Operator Decomposition of Graphs
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Abstract: In this paper we introduce a new form of decomposition of graphs, the (P, Q)-decomposition. We first give an optimal algorithm for finding the 1-decomposition of a graph which is a special case of the (P, Q)-decomposition which was first introduced in [21]. We then examine the connections between the 1-decomposition and well known forms of decomposition of graphs, namely, modular and homogeneous decomposition. The characterization of graphs totally decomposable by 1-decomposition is also given. The last part of our paper is devoted to a generalization of the 1-decomposition. We first show that some basic properties of modular decomposition can be extended in a new form of decomposition of graphs that we called operator decomposition. We introduce the notion of a (P, Q)-module, where P and Q are hereditary graph-theoretic properties, the notion of a (P, Q)-split graph and the closed hereditary class (P, Q) of graphs (P and Q are closed under the operations of join of graphs and disjoint union of graphs, respectively). On this base, we construct a special case of the operator decomposition that is called (P, Q)-decomposition. Such decomposition is uniquely determined by an arbitrary minimal nontrivial (P, Q)-module in G. In particular, if G \not\in (P, Q), then G has the unique canonical (P, Q)-decomposition.

Keywords: Graph decomposition, hereditary class, split graph.

Received October 28, 2004; accepted August 8, 2005

1. Introduction

All graphs considered are finite, undirected, without loops and multiple edges. For all notions not defined here the reader is referred to [3]. The vertex and the edge sets of a graph G are denoted by V(G) and E(G), respectively, while n denotes the cardinality of V(G) and m the cardinality of E(G). We write u \sim v (u \not\sim v) if vertices u and v are adjacent (non-adjacent). For the subsets U, W \subseteq V(G) the notation U \sim W means that u \sim w for all vertices u \in U and w \in W. To shorten notation, we write u \sim W (u \not\sim W) instead of \{u\} \sim W (\{u\} \not\sim W). The subgraph of G induced by a set A \subseteq V(G) is denoted by G[A]. We write \overline{G} for the complement graph of G. The neighborhood of a vertex v in the graph G is denoted by N_G(v) (or N(v)), \overline{N}_G(v) = V(G) \backslash v \backslash N_G(v).

One type of graph decomposition based on the well-known notion of split graphs is investigated. A triad T = (G, A, B), where G is a graph and (A, B) is an ordered bipartition of V(G) into a clique A and a stable set B, is considered as an operator acting on the set of graphs. An operator T acts on a graph H by formula:

\[ TH = G \cup H \cup \{ax / a \in A, x \in V(H)\} \]  

(1)

(all edges of the complete bipartite graph with the parts A and V(H) are added to the disjoint union G \cup H).

An isomorphism of triads is defined as an isomorphism of 2-colored graphs. Denote by Tr the set of triads distinguished up to isomorphism of triads. The action (1) induces the associative binary operation on Tr. So the set Tr becomes a semigroup of operators with the exact action on the set of graphs. The semigroup Tr was introduced in [21]. The following structure theorem of the decomposition was presented in the same paper.

A graph F is called decomposable if there exist a triad T and a graph H such that F = TH, otherwise F is indecomposable. The decomposition theorem asserts that every decomposable graph F can be uniquely represented in the form:

\[ F = T_1T_2...T_kF_0 \]

Where T_i is indecomposable element of the semigroup Tr and F_0 is indecomposable graph. This theorem occurs to be useful instrument for the characterization and enumeration of several graph classes [19, 22]. On the base of the theorem, the Kelly-Ulam reconstruction conjecture was proved for the class of decomposable graphs. A criterion of decomposability of graphs was presented in [23]. In the same paper on the base of the decomposition theorem an exhaustive description of unigraphs was obtained. (A graph is called a unigraph if it is determined uniquely up to isomorphism by its degree sequence). Namely, it was proved that a graph is a unigraph if and only if all its indecomposable components are unigraphs, and the catalogue of indecomposable unigraphs was given.

In this paper, a decomposition theory is developed. In section 2, we present the 1-decomposition which