# Ch 13. Magnetically Coupled Circuits

Note Title 9/13/2014

## 13.2 Mutual Inductance:

* Single Inductor case:

Faraday's Law:

(# of turns)

$$v = N\frac{d\phi}{dt}$$

$$v = N\frac{d\phi}{di}\frac{di}{dt}$$ (self-inductance (L))

$$v = L\frac{di}{dt}$$

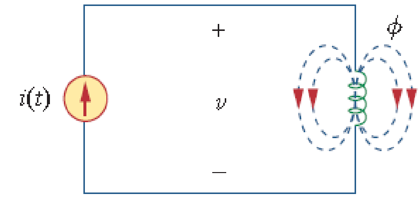



**Figure 13.1**
Magnetic flux produced by a single coil with $N$ turns.

* Two Inductors case:

$L_1$ & $L_2$ are magnetically coupled

a) $i_2 = 0$ :

$$\phi_1 = \phi_{11} + \phi_{12}$$

$$v_1 = N_1\frac{d\phi_1}{di_1}\frac{di_1}{dt} = L_1\frac{di_1}{dt}$$

$$v_2 = N_2\frac{d\phi_{12}}{di_1}\frac{di_1}{dt} = M_{21}\frac{di_1}{dt}$$

(mutual inductance)

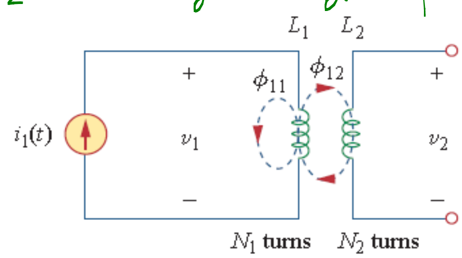



**Figure 13.2**
Mutual inductance $M_{21}$ of coil 2 with respect to coil 1.

b) $i_1 = 0$ :

$$\phi_2 = \phi_{21} + \phi_{22}$$

$$v_2 = N_2\frac{d\phi_2}{dt} = N_2\frac{d\phi_2}{di_2}\frac{di_2}{dt} = L_2\frac{di_2}{dt}$$

$$v_1 = N_1\frac{d\phi_{21}}{dt} = N_1\frac{d\phi_{21}}{di_2}\frac{di_2}{dt} = M_{12}\frac{di_2}{dt}$$

(mutual inductance)

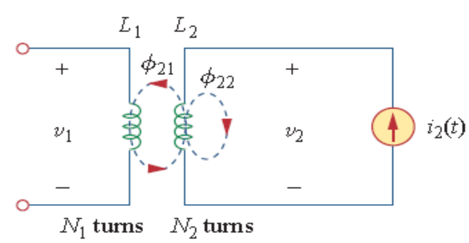



**Figure 13.3**
Mutual inductance $M_{12}$ of coil 1 with respect to coil 2.

$$M_{12} = M_{21} = M$$ → measures in Henrys (H)

**Mutual inductance** is the ability of one inductor to induce a voltage across a neighboring inductor, measured in henrys (H).

## * Dot Convention:

If a current **enters** the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is **positive** at the dotted terminal of the second coil.

If a current **leaves** the dotted terminal of one coil, the reference polarity of the mutual voltage in the second coil is **negative** at the dotted terminal of the second coil.

