In this paper, we investigate the problem of existence of positive solutions for the nonlinear $q$-boundary value problem or quantum boundary value problem:

$$D_q^n u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

satisfying three kinds of $q$-different boundary value conditions. Our analysis relies on Krasnoselskii’s fixed point theorem of cone.

**ABSTRACT**

In this paper, we investigate the problem of existence of positive solutions for the nonlinear $q$-boundary value problem or quantum boundary value problem:

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**Keywords:** $q$-difference equations; Fixed point theorem; Boundary value problem; Positive solution

**I. INTRODUCTION:**

There is currently a great deal of interest in positive solutions for several types of boundary value problems. A large part of the literature on positive solutions to boundary value problems seems to be traced back to Krasnoselskii’s work on nonlinear operator equations [15], especially the part dealing with the theory of cones in Banach spaces. In 1994, Erbe and Wang [6] applied Krasnoselskii’s work to eigenvalue problems to establish intervals of the parameter $\lambda$ for which there is at least one positive solution. In 1995, Eloe and Henderson [2] obtained the solutions that are positive to a cone for the boundary value problem

$$u^{(i)}(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

$$u^{(i)}(0) = u^{(n-2)}(1) = 0, \quad 0 \leq i \leq n - 2.$$

Since this pioneering works, a lot research has been done in this area [3, 6, 11, 16, 19, 20]. In 2008, EL-Shahed [4] obtained the existence of positive solutions to nonlinear $n$th order boundary value problems

$$u^{(i)}(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = u(0) = u(0) = \ldots = u^{(n-1)}(0) = 0, \quad u(1) = 0,$$

$$u(0) = u(0) = u(0) = \ldots = u^{(n-2)}(0) = 0, \quad u(1) = 0,$$

$$u(0) = u(0) = u(0) = \ldots = u^{(n-2)}(0) = 0, \quad u(1) = 0.$$

El-Shahed and Hassan [5] studied the existence of positive solutions of the $q$-difference boundary value problem:

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The purpose of this paper is to establish the existence of positive solutions to nonlinear nth order q-boundary value problems:

\[ D_q^n u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \]

\[ u(0) = D_q u(0) = D_q^2 u(0) = \ldots = D_q^{n-1} u(0) = 0, D_q u(1) = 0, \]

where \( \lambda \) is a positive parameter. Throughout the paper, we assume that

\[ C1: f : [0, \infty) \to [0, \infty) \text{ is continuous} \]

\[ C2: a : (0, 1) \to [0, \infty) \text{ is continuous function such that } \int_0^1 a(t)d_q t > 0. \]

2. PRELIMINARIES:

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof our main results.

Let \( q \in (0, 1) \) and defined [14]

\[ [a]_q = \frac{q^a - 1}{q - 1} = q^{a-1} + \ldots + 1, \quad a \in \mathbb{R}. \]

The q-analogue of the power function \((a - b)^n\) with \( n \in \mathbb{R} \) is

\[ (a - b)^0 = 1, \quad (a - b)^n = \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \mathbb{R}, \quad n \in \mathbb{R}. \]

More generally, if \( \alpha \in \mathbb{R} \), then

\[ (a - b)^{(\alpha)} = a^\alpha \prod_{i=0}^{\infty} \frac{(a - bq^i)}{(a - bq^{\alpha+i})}. \]

Note that, if \( b = 0 \) then \( a^{(\alpha)} = a^\alpha \). The q-gamma function is defined by

\[ \Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}, \quad 0 < q < 1, \]

and satisfies \( \Gamma_q(x+1) = [x]_q \Gamma_q(x) \).

