Inference on Stress-Strength Reliability from Topp-Leone Distributions

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Abstract. Our goal is to estimate the probability \( R = P(Y < X) \) when \( X \) and \( Y \) are independent random variables following Topp-Leone distribution with different scale parameters. We study the ML estimator \( \hat{R} \) of \( R \), establish its asymptotic properties and construct confidence intervals. Assuming that the shape parameter is known, we study both the ML and Bayes estimators of \( R \), and find confidence intervals. Simulation study has been presented for illustrative purposes.

Keywords: Topp-Leone distribution, Maximum likelihood estimator, Bayes estimator, Stress-Strength model

1. Introduction

Topp-Leone distribution is a continuous unimodal distribution with bounded support. It is a two-parametric family continuous distribution proposed by Topp & Leone\cite{1}. Such a distribution is useful for modelling lifetime phenomena, different aspects of this class of distributions have been studied e.g. by Nadarajah and Kotz\cite{2}. We say that a random variable \( X \) with range of values \((0, \lambda)\) has a Topp-Leone distribution, and write \( X \sim TL(\alpha, \lambda) \), if the cumulative distribution function (cdf) is

\[
F(x) = \left(\frac{x}{\lambda}\right)^{\alpha} \left(2 - \frac{x}{\lambda}\right)^{\alpha}, \quad 0 < x < 1,
\]

\[
\alpha, \lambda > 0,
\]

and the density function (pdf) is
The Topp-Leone distribution is known as the J-shaped distribution. This is due to the fact that $f(x) > 0$, $f'(x) < 0$, and $f''(x) > 0$ for all $0 < x < \lambda$, where $f'$ and $f''$ are the first and second derivatives of $f$ respectively. Nadarajah and Kotz [2], and Ghitany et al [3], derived and studied the properties of the failure rate, mean residual lifetime, reversed failure rate, and mean inactivity time for a random variable $X$, where $X \sim TL (\alpha, \lambda)$, Kotz and Seier [4], explore the kurtosis of the Topp-Leone distributions.

This paper focuses on estimation of $R = P(Y < X)$, where $X$ and $Y$ are two independent random variables which follow TL distribution with different scale parameters but having the same shape parameter. Estimation of $R$, when $X$ and $Y$ follow a specified distribution, has been discussed extensively in the literature. For example, the estimate of $R$, when $X$ and $Y$ are independent exponential random variables, has been discussed by Enis and Geisser [5], Kelley et al. [6], and Tong, [7]. Kundu and Gupta [8], considered the estimation of $R$ when $X$ and $Y$ are independent generalized exponential random variables.

The remainder of the paper is organized as follows. In Section 2, we obtain the MLE of $R$ and the asymptotic distribution of $\hat{R}$. The MLE and Bayes estimators of $R$, when $\lambda$ is known, are considered in Section 3. Simulation study is presented in Section 4 for illustrative purposes.

2. Estimation of $R$ with Common Scale Parameter

This section investigates the properties of $R$ when the scale parameters are different and the common shape parameter $\lambda$ is constant.

2.1 Maximum Likelihood Estimator of $R$

Let $X \sim TL (\alpha, \lambda)$, and $Y \sim TL (\beta, \lambda)$, where $X$ and $Y$ are two independent random variables. All three parameters, $\alpha, \beta$ and $\lambda$ are unknown to us. Then it can be shown that

$$R = P(Y < X) = \int_{0 < y < x} f(x, y) \, dx \, dy = \frac{\alpha}{\alpha + \beta}$$