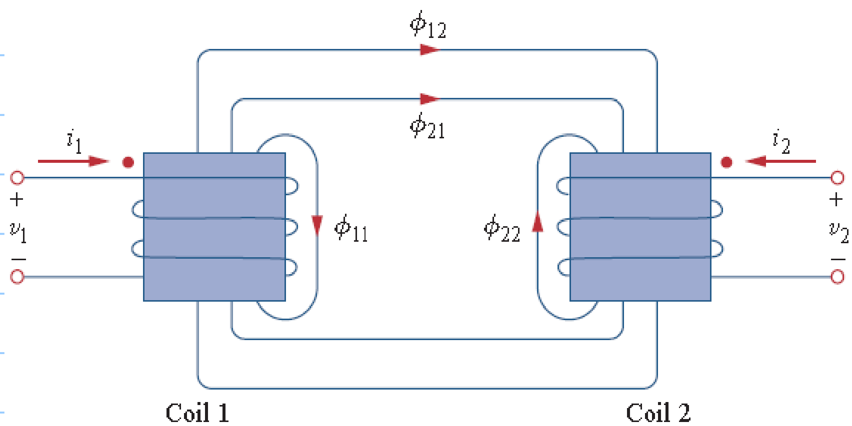



**Figure 13.4**
Illustration of the dot convention.

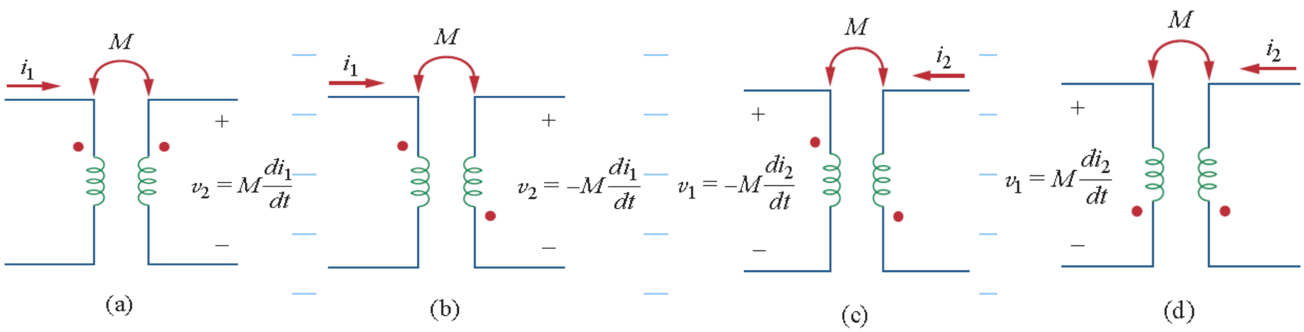



**Figure 13.5**
Examples illustrating how to apply the dot convention.

$$L = L_1 + L_2 + 2M \quad \text{(Series-aiding connection)}$$

$$L = L_1 - L_2 - 2M \quad \text{(Series-opposing connection)}$$

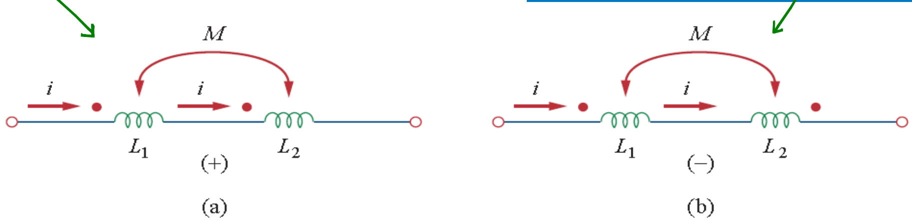



**Figure 13.6**
Dot convention for coils in series; the sign indicates the polarity of the mutual voltage: (a) series-aiding connection, (b) series-opposing connection.

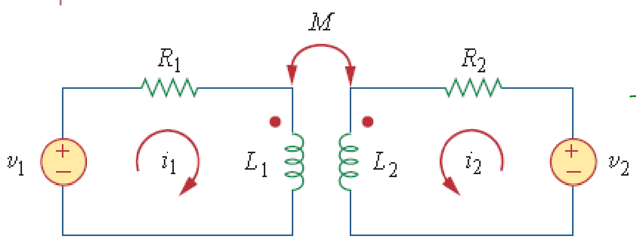



**Figure 13.7**
Time-domain analysis of a circuit containing coupled coils.

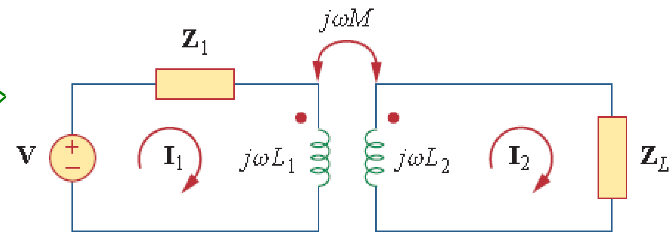



**Figure 13.8**
Frequency-domain analysis of a circuit containing coupled coils.

## Example 13.1

Calculate the phasor currents $\mathbf{I}_1$ and $\mathbf{I}_2$ in the circuit of Fig. 13.9.