The q-derivative of a function \( f \) is here defined by

\[ D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x}, \]
and q-derivatives of higher order by

$$D_q^n f(x) = \begin{cases} f(x) & \text{if } n = 0, \\ D_q D_q^{n-1} f(x) & \text{if } n \in \mathbb{N}. \end{cases}$$

The q-integral of a function \( f \) defined in the interval \([0, b]\) is given by

$$\int_0^b f(t) d_q t = x (1-q) \sum_{n=0}^\infty f(x q^n) q^n, \quad 0 \leq |q| < 1, \quad x \in [0, b].$$

If \( a \in [0, b] \) and \( f \) defined in the interval \([0, b]\), its integral from \( a \) to \( b \) is defined by

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Similarly as done for derivatives, it can be defined an operator \( I_q^n \), namely,

$$\left(I_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(I_q^n f\right)(x) = I_q \left(I_q^{n-1} f\right)(x), \quad n \in \mathbb{N}.$$  

The fundamental theorem of calculus applies to these operators \( I_q \) and \( D_q \), i.e.,

$$\left(D_q I_q^n f\right)(x) = f(x),$$

and if \( f \) is continuous at \( x = 0 \), then

$$\left(I_q D_q^n f\right)(x) = f(x) - f(0).$$

Basic properties of the two operators can be found in the book [14]. We now point out three formulas that will be used later (\( D_q^i \) denotes the derivative with respect to variable \( i \)) [8]

$$\left[a(t-s)\right]^{(\alpha)} = a^\alpha (t-s)^{\alpha}, \quad (5)$$

$$D_q (t-s)^{\alpha} = \left[\alpha\right]_q (t-s)^{\alpha-1}, \quad (6)$$

$$\left(D_q \int_0^x f(t) d_q t\right)(x) = \int_0^x D_q f(t) d_q t + f(qx, x). \quad (7)$$

**Remark:** 2.1. We note that if \( \alpha > 0 \) and \( a \leq b \leq t \), then \((t - a)^{(\alpha)} \geq (t - b)^{(\alpha)}\) [8].

**Definition:** 2.1. Let \( \alpha \geq 0 \) and \( f \) be a function defined on \([0, 1]\). The fractional q-integral of the Riemann–Liouville type is \((RL I_q^0 f)(x) = f(x)\) and

$$\left(RL I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^x (x - qt)^{(\alpha-1)} f(t) d_q t, \quad \alpha \in \mathbb{R}^+, \ x \in [0, 1].$$

**Definition:** 2.2. [18] The fractional q-derivative of the Riemann–Liouville type of order \( \alpha \geq 0 \) is defined by

$$\left(RL D_q^0 f\right)(x) = f(x) \quad \text{and} \quad \left(RL D_q^\alpha f\right)(x) = (D_q^{|\alpha|} I_q^{|\alpha| - \alpha} f)(x), \quad \alpha > 0,$$

where \([\alpha]\) is the smallest integer greater than or equal to \(\alpha\).
Definition: 2.3. [18] The fractional q-derivative of the Caputo type of order \(\alpha \geq 0\) is defined by
\[
\left( cD_q^{[\alpha]} f \right)(x) = \left( I_q^{[\alpha]} D_q^{[\alpha]} f \right)(x), \quad \alpha > 0,
\]
where \([\alpha]\) is the smallest integer greater than or equal to \(\alpha\).

Lemma: 2.1. Let \(\alpha, \beta \geq 0\) and \(f\) be a function defined on \([0, 1]\). Then, the next formulas hold:

1. \((f Q I_q^\beta f)(x) = (f Q Q I_q^\alpha f)(x)\),
2. \((D_q^\alpha I_q^\beta f)(x) = f(x)\).

The next result is important in the sequel. It was proved in a recent work by the author [8].

Theorem: 2.1. Let \(\alpha > 0\) and \(p\) be a positive integer. Then, the following equality holds:
\[
\left( R L I_q^\alpha R L D_q^p f \right)(x) = \left( D_q^p I_q^\alpha f \right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k+1)} (D_q^k f)(0).
\]

Theorem: 2.2. [18] Let \(x > 0\) and \(\alpha \in \mathbb{R}^+ \setminus \{0\}\). Then, the following equality holds:
\[
\left( I_q^\alpha C D_q^p f \right)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{x^k}{\Gamma_q(k+1)} (D_q^k f)(0).
\]

Definition: 2.4. Let \(X\) be a real Banach space. A nonempty closed convex set \(P \subset X\) is called cone of \(X\) if it satisfies the following conditions

1. \(x \in P, \sigma \geq 0\) Implies \(\sigma x \in P\);
2. \(x \in P, -x \in P\) Implies \(x = 0\)

