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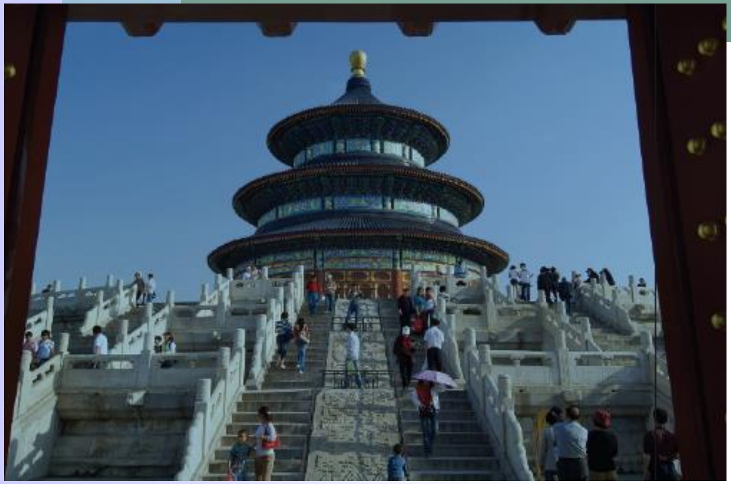
Introduction to Probability



Chapter 4

Learning Objectives

LO1 Appreciate the role probability information plays in the decision-making process.

LO2 Understand probability as a numerical measure of the likelihood of occurrence.

LO3 Appreciate the two methods commonly used for assigning probabilities and understand when they should be used.

LO4 Use the laws that are available for computing the probabilities of events.

LO5 Understand how new information can be used to revise initial (prior) probability estimates using Bayes' theorem.

Introduction

Managers often base their decisions on an analysis of uncertainties such as the following:

1. What are the chances that sales will decrease if we increase prices?

2. What is the likelihood a new assembly method will increase productivity?

3. How likely is it that the project will be finished on time?

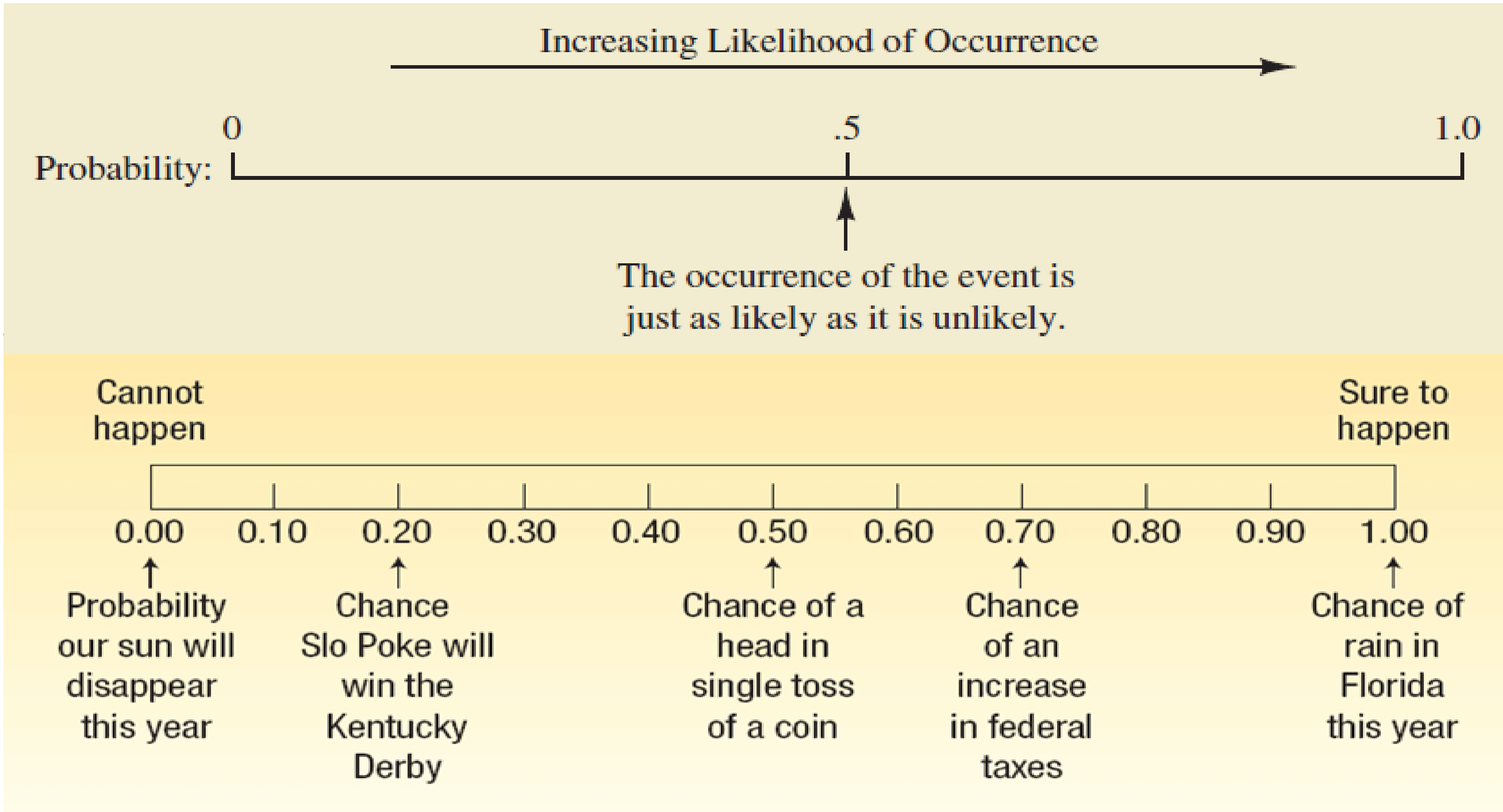
4. What is the chance that a new investment will be profitable?