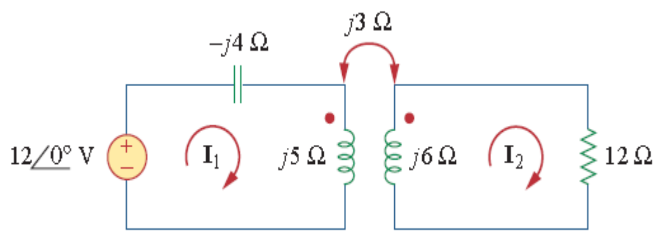



**Figure 13.9**
For Example 13.1.

**Solution:**
For coil 1, KVL gives

$$-12 + (-j4 + j5)\mathbf{I}_1 - j3\mathbf{I}_2 = 0$$

or

$$j\mathbf{I}_1 - j3\mathbf{I}_2 = 12 \qquad \textbf{(13.1.1)}$$

For coil 2, KVL gives.

$$-j3\mathbf{I}_1 + (12 + j6)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = \frac{(12 + j6)\mathbf{I}_2}{j3} = (2 - j4)\mathbf{I}_2 \qquad \textbf{(13.1.2)}$$

Substituting this in Eq. (13.1.1), we get

$$(j2 + 4 - j3)\mathbf{I}_2 = (4 - j)\mathbf{I}_2 = 12$$

or

$$\mathbf{I}_2 = \frac{12}{4 - j} = 2.91\underline{/14.04^\circ}\text{ A}$$

From Eqs. (13.1.2) and (13.1.3),

$$\mathbf{I}_1 = (2 - j4)\mathbf{I}_2 = (4.472\underline{/-63.43^\circ})(2.91\underline{/14.04^\circ})$$
$$= 13.01\underline{/-49.39^\circ}\text{ A}$$

## Example 13.2

Calculate the mesh currents in the circuit of Fig. 13.11.

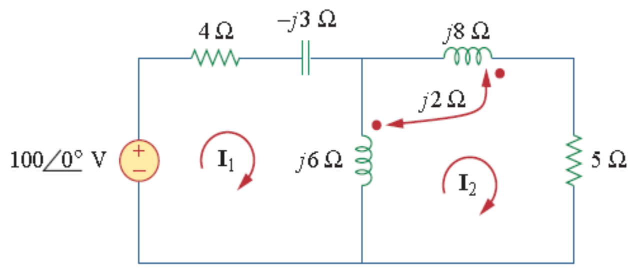



**Figure 13.11**
For Example 13.2.

## Solution:

Thus, for mesh 1 in Fig. 13.11, KVL gives

$$-100 + \mathbf{I}_1(4 - j3 + j6) - j6\mathbf{I}_2 - j2\mathbf{I}_2 = 0$$

or

$$100 = (4 + j3)\mathbf{I}_1 - j8\mathbf{I}_2 \qquad \textbf{(13.2.1)}$$

Therefore, for mesh 2 in Fig. 13.11, KVL gives

$$0 = -2j\mathbf{I}_1 - j6\mathbf{I}_1 + (j6 + j8 + j2 \times 2 + 5)\mathbf{I}_2$$

or

$$0 = -j8\mathbf{I}_1 + (5 + j18)\mathbf{I}_2 \qquad \textbf{(13.2.2)}$$

Putting Eqs. (13.2.1) and (13.2.2) in matrix form, we get

$$\begin{bmatrix} 100 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{bmatrix} \begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix}$$

The determinants are

$$\Delta = \begin{vmatrix} 4 + j3 & -j8 \\ -j8 & 5 + j18 \end{vmatrix} = 30 + j87$$

$$\Delta_1 = \begin{vmatrix} 100 & -j8 \\ 0 & 5 + j18 \end{vmatrix} = 100(5 + j18)$$

$$\Delta_2 = \begin{vmatrix} 4 + j3 & 100 \\ -j8 & 0 \end{vmatrix} = j800$$

Thus, we obtain the mesh currents as

$$\mathbf{I}_1 = \frac{\Delta_1}{\Delta} = \frac{100(5 + j18)}{30 + j87} = \frac{1{,}868.2\underline{/74.5^\circ}}{92.03\underline{/71^\circ}} = 20.3\underline{/3.5^\circ} \text{ A}$$

$$\mathbf{I}_2 = \frac{\Delta_2}{\Delta} = \frac{j800}{30 + j87} = \frac{800\underline{/90^\circ}}{92.03\underline{/71^\circ}} = 8.693\underline{/19^\circ} \text{ A}$$

# 13.3 Energy in a Coupled Circuit:

Energy in one coil → $w = \frac{1}{2}Li^2$

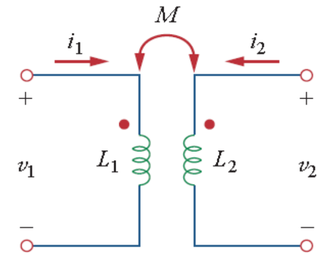



**Figure 13.14**
The circuit for deriving energy stored in a coupled circuit.

Total energy:

If $i_1$ & $i_2$ enter or leave the dots

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

$i_1 \rightarrow \bullet$, $i_2 \leftarrow \bullet$ | $i_1 \leftarrow \bullet$, $i_2 \rightarrow \bullet$

$$M \leq \sqrt{L_1 L_2}$$

→ geometric mean

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{\phi_{12}}{\phi_1} = \frac{\phi_{12}}{\phi_{11} + \phi_{12}}$$

If the entire flux produced by one coil links another coil, then $k = 1$ and we have 100 percent coupling, or the coils are said to be *perfectly coupled*. For $k < 0.5$, coils are said to be *loosely coupled*; and for $k > 0.5$, they are said to be *tightly coupled*. Thus,

> The **coupling coefficient** $k$ is a measure of the magnetic coupling between two coils; $0 \leq k \leq 1$.

## Example 13.3

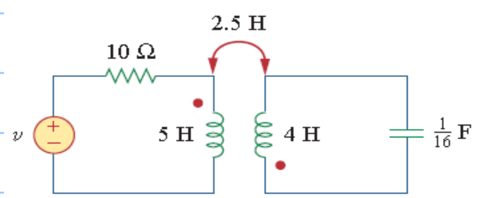



**Figure 13.16**
For Example 13.3.

Consider the circuit in Fig. 13.16. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time $t = 1$ s if $v = 60\cos(4t + 30°)$ V.

**Solution:**
The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

indicating that the inductors are tightly coupled. To find the energy stored, we need to calculate the current. To find the current, we need to obtain the frequency-domain equivalent of the circuit.

$$\begin{aligned}
60\cos(4t + 30°) &\Rightarrow 60\angle 30°, \quad \omega = 4 \text{ rad/s} \\
5 \text{ H} &\Rightarrow j\omega L_1 = j20\ \Omega \\
2.5 \text{ H} &\Rightarrow j\omega M = j10\ \Omega \\
4 \text{ H} &\Rightarrow j\omega L_2 = j16\ \Omega \\
\frac{1}{16} \text{ F} &\Rightarrow \frac{1}{j\omega C} = -j4\ \Omega
\end{aligned}$$

The frequency-domain equivalent is shown in Fig. 13.17. We now apply mesh analysis. For mesh 1,

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\angle 30^\circ \qquad \textbf{(13.3.1)}$$

For mesh 2,

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 \qquad \textbf{(13.3.2)}$$

Substituting this into Eq. (13.3.1) yields

$$\mathbf{I}_2(-12 - j14) = 60\angle 30^\circ \quad \Rightarrow \quad \mathbf{I}_2 = 3.254\angle 160.6^\circ \text{ A}$$

and

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\angle -19.4^\circ \text{ A}$$

In the time-domain,

$$i_1 = 3.905\cos(4t - 19.4^\circ), \qquad i_2 = 3.254\cos(4t + 160.6^\circ)$$

At time $t = 1$ s, $4t = 4$ rad $= 229.2^\circ$, and

$$i_1 = 3.905\cos(229.2^\circ - 19.4^\circ) = -3.389 \text{ A}$$
$$i_2 = 3.254\cos(229.2^\circ + 160.6^\circ) = 2.824 \text{ A}$$

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$
$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73 \text{ J}$$

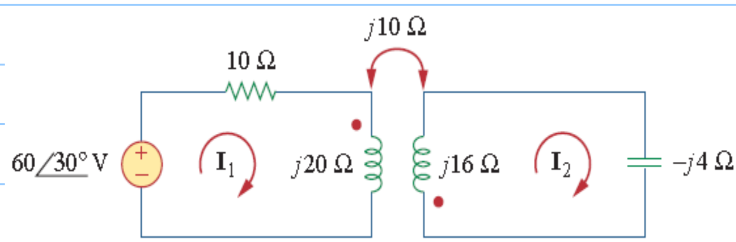



**Figure 13.17**
Frequency-domain equivalent of the circuit in Fig. 13.16.

## 13.4 Linear Transformer:

A **transformer** is generally a four-terminal device comprising two (or more) magnetically coupled coils.