Definition: 2.5. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Theorem: 2.3. [10,15] Let \(X\) be a Banach space and \(P \subset X\) is a cone in \(X\). Assume that \(\Omega_1\) and \(\Omega_2\) are open subsets in \(X\) of with \(0 \in \Omega_1\) and \(\overline{\Omega_1} \subset \Omega_2\). Let \(T: P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P\) be completely continuous operator. In addition suppose either:

H1: \(\|Tu\| \leq \|u\|, u \in P \cap \partial \Omega_1\) and \(\|Tu\| \geq \|u\|, u \in P \cap \partial \Omega_2\) or
H2: \(\|Tu\| \leq \|u\|, u \in P \cap \partial \Omega_2\) and \(\|Tu\| \geq \|u\|, u \in P \cap \partial \Omega_1\),

holds. Then \(T\) has a fixed point in \(P \cap (\overline{\Omega_2} \setminus \Omega_1)\).

3. GREEN FUNCTIONS AND THEIR PROPERTIES:

Lemma: 3.1. Let \(y \in C[0,1]\), then the boundary value problem
\[
D_q^n u_2(t) + y(t) = 0, \quad 0 < t < 1,
\]
\[
u_2(0) = D_q^2 u_2(0) = D_q^3 u_2(0) = \ldots = D_q^{n-1} u_2(0) = 0, D_q^n u_2(1) = 0,
\]
has a unique solution.

\[ u_2 (t) = \int_0^1 G_2 (t, qs) y(s) \, d_q s, \]

where

\[
G_2 (t, s) = \begin{cases}
\frac{(t-qs)^{n-2}}{\Gamma_q (n-1)} - \frac{(t-s)^{n-1}}{\Gamma_q (n)}, & 0 \leq s \leq t \leq 1, \\
\frac{t (1-s)^{n-2}}{\Gamma_q (n-1)}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

**Proof:** We may apply Lemma 2.1 and Theorem 2.2, we see that

\[ u_2 (t) = u_2 (0) + \frac{D_q u_2 (0)}{\Gamma_q (2)} t + \frac{D_q^2 u_2 (0)}{\Gamma_q (3)} t^2 + \frac{D_q^3 u_2 (0)}{\Gamma_q (4)} t^3 + \ldots + \frac{D_q^{n-1} u_2 (0)}{\Gamma_q (n)} t^{n-1} - I_q^a y (t). \]

By using the boundary conditions \( u_2 (0) = D_q^2 u_2 (0) = D_q^3 u_2 (0) = \ldots = D_q^{n-1} u_2 (0) = 0 \), we get

\[ u_2 (t) = B_2 t - \int_0^t \frac{(t-qs)^{n-1}}{\Gamma_q (n)} y(s) \, d_q s. \]

Differentiating both sides of (8) one obtain, with the help (6) and (7),

\[ (D_q^2 u_2) (t) = B_2 - \int_0^t \frac{(n-1) \Gamma_q (t-qs)^{n-2}}{\Gamma_q (n)} y(s) \, d_q s. \]

then by the condition \( D_q^2 u_2 (1) = 0 \), we have

\[ B_2 = \frac{1}{\int_0^1 \frac{(1-qs)^{n-2}}{\Gamma_q (n-1)} y(s) \, d_q s}, \]

the proof is complete.

**Lemma 3.2.** Function \( G_2 \) defined above satisfies the following conditions:

\[ G_2 (t, qs) \geq 0 \text{ and } G_2 (t, qs) \leq G_2 (1, qs), \quad 0 \leq t, s \leq 1, \]

(9)

\[ G_2 (t, qs) \geq \eta_2 (t) G_2 (1, qs), \quad 0 \leq t, s \leq 1 \quad \text{with} \quad \eta_2 (t) = t. \]

(10)

**Proof:** We start by defining two functions

\[ g_1 (t, s) = \frac{t (1-s)^{n-2}}{\Gamma_q (n-1)} - \frac{(t-s)^{n-1}}{\Gamma_q (n)}, \quad 0 \leq s \leq t \leq 1, \]

and

\[ g_2 (t, s) = \frac{t (1-s)^{n-2}}{\Gamma_q (n-1)}, \quad 0 \leq t \leq s \leq 1. \]