To compute the MLE of $R$, suppose that $X_1, X_2, ..., X_n$ is a random sample from $TL (\alpha, \lambda)$ and $Y_1, Y_2, ..., Y_m$ is another random sample from $TL (\beta, \lambda)$. Then the log-likelihood function of the observed sample is
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\[ L(x, y, \alpha, \beta, \lambda) = (n + m) \ln \frac{2}{\lambda} + n \ln \alpha + m \ln \beta + \sum_{i=1}^{n} \ln \left(1 - \frac{x_i}{\lambda}\right) \]

\[ + (\alpha - 1) \left\{ \sum_{i=1}^{n} \ln \left(\frac{x_i}{\lambda}\right) + \sum_{i=1}^{n} \ln \left(2 - \frac{x_i}{\lambda}\right) \right\} \]

\[ + \sum_{j=1}^{m} \ln \left(1 - \frac{y_j}{\lambda}\right) + (\beta - 1) \left\{ \sum_{j=1}^{m} \ln \left(\frac{y_j}{\lambda}\right) + \sum_{j=1}^{m} \ln \left(2 - \frac{y_j}{\lambda}\right) \right\}. \]

The MLEs of \(\alpha, \beta\) and \(\lambda\) say \(\hat{\alpha}, \hat{\beta}\) and \(\hat{\lambda}\), respectively, can be obtained as the solutions of the following equations:

\[ \frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln \left(\frac{x_i}{\lambda}\right) + \sum_{i=1}^{n} \ln \left(2 - \frac{x_i}{\lambda}\right) = 0 \]

\[ \frac{\partial L}{\partial \beta} = \frac{m}{\alpha} + \sum_{j=1}^{m} \ln \left(\frac{y_j}{\lambda}\right) + \sum_{j=1}^{m} \ln \left(2 - \frac{y_j}{\lambda}\right) = 0 \]

\[ \frac{\partial L}{\partial \lambda} = \frac{-2(n + m)}{\lambda} + \sum_{i=1}^{n} \frac{1}{\lambda - x_i} + \sum_{j=1}^{m} \frac{1}{\lambda - y_j} \]

\[ + (\alpha - 1) \left\{ \sum_{i=1}^{n} \frac{2}{2\lambda - x_i} - \frac{2n}{\lambda} \right\} \]

\[ + (\beta - 1) \left\{ \sum_{j=1}^{m} \frac{2}{2\lambda - y_j} - \frac{2m}{\lambda} \right\} = 0 \]

We obtain

\[ \hat{\alpha} = \frac{-n}{\sum_{i=1}^{n} \ln \left(\frac{x_i}{\lambda}\right) + \sum_{i=1}^{n} \ln \left(2 - \frac{x_i}{\lambda}\right)}, \quad (3) \]

\[ \hat{\beta} = \frac{-m}{\sum_{j=1}^{m} \ln \left(\frac{y_j}{\lambda}\right) + \sum_{j=1}^{m} \ln \left(2 - \frac{y_j}{\lambda}\right)}, \quad (4) \]

and \(\hat{\lambda}\) can be obtained as the solution of the nonlinear equation
\[ f(\lambda) = \frac{-2(n+m)}{\lambda} + \sum_{i}^{n} \frac{1}{\lambda - x_i} + \sum_{j}^{m} \frac{1}{\lambda - y_j} \]
\[ + \frac{-n}{\sum_{i}^{n} \ln \left( \frac{x_i}{\lambda} \right) + \sum_{i}^{n} \ln \left( 2 - \frac{x_i}{\lambda} \right)} - 1 \times \frac{\sum_{i}^{n} \frac{2}{2\lambda - x_i} - \frac{2n}{\lambda}}{\lambda} \]
\[ + \frac{-m}{\sum_{j}^{m} \ln \left( \frac{y_j}{\lambda} \right) + \sum_{j}^{m} \ln \left( 2 - \frac{y_j}{\lambda} \right)} - 1 \times \frac{\sum_{j}^{m} \frac{2}{2\lambda - y_j} - \frac{2m}{\lambda}}{\lambda} \]
\[ = 0 \]