In discussing probability, we define an **experiment** as a process that generates well-defined outcomes. On any single experiment, one and only one of the possible experimental outcomes will occur. Some examples of experiments and their associated outcomes follow.

Experiment	Experimental Outcomes
Toss a coin	Head, tail
Select a part for inspection	Defective, nondefective
Conduct a sales call	Purchase, no purchase
Roll a die	1, 2, 3, 4, 5, 6
Play a football game	Win, lose, tie

Probability

PROBABILITY is a numerical value between zero and one, inclusive, describing the relative possibility (chance or likelihood) an event will occur.



Experiment, Outcome, Event: Defined

- An **experiment** is a chance process that leads to the occurrence of one and only one of several possible outcomes.
- An **outcome** or sample point is the result of a single trial of an experiment.
- A **sample space** is the set of all possible outcomes of an experiment.
- An **event** is the collection of one or more outcomes of an experiment.

		
Experiment	Roll a die	Count the number of members of the board of directors for Fortune 500 companies who are over 60 years of age
All possible outcomes	Observe a 1 Observe a 2 Observe a 3 Observe a 4 Observe a 5 Observe a 6	None are over 60 One is over 60 Two are over 60 ... 29 are over 60 ... ... 48 are over 60 ...
Some possible events	Observe an even number Observe a number greater than 4 Observe a number 3 or less	More than 13 are over 60 Fewer than 20 are over 60

Counting, Permutation and Combination

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

Multiple-step experiments The first counting rule applies to multiple-step experiments. The counting rule for multiple-step experiments makes it possible to determine the number of experimental outcomes without listing them.

COUNTING RULE FOR MULTIPLE-STEP EXPERIMENTS

If an experiment can be described as a sequence of k steps with n_1 possible outcomes on the first step, n_2 possible outcomes on the second step, and so on, then the total number of experimental outcomes is given by $(n_1)\,(n_2)\,.\,.\,.\,(n_k)$.