It is clear that \( g_2 (t, qs) \geq 0 \). Now, \( g_1 (0, qs) = 0 \) and, in view of Remark 2.1, for \( t \neq 0 \)
\[ g_1(t, qs) = \frac{(1-qs)^{n-2} - (t-qs)^{n-1}}{\Gamma_q(n-1)} \geq 0, \]

therefore, \( G_2(t, qs) \geq 0 \). Moreover, for fixed \( s \in [0, 1] \),

\[ D_q^\alpha g_1(t, qs) = \frac{(1-qs)^{n-2} - (t-qs)^{n-1}}{\Gamma_q(n-1)} \geq 0, \]

i.e., \( g_1(t, qs) \) is an increasing function of \( t \). Obviously, \( G_2(t, qs) \) is increasing in \( t \), therefore \( G_2(t, qs) \) is an increasing function of \( t \) for fixed \( s \in [0, 1] \). This concludes the proof of (9).

Suppose now that \( t \geq qs \), then

\[ G_2(t, qs) = \frac{t(1-qs)^{n-2} - (t-qs)^{n-1}}{\Gamma_q(n-1)} \geq 0. \]

If \( t \leq qs \), then

\[ G_2(t, qs) = \frac{t(1-qs)^{n-2} - (t-qs)^{n-1}}{\Gamma_q(n-1)} = t, \]

and this finishes the proof of (10).

**Lemma 3.3.** Let \( y \in C[0, 1] \), then the q-boundary value problem

\[ D_q^\alpha u_3(t) + y(t) = 0, \quad 0 < t < 1, \]

\[ u_3(0) = D_q u_3(0) = D_q^2 u_3(0) = \ldots = D_q^{n-2} u_3(0) = 0, D_q u_3(1) = 0, \]

has a unique solution

\[ u_3(t) = \int_0^t G_3(t, qs) y(s) d_q s. \]
where
\[
G_3(t, s) = \begin{cases} 
\frac{t^{n-1} (1-s)^{n-2}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t \leq 1, \\
\frac{t^{n-1} (1-s)^{n-2}}{\Gamma_q(n)}, & 0 \leq t \leq s \leq 1.
\end{cases}
\]

**Proof:** We may apply Lemma 2.1 and Theorem 2.2, we see that
\[
u_3(t) = \frac{D_q u_3(0)}{\Gamma_q(2)} t + \frac{D_q^2 u_3(0)}{\Gamma_q(3)} t^2 + \ldots + \frac{D_q^{n-1} u_3(0)}{\Gamma_q(n)} t^{n-1} - \int_0^t y(s) \, dq_s.
\]
By using the boundary conditions \( u_3(0) = D_q u_3(0) = D_q^2 u_3(0) = \ldots = D_q^{n-2} u_3(0) = 0 \), we get
\[
u_3(t) = B_3 t^{n-1} \int_0^t \frac{(t-qs)^{n-2}}{\Gamma_q(n)} y(s) \, dq_s.
\]
Differentiating both sides of (11) one obtain, with the help (6) and (7),
\[
(D_q \nu_3)(t) = B_3 \left[ n-1 \right] t^{n-2} - \int_0^t \frac{[n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} y(s) \, dq_s,
\]
then by the condition \( D_q \nu_3(1) = 0 \), we have
\[
B_3 = \int_0^1 \frac{(1-qs)^{n-2}}{\Gamma_q(n)} y(s) \, dq_s,
\]
the proof is complete.

