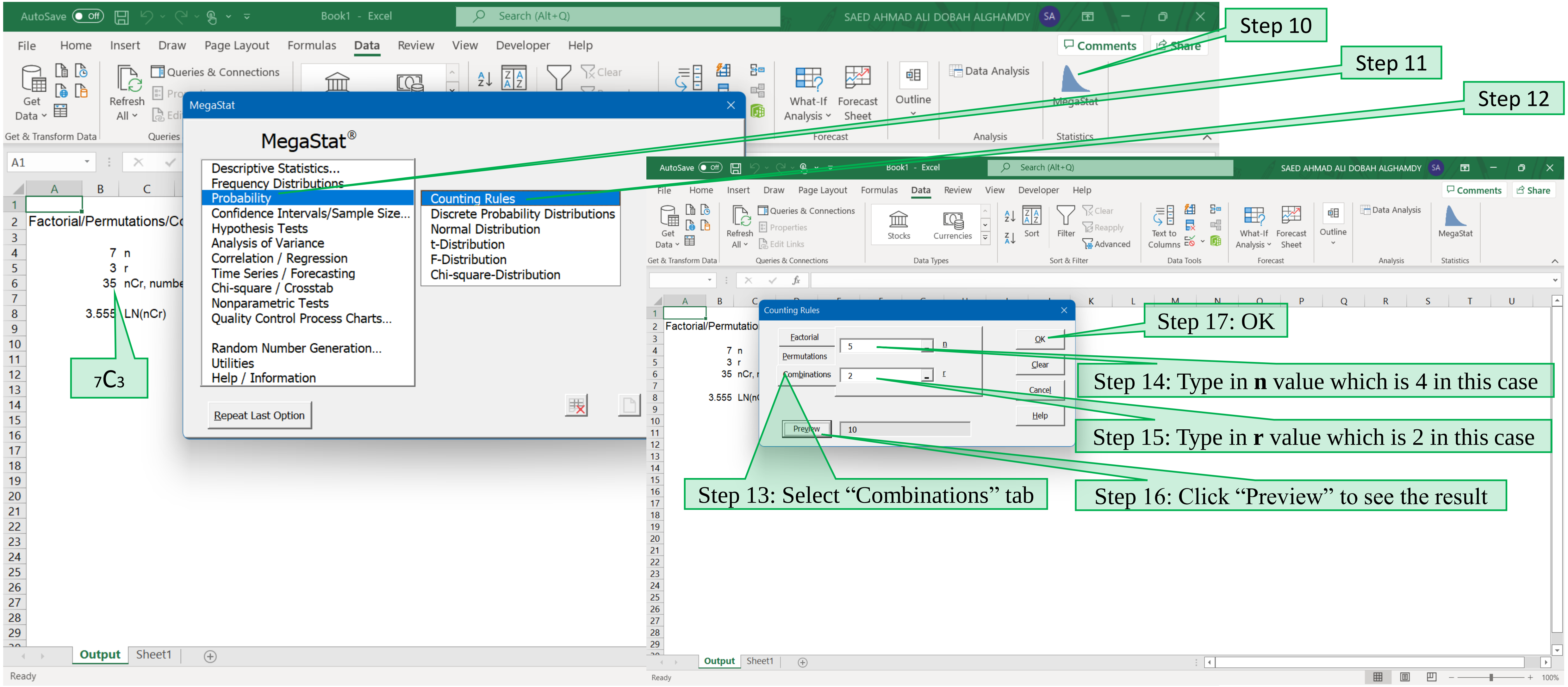
Combinations

EXAMPLE Selecting 3 from 7 and 2 from 5 regardless of order



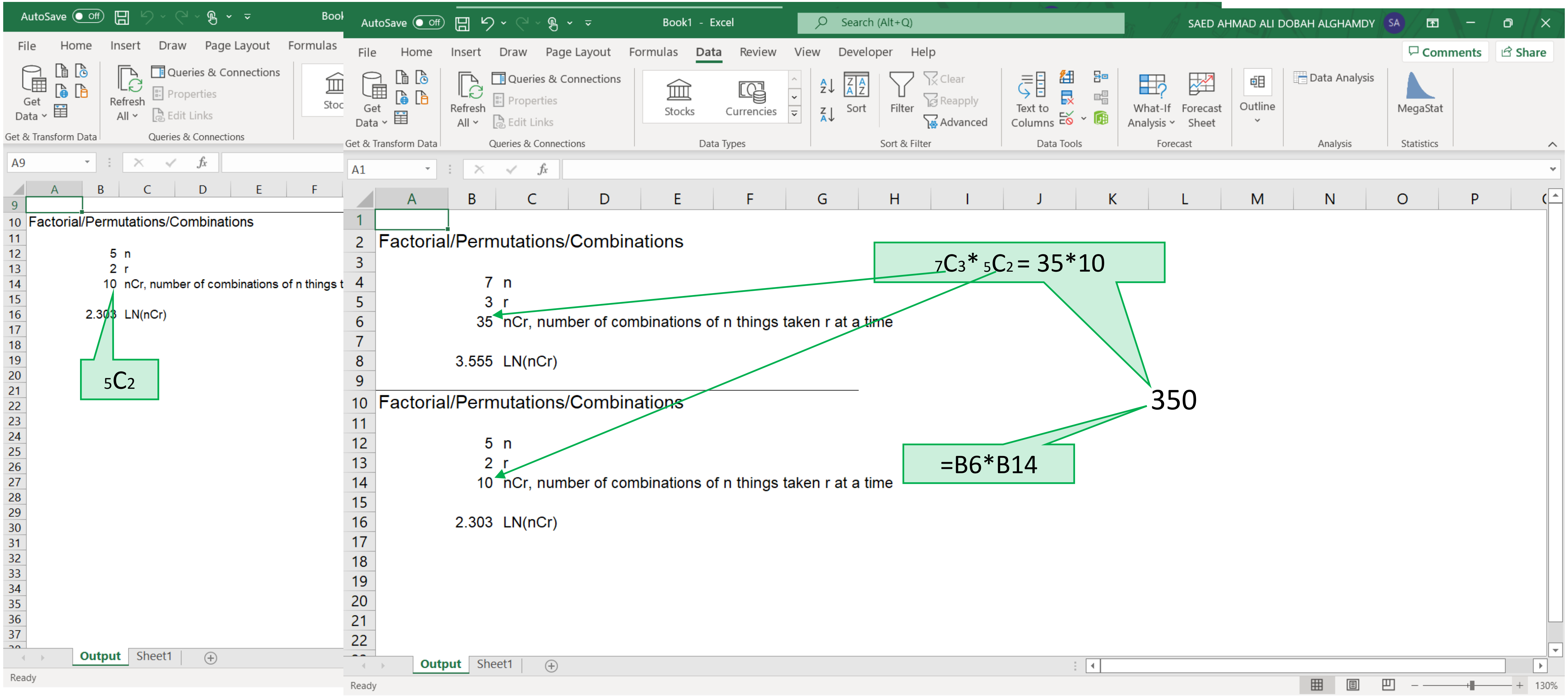
Combinations

EXAMPLE Selecting 3 from 7 and 2 from 5 regardless of order



Combinations

EXAMPLE Selecting 3 from 7 and 2 from 5 regardless of order



Summary

- ✓ The two types of probability are *classical* and *empirical*.
- ✓ Classical probability uses sample spaces and assumes that all outcomes in the sample space are equally likely to occur.
- ✓ Empirical probability uses frequency distributions and is based on observations.
- ✓ Two events are said to be mutually exclusive if they cannot occur together at the same time.
- ✓ Events can be independent or dependent if they occur in sequence.
- ✓ If events are independent, whether or not the first event occurs does not affect the probability of the next event occurring.
- ✓ If the probability of the second event occurring is changed by the occurrence of the first event, then the events are dependent.

Summary

Rule	Definition
Multiplication rule $k_1 \cdot k_2 \cdot k_3 \cdots \cdot k_n$	The number of ways a sequence of *n* events can occur; if the first event can occur in k_1 ways, the second event can occur in k_2 ways, etc.
Permutation rule ${}_nP_r = \frac{n!}{(n-r)!}$	The arrangement of *n* objects in a specific order using *r* objects at a time (order is important)
Combination rule ${}_nC_r = \frac{n!}{(n-r)!r!}$	The number of combinations of *r* objects selected from *n* objects (order is not important)

Review Examples

A combination lock consists of the 26 letters of the alphabet. If a 3-letter combination is needed and the same letter can be used more than once.

Find the probability that the combination will consist of the letters ABC in that order.

Find the probability that the combination will consist of the letters ABC.

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen according to their abilities. How many different possibilities are there?

There are 8 married couples in a tennis club. If a man and a woman are selected at random to plan the summer tournament, find the probability that they are married to each other.