EXAMPLE

An experiment has three steps with three outcomes possible for the first step, two outcomes possible for the second step, and four outcomes possible for the third step. How many experimental outcomes exist for the entire experiment?

$$3 \times 2 \times 4 = 12 \text{ outcomes}$$

Counting, Permutation and Combination

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

A **combination** is the number of ways to choose *n* objects from a group of *N* objects without regard to order.

COUNTING RULE FOR COMBINATIONS

The number of combinations of N objects taken n at a time is

$$C_n^N = \binom{N}{n} = \frac{N!}{n!(N-n)!}$$

where

$$N! = N(N-1)(N-2)\cdots(2)(1)$$
$$n! = n(n-1)(n-2)\cdots(2)(1)$$

and, by definition,

$$0! = 1$$

EXAMPLE

Consider a quality control procedure in which an inspector randomly selects two of five parts to test for defects. In a group of five parts, In how many ways of two parts can be selected?

$$C_2^5 = \binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{(5)(4)(3)(2)(1)}{(2)(1)(3)(2)(1)} = \frac{120}{12} = 10$$

Counting, Permutation and Combination

Being able to identify and count the experimental outcomes is a necessary step in assigning probabilities. We now discuss three useful counting rules.

A **permutation** is any arrangement of *r* objects selected from *n* possible objects. The order of arrangement is important in permutations.

COUNTING RULE FOR PERMUTATIONS

The number of permutations of N objects taken n at a time is given by

$$P_n^N = n!\binom{N}{n} = \frac{N!}{(N-n)!}$$

EXAMPLE

Consider again the quality control procedure in which an inspector randomly selects two of five parts to test for defects but when the order of selection must be taken into account. In a group of five parts, In how many ways of two parts can be selected?

$$P_2^5 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{(5)(4)(3)(2)(1)}{(3)(2)(1)} = \frac{120}{6} = 20$$

Assigning Probabilities

BASIC REQUIREMENTS FOR ASSIGNING PROBABILITIES

1. The probability assigned to each experimental outcome must be between 0 and 1, inclusively. If we let E_i denote the ith experimental outcome and $P(E_i)$ its probability, then this requirement can be written as

$$0 \leq P(E_i) \leq 1 \text{ for all } i$$

2. The sum of the probabilities for all the experimental outcomes must equal 1.0. For n experimental outcomes, this requirement can be written as

$$P(E_1) + P(E_2) + \cdots + P(E_n) = 1$$

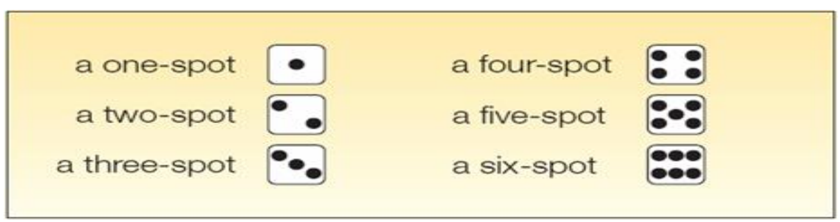
Classical Probability

- ***Classical probability*** uses sample spaces to determine the numerical probability that an event will happen. It assumes that all outcomes in the sample space are equally likely to occur (events have the same probability of occurring).
- The probability of an event *E* can be defined as

$$P(E) = \frac{n(E)}{n(S)} = \frac{\text{Number of outcomes in E}}{\text{number of outcomes in the sample space}}$$

EXAMPLE

Consider an experiment of rolling a six-sided die. What is the probability of the event “an even number of spots appear face up”? The possible outcomes are:



There are three “favorable” outcomes (a two, a four, and a six) in the collection of six equally likely possible outcomes.