Therefore, \( \hat{\lambda} \) can be obtained as solution of the nonlinear equation of the form \( g(\lambda) = \lambda \), where

\[ g(\lambda) = 2(n+m) \left[ \sum_{i}^{n} \frac{1}{\lambda - x_i} \right] \]
\[ + \frac{-n}{\sum_{i}^{n} \ln \left( \frac{x_i}{\lambda} \right) + \sum_{i}^{n} \ln \left( 2 - \frac{x_i}{\lambda} \right)} - 1 \times \frac{\sum_{i}^{n} \frac{2}{2\lambda - x_i} - \frac{2n}{\lambda}}{\lambda} \]
\[ + \sum_{j}^{m} \frac{1}{\lambda - y_j} \]
\[ + \frac{-m}{\sum_{j}^{m} \ln \left( \frac{y_j}{\lambda} \right) + \sum_{j}^{m} \ln \left( 2 - \frac{y_j}{\lambda} \right)} - 1 \times \frac{\sum_{j}^{m} \frac{2}{2\lambda - y_j} - \frac{2m}{\lambda}}{\lambda} \]
\[ = 0 \]  

(5)

Since, \( \hat{\lambda} \) is a fixed-point solution of the non-linear equation (5), therefore, it can be obtained by using an iterative scheme as follows:

\[ g(\lambda_j) = \lambda_{j+1} \]  

(6)
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where $\lambda_j$ is the $j$th iterate of $\hat{\lambda}$. The iteration procedure should be stopped when $|\lambda_j - \lambda_{j+1}|$ is sufficiently small. Once we obtain $\hat{\lambda}$, then $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from (3) and (4), respectively. Since ML estimators are invariant, so the MLE of $R$ becomes

$$\hat{R} = \frac{\hat{\alpha}}{\hat{\alpha} + \hat{\beta}} \quad (7)$$

### 2.2 Asymptotic Distribution

In this section, the asymptotic distributions of $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and the asymptotic distribution of $\hat{R}$ are obtained. The Fisher information matrix of $\theta$, denoted by $J(\theta) = E(I, \theta)$, where $I = [I_{i,j}]_{i,j=1,2,3}$ is the observed information matrix i.e.,

$$I = - \begin{bmatrix} \frac{\partial^2 L}{\partial \alpha^2} & \frac{\partial^2 L}{\partial \alpha \partial \beta} & \frac{\partial^2 L}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 L}{\partial \beta \partial \alpha} & \frac{\partial^2 L}{\partial \beta^2} & \frac{\partial^2 L}{\partial \beta \partial \lambda} \\ \frac{\partial^2 L}{\partial \lambda \partial \alpha} & \frac{\partial^2 L}{\partial \lambda \partial \beta} & \frac{\partial^2 L}{\partial \lambda^2} \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

where

$$I_{11} = \frac{n}{\alpha^2}, \quad I_{12} = I_{21} = 0, \quad I_{13} = I_{31} = \frac{2n}{\lambda} - \sum_i^{n} \frac{2}{2\lambda - x_i},$$

$$I_{22} = \frac{m}{\beta^2}, \quad I_{23} = I_{32} = \frac{2m}{\lambda} - \sum_j^{m} \frac{2}{2\lambda - y_j},$$

and

$$I_{33} = -\frac{2(n + m)}{\lambda^2}$$

$$- \sum_i^{n} \frac{1}{(\lambda - x_i)^2}$$

$$- \sum_j^{m} \frac{1}{(\lambda - y_j)^2} + (\alpha - 1) \left( \sum_i^{n} \frac{-4}{(2\lambda - x_i)^2} + \frac{2n}{\lambda^2} \right) + (\beta$$

$$- 1) \left( \sum_j^{m} \frac{-4}{(2\lambda - y_j)^2} + \frac{2m}{\lambda^2} \right)$$

Using the incomplete gamma function, $Be(a, b, c) = \int_0^c x^{a-1} (1 - x)^{b-1} \, dx$, we have the following:
\[
J_{11} = \frac{n}{\alpha^2}, \quad J_{22} = \frac{m}{\beta^2}, \quad J_{12} = J_{21} = 0.
\]
\[
J_{13} = J_{31} = \frac{n}{\lambda} \left[ 2 - \alpha 2^{2\alpha} \left\{ Be\left( \alpha, \alpha - 1, \frac{1}{2} \right) - 2 Be\left( \alpha + 1, \alpha - 1, \frac{1}{2} \right) \right\} \right]
\]
\[
J_{23} = J_{32} = \frac{m}{\lambda} \left[ 2 - \beta 2^{2\beta} \left\{ Be\left( \beta, \beta - 1, \frac{1}{2} \right) - 2 Be\left( \beta + 1, \beta - 1, \frac{1}{2} \right) \right\} \right]
\]

and \( J_{33} \) can take different forms depending on the values of \( \alpha \) and \( \beta \).