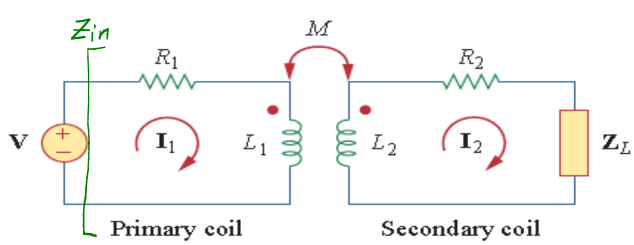
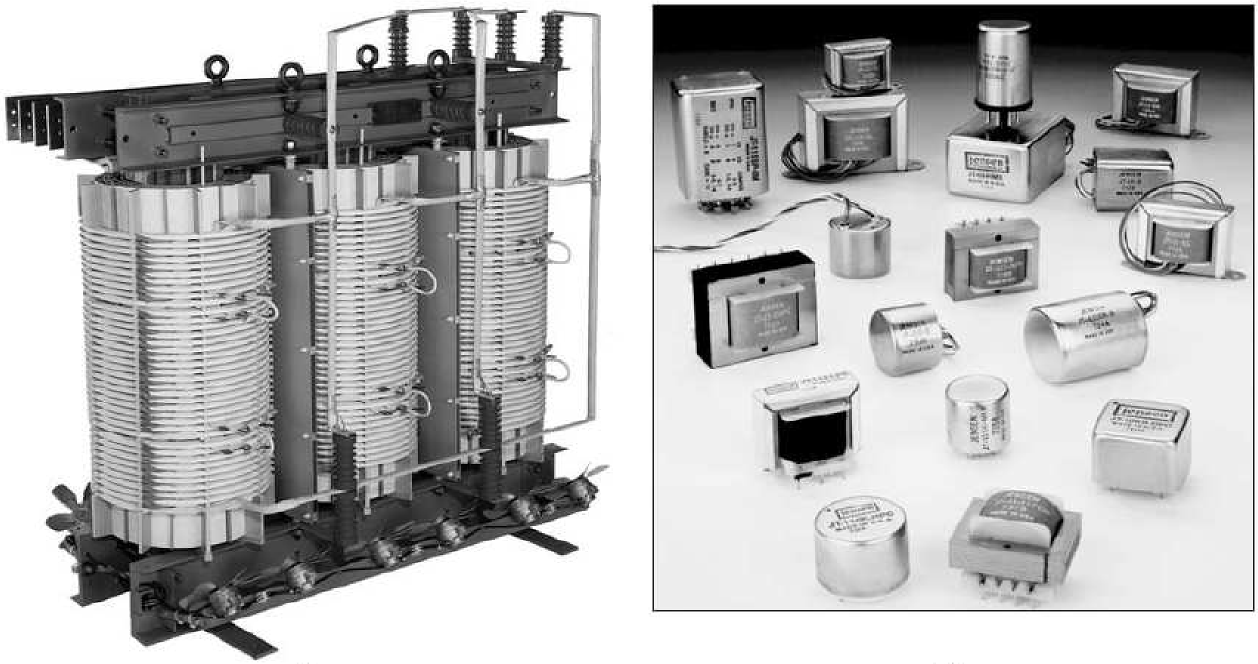



**Figure 13.19**
A linear transformer.



(a) (b)

**Figure 13.20**
Different types of transformers: (a) copper wound dry power transformer, (b) audio transformers.
Courtesy of: (a) Electric Service Co., (b) Jensen Transformers.

$$\mathbf{Z}_{in} = \frac{\mathbf{V}}{\mathbf{I}_1} = R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}$$

$$\mathbf{Z}_R = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + \mathbf{Z}_L}$$

(no dot rule)
reflected impedance
or coupled ~

## Example 13.4

In the circuit of Fig. 13.24, calculate the input impedance and current $\mathbf{I}_1$. Take $\mathbf{Z}_1 = 60 - j100\ \Omega$, $\mathbf{Z}_2 = 30 + j40\ \Omega$, and $\mathbf{Z}_L = 80 + j60\ \Omega$.