Empirical Probability

- ***Empirical probability*** relies on actual experience to determine the likelihood of outcomes.
- The empirical approach to probability is based on the law of large numbers. The key to establishing probabilities empirically is that more observations will provide a more accurate estimate of the probability.
- Given a frequency distribution, the probability of an event being in a given class is the relative frequency of that class:

$$P(E) = \frac{\text{frequency of the class}}{\text{total frequency of the distribution}} = \frac{f}{n}$$

EXAMPLE:

On February 1, 2003, the Space Shuttle Columbia exploded. This was the second disaster in 123 space missions for NASA. On the basis of this information, what is the probability that a future mission is successfully completed?

$$\text{Probability of a successful flight} = \frac{\text{Number of successful flights}}{\text{Total number of flights}}$$

$$= \frac{121}{123} = 0.98$$

Events and Their Probabilities

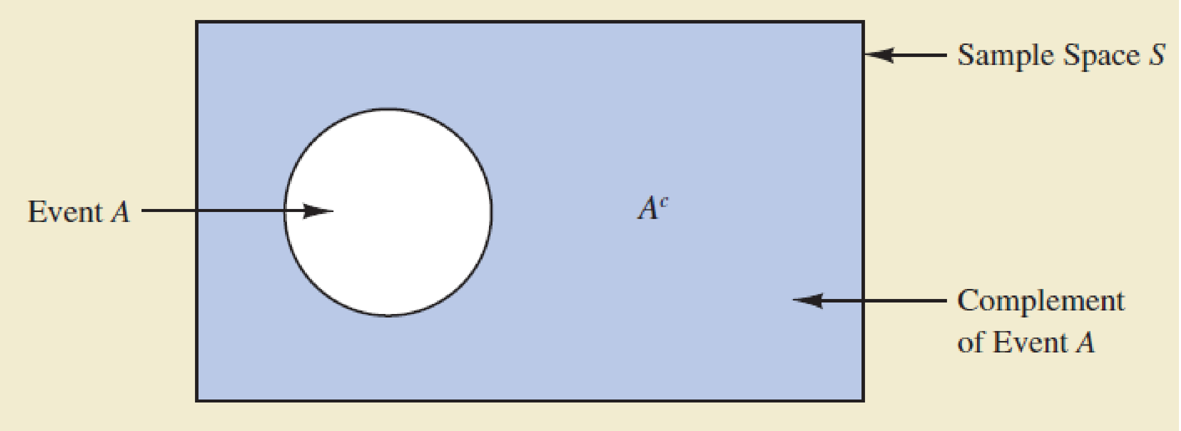
PROBABILITY OF AN EVENT

The probability of any event is equal to the sum of the probabilities of the sample points in the event.

The **complement of an event *A*** is defined to be the event consisting of all sample points that are *not* in ***A***.

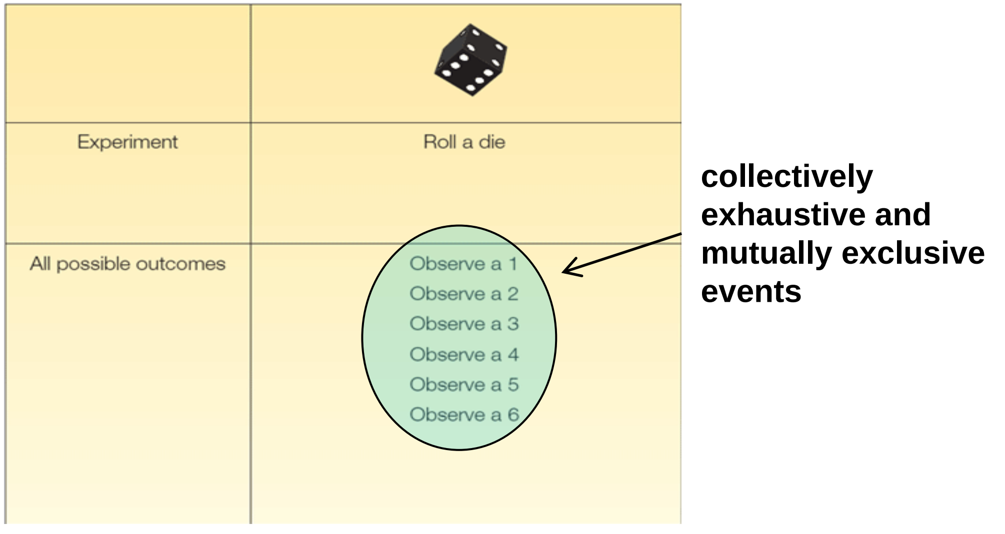
In any probability application, either event A or its complement A^c must occur. Therefore, we have