Review Examples

Given the letters A, B, C, and D, list the permutations and combinations for selecting two letters.

A school musical director can select 2 musical plays to present next year. One will be presented in the fall, and one will be presented in the spring. If she has 9 to pick from, how many different possibilities are there?

Suppose a business owner has a choice of 5 locations in which to establish his business. He decides to rank only the top 3 of the 5 locations. How many different ways can he rank them?

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

Review Examples

There are four blood types, A, B, AB, and O. Blood can also be +ve or -ve. Finally, a blood donor can be classified as either male or female. How many different ways can a donor have his or her blood labeled?

A Harris poll found that 46% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Find the probability of getting identical number of spots when rolling two dice.

Review Examples

The probability of a person driving with a friend is 0.32, the probability of a person having a driving accident is 0.09, and the probability of a person having a driving accident while driving with a friend is 0.15. What is the probability of a person driving with a friend or having a driving accident?

The corporate research and development centers for three local companies have the following number of employees:

U.S. Steel	110
Alcoa	750
Bayer Material Science	250

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

Review Examples

A city has 9 coffee shops: 3 Starbuck's, 2 Caribou Coffees, and 4 Crazy Mocha Coffees. If a person selects one shop at random to buy a cup of coffee, find the probability that it is either a Starbuck's or Crazy Mocha Coffees.

Hospital records indicated that knee replacement patients stayed in the hospital for the number of days shown in the distribution.

Number of days stayed	**3**	**4**	**5**	**6**	**7**	**Σ**
Frequency	**15**	**32**	**56**	**19**	**5**	**127**

Find these probabilities.

a. A patient stayed exactly 5 days.

b. A patient stayed at most 4 days.

c. A patient stayed less than 6 days.

d. A patient stayed at least 5 days.