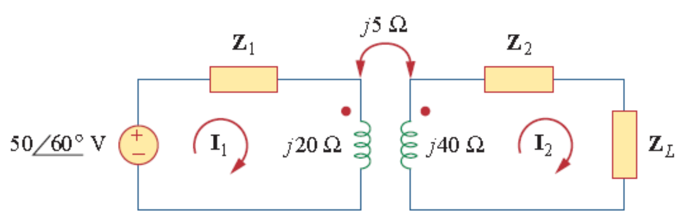



**Figure 13.24**
For Example 13.4.

**Solution:**

From Eq. (13.41),

$$\begin{aligned}\mathbf{Z}_{in} &= \mathbf{Z}_1 + j20 + \frac{(5)^2}{j40 + \mathbf{Z}_2 + \mathbf{Z}_L}\\ &= 60 - j100 + j20 + \frac{25}{110 + j140}\\ &= 60 - j80 + 0.14\angle{-51.84^\circ}\\ &= 60.09 - j80.11 = 100.14\angle{-53.1^\circ}\ \Omega\end{aligned}$$

Thus,

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_{in}} = \frac{50\angle 60^\circ}{100.14\angle{-53.1^\circ}} = 0.5\angle 113.1^\circ\ \text{A}$$

* Convert the linear transformer into T or Π equivalent circuit:

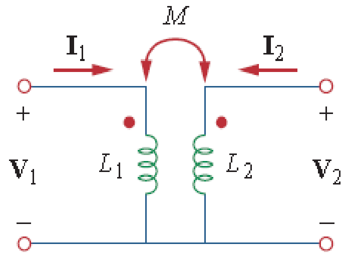



**Figure 13.21**
Determining the equivalent circuit of a linear transformer.

T- Model

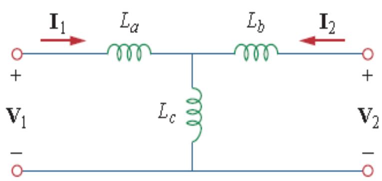



**Figure 13.22**
An equivalent T circuit.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \dfrac{L_2}{j\omega(L_1L_2 - M^2)} & \dfrac{-M}{j\omega(L_1L_2 - M^2)} \\ \dfrac{-M}{j\omega(L_1L_2 - M^2)} & \dfrac{L_1}{j\omega(L_1L_2 - M^2)} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} j\omega(L_a + L_c) & j\omega L_c \\ j\omega L_c & j\omega(L_b + L_c) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$L_a = L_1 - M, \qquad L_b = L_2 - M, \qquad L_c = M$$

Π - Model

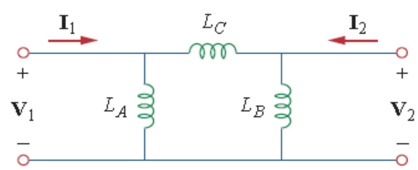



**Figure 13.23**
An equivalent Π circuit.

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{j\omega L_A} + \dfrac{1}{j\omega L_C} & -\dfrac{1}{j\omega L_C} \\ -\dfrac{1}{j\omega L_C} & \dfrac{1}{j\omega L_B} + \dfrac{1}{j\omega L_C} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$L_A = \frac{L_1L_2 - M^2}{L_2 - M}, \qquad L_B = \frac{L_1L_2 - M^2}{L_1 - M}$$

$$L_C = \frac{L_1L_2 - M^2}{M}$$

## Example 13.6

Solve for $I_1$, $I_2$, and $V_o$ in Fig. 13.27 (the same circuit as for Practice Prob. 13.1) using the T-equivalent circuit for the linear transformer.

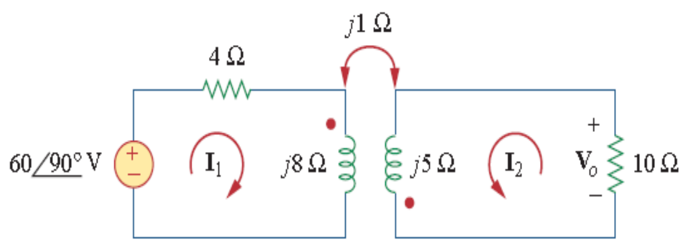
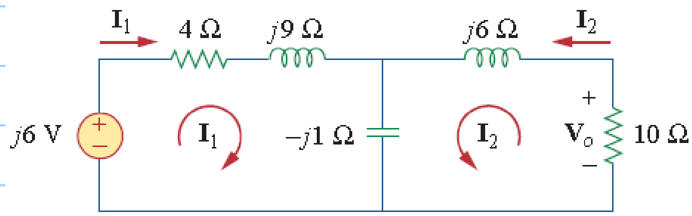