$$P(A) + P(A^c) = 1$$



Events and Their Probabilities

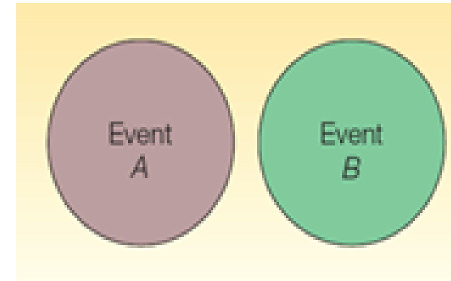
- Events are **mutually exclusive** if the occurrence of an event means that none of the others can occur at the same time.
- Events are **collectively exhaustive** if at least one of the events must occur when an experiment is conducted.
- The sum of all collectively exhaustive and mutually exclusive events is 1.0 (or 100%)



- Events are **independent** if the occurrence of one event does not affect the occurrence of another.

Addition Law

- Special Rule of Addition - If two events ***A*** and ***B*** are mutually exclusive, the probability of one *or* the other event’s occurring is



$$P(A \text{ or } B) = P(A) + P(B)$$

EXAMPLE

An automatic Shaw machine fills plastic bags with a Mixture of beans, broccoli, and other vegetables. Most of the bags contain the correct weight, but because of the variation in the size of the beans and other vegetables, a package might be under-weight or overweight. A check of 4,000 packages filled in the past month revealed:

What is the probability that a particular package will be either underweight or overweight?

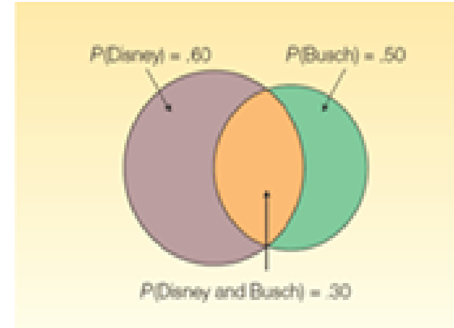
Weight	Event	Number of Packages	Probability of Occurrence	
Underweight	*A*	100	.025	← $\frac{100}{4,000}$
Satisfactory	*B*	3,600	.900	
Overweight	*C*	300	.075	
		4,000	1.000	

$$P(A \text{ or } C) = P(A) + P(C) = .025 + .075 = .10$$

Addition Law

- The General Rule of Addition - If ***A*** and ***B*** are two events that are not mutually exclusive, the probability of one *or* the other event's occurring is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$



EXAMPLE

The Venn Diagram shows the result of a survey of 200 tourists who visited Florida during the year. The survey revealed that 120 went to Disney World, 100 went to Busch Gardens and 60 visited both.

What is the probability a selected person visited either Disney World or Busch Gardens?

$$\begin{aligned} P(\text{Disney or Busch}) &= P(\text{Disney}) + P(\text{Busch}) - P(\text{both Disney and Busch}) \\ &= 120/200 + 100/200 - 60/200 \\ &= .60 + .50 - .30 = 0.8 \end{aligned}$$

Joint Probability and Multiplication Rules

The **JOINT PROBABILITY** is a probability that measures the likelihood of two or more events that will happen concurrently.

The special rule of multiplication is used to find the joint probability that two *independent* events will occur.