**Lemma: 3.4.** Function \( G_3 \) defined above satisfies the following conditions:
\[
G_3(t, qs) \geq 0 \text{ and } G_3(t, qs) \leq G_3(1, qs), \quad 0 \leq t, s \leq 1,
\]
\[
G_3(t, qs) \geq \eta_3(t) G_3(1, qs), \quad 0 \leq t, s \leq 1 \quad \text{with} \quad \eta_3(t) = t^{n-1}.
\]

**Proof:** We start by defining two functions
\[
g_3(t, s) = \frac{t^{n-1} (1-s)^{n-2}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \leq s \leq t \leq 1,
\]
and
\[
g_4(t, s) = \frac{t^{n-1} (1-s)^{n-2}}{\Gamma_q(n)}, \quad 0 \leq t \leq s \leq 1.
\]
It is clear that \( g_4(t, qs) \geq 0 \). Now, \( g_3(0, qs) = 0 \) and, in view of Remark 2.1, for \( t \neq 0 \)
\[
g_3(t, qs) \geq \frac{t^{n-1} (1-qs)^{n-2}}{\Gamma_q(n)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)}
\]
\[\geq \frac{t^{n-1} (1-qs)^{n-2}}{\Gamma_q(n)} - \frac{t^{n-1} (1-qs)^{n-2}}{\Gamma_q(n)}
\]
\[= \frac{t^{n-1}}{\Gamma_q(n)} \left[ (1-qs)^{n-2} - (1-qs)^{n-2} \right] \geq 0,
\]

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therefore, \( G_3(t, qs) \geq 0 \). Moreover, for fixed \( s \in [0, 1] \),

\[
\begin{align*}
\quad & \frac{tD_q g_3(t, qs)}{\Gamma_q(n)} = \frac{[n-1]_q t^{n-2} (1-qs)^{n-2} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)} \\
\geq & \frac{[n-1]_q t^{n-2} (1-qs)^{n-2} - [n-1]_q t^{n-2} (1-qs)^{n-2}}{\Gamma_q(n)} = 0,
\end{align*}
\]

i.e., \( g_3(t, qs) \) is an increasing function of \( t \). Obviously, \( g_4(t, qs) \) is increasing in \( t \), therefore \( G_3(t, qs) \) is an increasing function of \( t \) for fixed \( s \in [0, 1] \). This concludes the proof of (12).

Suppose now that \( t \geq qs \). Then

\[
\begin{align*}
\frac{G_3(t, qs)}{G_3(1, qs)} = & \frac{t^{n-1}(1-qs)^{n-2} - (t-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}} \\
\geq & \frac{t^{n-1}(1-qs)^{n-2} - t^{n-1}(1-qs)^{n-1}}{(1-qs)^{n-2} - (1-qs)^{n-1}} = t^{n-1}.
\end{align*}
\]

If \( t \leq qs \). Then

\[
\begin{align*}
\frac{G_3(t, qs)}{G_3(1, qs)} = & \frac{t^{n-1}(1-qs)^{n-2}/\Gamma_q(n)}{G_3(1, qs)} \\
> & \frac{t^{n-1}(1-qs)^{n-2}/\Gamma_q(n)}{G_3(1, qs)} = t^{n-1},
\end{align*}
\]

and this finishes the proof of (13).

**Lemma 3.5.** Let \( y \in C[0, 1] \), then the q-boundary value problem

\[
D_q^n u_4(t) + y(t) = 0, \quad 0 < t < 1,
\]

\[
u_4(0) = D_q u_4(0) = D_q^2 u_4(0) = \ldots = D_q^{n-2} u_4(0) = 0, D_q^2 u_4(1) = 0,
\]

has a unique solution

\[
u_4(t) = \int_0^1 G_4(t, qs) y(s) d_q s,
\]

where

\[
G_4(t, s) = \begin{cases}
\frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, & 0 \leq s \leq t < 1, \\
\frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)}, & 0 < t \leq s \leq 1.
\end{cases}
\]

The proof of Lemma 3.5 is very similar to that of Lemma 3.3 and therefore omitted.
Lemma 3.6. Function \( G_4 \) defined above satisfies the following conditions:

\[
G_4(t, qs) \geq 0 \quad \text{and} \quad G_4(t, qs) \leq G_4(1, qs), \quad 0 \leq t, s \leq 1, \quad (14)
\]

\[
G_4(t, qs) \geq \eta_4(t)G_4(1, qs), \quad 0 \leq t, s \leq 1 \quad \text{with} \quad \eta_4(t) = t^{n-1}. \quad (15)
\]