**Figure 13.27**
For Example 13.6.

**Solution:**

$$L_a = L_1 - (-M) = 8 + 1 = 9 \text{ H}$$

$$L_b = L_2 - (-M) = 5 + 1 = 6 \text{ H}, \qquad L_c = -M = -1 \text{ H}$$




$$j6 = \mathbf{I}_1(4 + j9 - j1) + \mathbf{I}_2(-j1) \qquad \textbf{(13.6.1)}$$

and

$$0 = \mathbf{I}_1(-j1) + \mathbf{I}_2(10 + j6 - j1) \qquad \textbf{(13.6.2)}$$

From Eq. (13.6.2),

$$\mathbf{I}_1 = \frac{(10 + j5)}{j}\mathbf{I}_2 = (5 - j10)\mathbf{I}_2 \qquad \textbf{(13.6.3)}$$

Substituting Eq. (13.6.3) into Eq. (13.6.1) gives

$$j6 = (4 + j8)(5 - j10)\mathbf{I}_2 - j\mathbf{I}_2 = (100 - j)\mathbf{I}_2 \simeq 100\mathbf{I}_2$$

Since 100 is very large compared with 1, the imaginary part of $(100 - j)$ can be ignored so that $100 - j \simeq 100$. Hence,

$$\mathbf{I}_2 = \frac{j6}{100} = j0.06 = 0.06\underline{/90^\circ} \text{ A}$$

From Eq. (13.6.3),

$$\mathbf{I}_1 = (5 - j10)j0.06 = 0.6 + j0.3 \text{ A}$$

and

$$\mathbf{V}_o = -10\mathbf{I}_2 = -j0.6 = 0.6\underline{/-90^\circ} \text{ V}$$

## 13.5 Ideal Transformers:

An **ideal transformer** is a unity-coupled, lossless transformer in which the primary and secondary coils have infinite self-inductances.

1. Coils have very large reactances ($L_1$, $L_2$, $M \rightarrow \infty$).
2. Coupling coefficient is equal to unity ($k = 1$).
3. Primary and secondary coils are lossless ($R_1 = 0 = R_2$).

Turns ratio: (Transformation ratio)

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{N_2}{N_1} = n$$

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{N_1}{N_2} = \frac{1}{n}$$

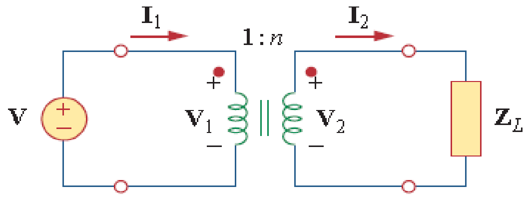


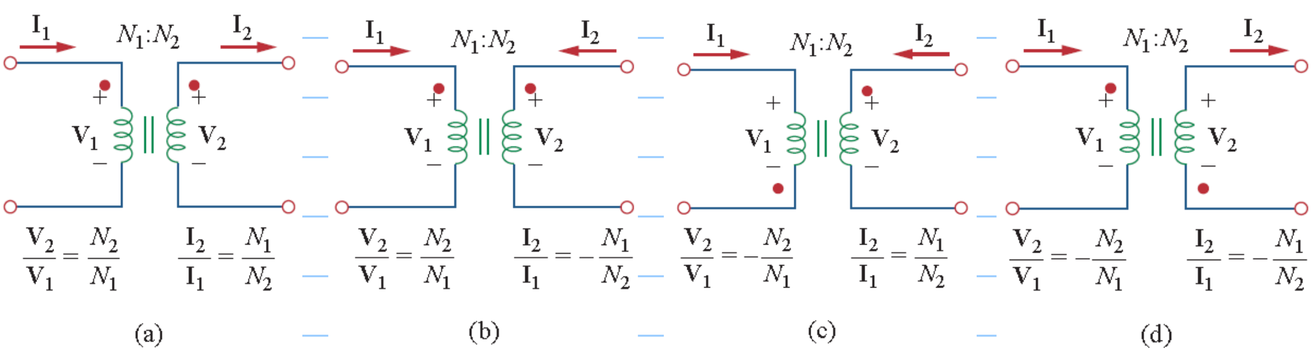

**Figure 13.31**
Relating primary and secondary quantities in an ideal transformer.

A **step-down transformer** is one whose secondary voltage is less than its primary voltage.

A **step-up transformer** is one whose secondary voltage is greater than its primary voltage.

## * Dot Rule:

1. If $\mathbf{V}_1$ and $\mathbf{V}_2$ are *both* positive or both negative at the dotted terminals, use $+n$ in Eq. (13.52). Otherwise, use $-n$.
2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ *both* enter into or both leave the dotted terminals, use $-n$ in Eq. (13.55). Otherwise, use $+n$.