- Two events *A* and *B* are **independent** if the occurrence of one has no effect on the probability of the occurrence of the other.

$$\boldsymbol{P(\text{A and B}) = P(\text{A})\, P(\text{B})}$$

EXAMPLE

A survey by the American Automobile association (AAA) revealed 60% of its members made airline reservations last year. Two members are selected at random. Since the number of AAA members is very large, we can assume that *M1 and M2 are independent*. What is the probability that *both* made airline reservations last year?

The probability that M1 made a reservation last year is $P(M_1) = 0.60$

The probability that M2 made a reservation last year is also $P(M_2) = 0.60$.

Thus, $\boldsymbol{P(M_1 \text{ and } M_2) = P(M_1)P(M_2) = (0.60)(0.60) = 0.36}$

Joint Probability and Multiplication Rules

The **general rule of multiplication** is used to find the joint probability that two *not independent* events will occur.

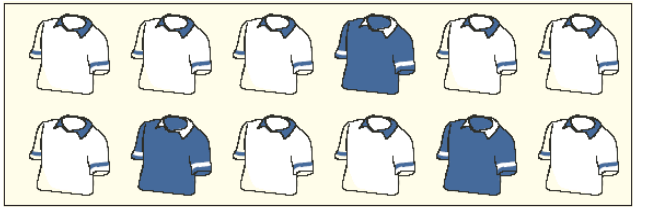
- Two events *A* and *B* are not independent if the occurrence of one has effect on the probability of the occurrence of the other.

$$P(\text{A and B}) = P(\text{A})\, P(\text{B}|\text{A})$$

EXAMPLE

A golfer has 12 golf shirts in his closet. Suppose 9 of these shirts are white and the others blue. He gets dressed in the dark, so he just grabs a shirt and puts it on. He plays golf two days in a row and does not do laundry.

What is the likelihood both shirts selected are white?



Let W_1 be the symbol for selecting the first shirt as white which has $P(W_1) = 9/12$.
Let W_2 be the symbol for selecting the second shirt as white.
The conditional probability that the second shirt selected is white, given that the first shirt selected is also white, is $P(W_2 \mid W_1) = 8/11$.
Thus, the probability of 2 white shirts being selected is

$$P(W_1 \text{ and } W_2) = P(W_1)P(W_2|W_1) = (9/12)(8/11) = 0.55$$

Bayes' Theorem

Prior Probabilities → New Information → Application of Bayes' Theorem → Posterior Probabilities

To find $P(B)$, we note that event B can occur in only two ways: ($A1 \;\; B$) or ($A2 \;\; B$). Therefore, we have

$$\boldsymbol{P(B) = P(A1 \cap B) + P(A2 \cap B) = P(A1)P(B \mid A1) + P(A2)P(B \mid A2)}$$

BAYES' THEOREM

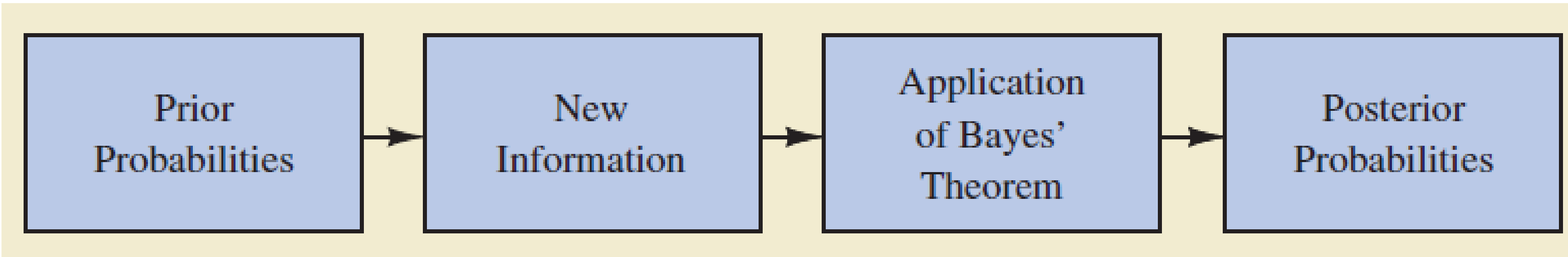
$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_n)P(B \mid A_n)}$$