**Proof:** We start by defining two functions

\[
g_5(t, s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)} - \frac{(t-s)^{n-1}}{\Gamma_q(n)}, \quad 0 \leq s \leq t \leq 1,
\]

and

\[
g_6(t, s) = \frac{t^{n-1}(1-s)^{n-3}}{\Gamma_q(n)}, \quad 0 \leq t \leq s \leq 1.
\]

It is clear that \( g_5(t, qs) \geq 0 \). Now, \( g_5(0, qs) = 0 \) and, in view of Remark 2.1, for \( t \neq 0 \)

\[
g_5(t, qs) = \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{(t-qs)^{n-1}}{\Gamma_q(n)}
\]

\[
\geq \frac{t^{n-1}(1-qs)^{n-3}}{\Gamma_q(n)} - \frac{t^{n-1}(1-qs)^{n-1}}{\Gamma_q(n)}
\]

\[
= \frac{t^{n-1}}{\Gamma_q(n)}[ (1-qs)^{n-3} - (1-qs)^{n-1} ] \geq 0,
\]

therefore, \( G_4(t, qs) \geq 0 \). Moreover, for fixed \( s \in [0, 1] \),

\[
D_qg_5(t, qs) = \frac{[n-1]_q t^{n-2} (1-qs)^{n-3} - [n-1]_q (t-qs)^{n-2}}{\Gamma_q(n)}
\]

\[
\geq \frac{t^{n-2} (1-qs)^{n-3} - t^{n-2} (1-qs)^{n-2}}{\Gamma_q(n-1)}
\]

\[
= \frac{t^{n-2}}{\Gamma_q(n-1)}[ (1-qs)^{n-3} - (1-qs)^{n-2} ] \geq 0,
\]

i.e., \( g_5(t, qs) \) is an increasing function of \( t \). Obviously, \( g_5(t, qs) \) is increasing in \( t \), therefore \( G_4(t, qs) \) is an increasing function of \( t \) for fixed \( s \in [0, 1] \). This concludes the proof of (14).

Suppose now that \( t \geq qs \), Then

\[
\frac{G_4(t, qs)}{G_4(1, qs)} = \frac{t^{n-1}(1-qs)^{n-3} - (t-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}}
\]

\[
\geq \frac{t^{n-1}(1-qs)^{n-3} - t^{n-1}(1-qs)^{n-1}}{(1-qs)^{n-3} - (1-qs)^{n-1}} = t^{n-1}.
\]

If \( t \leq qs \), Then
Moustafa El-Shahed and Maryam Al-Yami\textsuperscript{*}/Positive solutions of boundary value problems for nth order $q$-differential equations/

\[ G_a(t,qs) = \frac{t^{-1}(1-qs)^{n-3}/\Gamma_q(n)}{G_a(1,qs)} \]
\[ \frac{t^{-1}(1-qs)^{n-3}/\Gamma_q(n)}{G_a(1,qs)} > \frac{(1-qs)^{n-3}/\Gamma_q(n)}{(1-qs)^{n-3}/\Gamma_q(n)} = t^{-1}, \]

and this finishes the proof of (15).

4. Main results:
In this section, we will apply Krasnoselskii’s fixed point theorem to the eigenvalue problem (1), (i) (i=2,3,4).

Remark: 3.1: If we let $0 < \tau < 1$, then
\[ \min_{t \in [\tau,1]} G_i(t,qs) \geq \eta_i(\tau)G_i(1,qs), \quad \text{for} \quad s \in [0,1]. \]  \hspace{1cm} (16)

Let $X = C[0,1]$ be the Banach space endowed with norm $\|u_i\| = \max_{t \in [\tau,1]} |u_i(t)|$. Let $\tau = q^n$ for a given $n \in \mathbb{N}$ and define the cone $P \subset X$ by
\[ P = \left\{ u_i \in X : u_i(t) \geq 0, \min_{t \in [\tau,1]} u_i(t) \geq \eta_i(\tau)\|u_i\| \right\}. \]