\[
J_{33} = \frac{1}{\lambda^2} \left[ 2(n + m) - n\alpha \sum_{i=0}^{\infty} 2^{2\alpha + i} Be\left( \alpha + i, \alpha, \frac{1}{2} \right)
\right.
\]
\[
- m\beta \sum_{i=0}^{\infty} 2^{2\beta + i} Be\left( \beta + i, \beta, \frac{1}{2} \right)
\]
\[
+ \left. \frac{n\alpha (\alpha - 1) 2^{2\alpha} \left\{ Be\left( \alpha, \alpha - 2, \frac{1}{2} \right) - 2 Be\left( \alpha + 1, \alpha - 2, \frac{1}{2} \right) \right\}}{2} + 2n(\alpha - 1) \right)
\]
\[
- 2 Be\left( \alpha + 1, \alpha - 2, \frac{1}{2} \right) + 2m(\beta - 1)
\]
\[
- m\beta (\beta
\]
\[
- \left. \left( \frac{n\alpha (\alpha - 1) 2^{2\alpha} \left\{ Be\left( \beta, \beta - 2, \frac{1}{2} \right) - 2 Be\left( \beta + 1, \beta - 2, \frac{1}{2} \right) \right\}}{2} \right) \right]
\]

where \( \alpha \neq 2, \beta \neq 2 \)

\[
J_{33} = \frac{1}{\lambda^2} \left[ 2(n + m) - 2n \sum_{i=0}^{\infty} 2^{4 + i} Be\left( 2 + i, 2, \frac{1}{2} \right)
\right.
\]
\[
- m\beta \sum_{i=0}^{\infty} 2^{2\beta + i} Be\left( \beta + i, \beta, \frac{1}{2} \right)
\]
\[
- 32n \sum_{i=0}^{\infty} \left\{ Be\left( i + 2, 1, \frac{1}{2} \right) - 2 Be\left( i + 3, \frac{1}{2} \right) \right\} + 2n
\]
\[
+ 2m(\beta - 1) - m\beta (\beta
\]
\[
- \left. \left( \frac{n\alpha (\alpha - 1) 2^{2\alpha} \left\{ Be\left( \beta, \beta - 2, \frac{1}{2} \right) - 2 Be\left( \beta + 1, \beta - 2, \frac{1}{2} \right) \right\}}{2} \right) \right],
\]
where $\alpha = 2, \beta \neq 2$

$$J_{33} = \frac{1}{\lambda^2} \left[ 2(n + m) \right. - n\alpha \sum_{i=0}^{\infty} 2^{2\alpha + i} B_E \left( \alpha + i, \alpha, \frac{1}{2} \right) - 2m \sum_{i=0}^{\infty} 2^{4 + i} B_E \left( 2 + i, 2, \frac{1}{2} \right) + 2n(\alpha - 1) - n\alpha(\alpha - 1)2^{2\alpha} \left\{ B_E \left( \alpha, \alpha - 2, \frac{1}{2} \right) - 2B_E \left( \alpha + 1, \alpha - 2, \frac{1}{2} \right) \right\} + 2m - 32 \sum_{i=0}^{\infty} \left\{ B_E \left( i + 2, 1, \frac{1}{2} \right) - 2B_E \left( i + 3, 1, \frac{1}{2} \right) \right\} \left. \right]$$