BAYES' THEOREM (TWO-EVENT CASE)

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

Bayes' Theorem



As an application of Bayes' theorem, consider a manufacturing firm that receives shipments of parts from two different suppliers. Let $A1$ denote the event that a part is from supplier 1 and $A2$ denote the event that a part is from supplier 2. Currently, 65% of the parts purchased by the company are from supplier 1 and the remaining 35% are from supplier 2. Hence, if a part is selected at random, we would assign the prior probabilities $P(A1)=0.65$ and $P(A2)=0.35$.

The quality of the purchased parts varies with the source of supply. Historical data suggest that the quality ratings of the two suppliers are as the following where G for good and B for bad parts:

$$P(G \mid A1) = 0.98 \qquad P(B \mid A1) = 0.02$$
$$P(G \mid A2) = 0.95 \qquad P(B \mid A2) = 0.05$$

Bayes' Theorem

Prior Probabilities → New Information → Application of Bayes' Theorem → Posterior Probabilities

So,

(1) Events A_i	(2) Prior Probabilities $P(A_i)$	(3) Conditional Probabilities $P(B \mid A_i)$	(4) Joint Probabilities $P(A_i \cap B)$	(5) Posterior Probabilities $P(A_i \mid B)$
A_1	.65	.02	.0130	.0130/.0305 = .4262
A_2	.35	.05	.0175	.0175/.0305 = .5738
	1.00		$P(B)$ = .0305	1.0000

Thus, an overall probability of .0305 of finding a bad part from the combined shipments of the two suppliers.
Also, given information that the part is bad, the probability that the part is from supplier 1 drops to 0.4262. In fact, if the part is bad, it has better than a 50–50 chance that it came from supplier 2; that is, $P(A2 \mid B) = 0.5738$.

Summary

Probability A numerical measure of the likelihood that an event will occur.
Experiment A process that generates well-defined outcomes.
Sample space The set of all experimental outcomes.
Sample point An element of the sample space. A sample point represents an experimental outcome.
Basic requirements for assigning probabilities Two requirements that restrict the manner in which probability assignments can be made:
(1) for each experimental outcome Ei we must have $0 \leq P(Ei) \leq 1$;
(2) considering all experimental outcomes, we must have $P(E1)+P(E2)+...+P(En)=1.0$.
Classical method A method of assigning probabilities that is appropriate when all the experimental outcomes are equally likely.
Relative frequency method A method of assigning probabilities that is appropriate when data are available to estimate the proportion of the time the experimental outcome will occur if the experiment is repeated a large number of times.
Event A collection of sample points.
Complement of *A* The event consisting of all sample points that are not in A.

Summary

Addition law A probability law used to compute the probability of the union of two events. It is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. For mutually exclusive events, $P(A \cap B) = 0$.

Mutually exclusive events Events that have no sample points in common; that is, $A \cap B$ is empty and $P(A \cap B) = 0$.

Conditional probability The probability of an event given that another event already occurred. The conditional probability of A given B is $P(A|B)=P(A \cap B)/P(B)$

Joint probability The probability of two events both occurring; that is, the probability of the intersection of two events.

Independent events Two events A and B where $P(A|B)=P(A)$ or $P(B|A)=P(B)$ that is, the events have no influence on each other.

Multiplication law A probability law used to compute the probability of the intersection of two events. It is $P(A \cap B)=P(B)P(A \mid B)$ or $P(A \cap B)=P(A)P(B \mid A)$. For independent events it reduces to $P(A \cap B) = P(A)P(B)$.

Prior probabilities Initial estimates of the probabilities of events.

Posterior probabilities Revised probabilities of events based on additional information.

Bayes' theorem A method used to compute posterior probabilities.