Remark: 3.2: It follows from the non-negativeness and continuity of $G_i, a$ and $f$ that the operator $T : P \to X$ defined by
\[ Tu_i(t) = \lambda \int_{0}^{t} G_i(t,qs) a(s)f(u_i(s))dqs, \]
is completely continuous. Moreover, for $u_i \in P, (Tu_i(t)) \geq 0$ on $[0,1]$ and
\[ \min_{t \in [\tau,1]} (Tu_i)(t) = \min_{t \in [\tau,1]} \lambda \int_{0}^{t} G_i(t,qs) a(s)f(u_i(s))dqs \]
\[ \geq \eta_i(\tau) \int_{0}^{t} G_i(1,qs) a(s)f(u_i(s))dqs \]
\[ = \eta_i(\tau)\|Tu_i\|, \]
that is, $T(P) \subset P$.

For our purposes, let us define two constants
\[ \gamma = \lambda \int_{0}^{1} G_i(1,qs) a(s)dqs^{-1} \quad \text{and} \quad \beta = \eta_i(\tau)\lambda \int_{\tau}^{1} G_i(1,qs) a(s)dqs^{-1}. \]

Our existence result is now presented.

Theorem: 3.1. Let $\tau = q^n$ with $n \in \mathbb{N}$. Suppose that $f(u_i)$ is a nonnegative continuous function on $[0,1] \times [0,\infty)$. If there exist two positive constants $R > r > 0$ such that
\[ \max_{(s,u_i) \in [0,1] \times [0,r]} f(u_i(t)) \leq \gamma u_i, \]  \hspace{1cm} (17)
\[ \min_{(s,u_i) \in [\tau,1] \times [\eta_i(\tau)R,R]} f(u_i(t)) \geq \beta u_i, \]  \hspace{1cm} (18)
then problem (1)–(4) has a solution \( u_i \) satisfying \( u_i(t) > 0 \) for \( t \in (0,1) \).

**Proof:** Since the operator \( T : P \to X \) is completely continuous we only have to show that the operator equation \( u_i = Tu_i \) has a solution satisfying \( u_i(t) > 0 \) for \( t \in (0,1) \).

Let \( \Omega_i = \{ u_i \in X : \| u_i \| < r \} \). For \( u_i \in P \cap \partial \Omega_i \), we have \( 0 \leq u_i(t) \leq r \) on \( [0,1] \). Using (9),(12),(14) and (17) we obtain,

\[
\| Tu_i \| = \max_{t \in [0,1]} \int_0^1 G_i(t,qs) a(s) f(u_i(s))d_q s \\
\leq \lambda \int_0^1 G_i(1,qs) a(s) \max_{(r,u_i) \in [0,1]} f(u_i(s))d_q s \\
\leq r \int_0^1 G_i(1,qs) a(s)d_q s \\
= r = \| u_i \|.
\]

Let \( \Omega_2 = \{ u_i \in X : \| u_i \| < R \} \). For \( u_i \in P \cap \partial \Omega_2 \), we have \( \eta_i(\tau) R \leq u_i(t) \leq R \) on \( [\tau,1] \). Using (16) and (18), and the fact that \( \tau = q^n[9] \), we obtain

\[
\| Tu_i \| = \max_{t \in [0,1]} \int_0^1 G_i(t,qs) a(s) f(u_i(s))d_q s \\
\geq \lambda \int_\tau^1 G_i(1,qs) a(s) \min_{(r,u_i) \in [\tau,1]} f(u_i(s))d_q s \\
\geq \beta \eta_i(\tau) R \lambda \int_0^1 G_i(1,qs) a(s)d_q s \\
= R = \| u_i \|.
\]

Now, Theorem 3.1 assures the existence of a fixed point \( u_i \) of \( T \) such that \( r \leq \| u_i \| \leq R \). To finish the proof, note that by (10),(13) and (15)

\[
u_i(t) = \lambda \int_0^1 G_i(t,qs) a(s) f(u_i(s))d_q s \\
\geq \eta_i(t) \lambda \int_0^1 G_i(1,qs) a(s) f(u_i(s))d_q s \\
= \eta_i(t) \| u_i \|,
\]

which implies that \( u_i(t) \geq \eta_i(t) r \). Therefore, \( u_i(t) > 0 \) for \( t \in (0,1) \) and the proof is done.

**References:**


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