## * The Complex Power:

$$\mathbf{S}_1 = \mathbf{V}_1\mathbf{I}_1^* = \frac{\mathbf{V}_2}{n}(n\mathbf{I}_2)^* = \mathbf{V}_2\mathbf{I}_2^* = \mathbf{S}_2$$

## * The Input Impedance: (Reflected Impedance)

$$\mathbf{Z}_{in} = \frac{\mathbf{V}_1}{\mathbf{I}_1} = \frac{1}{n^2}\frac{\mathbf{V}_2}{\mathbf{I}_2}$$

or

$$\mathbf{Z}_{in} = \frac{\mathbf{Z}_L}{n^2}$$

## * Equivalent Circuit:

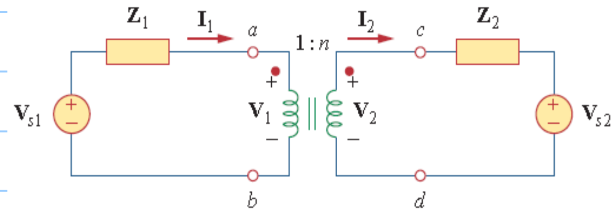
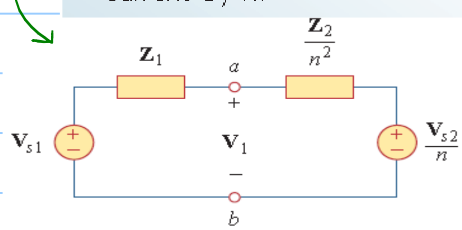
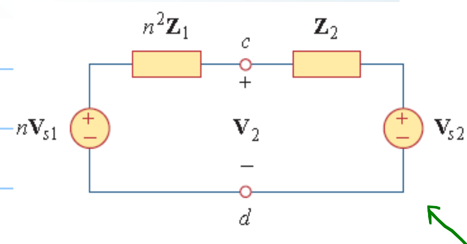



**Figure 13.33**
Ideal transformer circuit whose equivalent circuits are to be found.

The general rule for eliminating the transformer and reflecting the secondary circuit to the primary side is: divide the secondary impedance by $n^2$, divide the secondary voltage by $n$, and multiply the secondary current by $n$.




The rule for eliminating the transformer and reflecting the primary circuit to the secondary side is: multiply the primary impedance by $n^2$, multiply the primary voltage by $n$, and divide the primary current by $n$.

## Example 13.8

For the ideal transformer circuit of Fig. 13.37, find: (a) the source current $\mathbf{I}_1$, (b) the output voltage $\mathbf{V}_o$, and (c) the complex power supplied by the source.

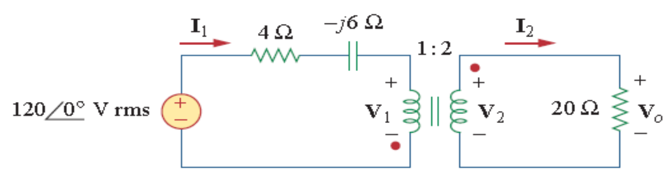



**Figure 13.37**
For Example 13.8.

**Solution:**

(a) The 20-Ω impedance can be reflected to the primary side and we get

$$\mathbf{Z}_R = \frac{20}{n^2} = \frac{20}{4} = 5\ \Omega$$

Thus,

$$\mathbf{Z}_{in} = 4 - j6 + \mathbf{Z}_R = 9 - j6 = 10.82\underline{/-33.69^\circ}\ \Omega$$

$$\mathbf{I}_1 = \frac{120\underline{/0^\circ}}{\mathbf{Z}_{in}} = \frac{120\underline{/0^\circ}}{10.82\underline{/-33.69^\circ}} = 11.09\underline{/33.69^\circ}\ \text{A}$$

(b) Since both $\mathbf{I}_1$ and $\mathbf{I}_2$ leave the dotted terminals,

$$\mathbf{I}_2 = -\frac{1}{n}\mathbf{I}_1 = -5.545\underline{/33.69^\circ}\ \text{A}$$

$$\mathbf{V}_o = 20\mathbf{I}_2 = 110.9\underline{/213.69^\circ}\ \text{V}$$

(c) The complex power supplied is

$$\mathbf{S} = \mathbf{V}_s\mathbf{I}_1^* = (120\underline{/0^\circ})(11.09\underline{/-33.69^\circ}) = 1{,}330.8\underline{/-33.69^\circ}\ \text{VA}$$