where $\alpha \neq 2, \beta = 2$

$$J_{33} = \frac{1}{\lambda^2} \left[ 2(n + m) \right. - 2n \sum_{i=0}^{\infty} 2^{4 + i} B_E \left( 2 + i, 2, \frac{1}{2} \right) - 2m \sum_{i=0}^{\infty} 2^{4 + i} B_E \left( 2 + i, 2, \frac{1}{2} \right) + 32 \sum_{i=0}^{\infty} \left\{ B_E \left( i + 2, 1, \frac{1}{2} \right) - 2B_E \left( i + 3, 1, \frac{1}{2} \right) \right\} + 2n + 2m - 32 \sum_{i=0}^{\infty} \left\{ B_E \left( i + 2, 1, \frac{1}{2} \right) - 2B_E \left( i + 3, 1, \frac{1}{2} \right) \right\} \left. \right]$$

where $\alpha = 2, \beta = 2$.

The Topp-Leone family distributions satisfy all the regularity conditions see e.g. Nadarajah and Kotz\(^2\). Therefore, we can formulate the following result:

**Theorem 2.1** As $n \to \infty, m \to \infty, \frac{n}{m} \to p$, then

$$\left[ \sqrt{n}(\hat{\alpha} - \alpha), \sqrt{m}(\hat{\beta} - \beta), \sqrt{n}(\hat{\lambda} - \lambda) \right] \to N_3(0, A^{-1}(\alpha, \beta, \lambda)), $$
where

\[ A(\alpha, \beta, \lambda) = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \]

and

\[
\begin{align*}
    a_{11} &= \frac{1}{\alpha^2}, \\
    a_{13} &= a_{31} = \frac{1}{n} J_{13}, \\
    a_{23} &= a_{32} = \frac{\sqrt{p}}{n} J_{23}, \\
    a_{33} &= \lim_{n,m \to \infty} \frac{1}{n} J_{33}
\end{align*}
\]

Note that \( a_{33} \) can be written as follows:

\[
\begin{align*}
    a_{33} &= \frac{2(p + 1)}{p \lambda^2} - \frac{\alpha}{\lambda^2} \sum_{i=0}^{\infty} 2^{2\alpha+i} \text{Be} \left( \alpha + i, \alpha, \frac{1}{2} \right) \\
    &\quad - \frac{\beta}{p \lambda^2} \sum_{i=0}^{\infty} 2^{2\beta+i} \text{Be} \left( \beta + i, \beta, \frac{1}{2} \right) + \frac{2(\alpha - 1)}{\lambda^2} \\
    &\quad - \frac{\alpha(\alpha - 1)}{\lambda^2} 2^{2\alpha} \left\{ \text{Be} \left( \alpha, \alpha - 2, \frac{1}{2} \right) \right\} \\
    &\quad - 2 \text{Be} \left( \alpha + 1, \alpha - 2, \frac{1}{2} \right) + \frac{2(\beta - 1)}{p \lambda^2} \\
    &\quad - \frac{\beta(\beta - 1)}{p \lambda^2} 2^{2\beta} \left\{ \text{Be} \left( \beta, \beta - 2, \frac{1}{2} \right) \right\} \\
    &\quad - 2 \text{Be} \left( \beta + 1, \beta - 2, \frac{1}{2} \right)
\end{align*}
\]

where \( \alpha \neq 2, \beta \neq 2 \)

Similarly, \( a_{33} \) can be written according to the other cases of \( J_{33} \).

**Proof.** The proof follows from the asymptotic normality of MLE.

**Theorem 2.2** As \( n \to \infty, m \to \infty \) so that \( \frac{n}{m} \to p \), then

\( \sqrt{n}(\hat{R} - R) \to N(0, B) \),

where

\[
B = \frac{1}{k(\alpha + \beta)^4} \left[ \beta^2 (a_{22}a_{33} - a_{23}^2) - 2\alpha\beta \sqrt{p} a_{23}a_{31} + \alpha^2 p(a_{11}a_{33} - a_{13}^2) \right]
\]

and

\[
k = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}.
\]

**Proof.** The proof is obvious from Theorem 2.

Note that Theorem 2 can be used to construct asymptotic confidence intervals. To compute the confidence interval of \( R \), the variance \( B \) needs to be estimated. It is clear that the confidence intervals based on the asymptotic results do not perform very well for small sample size. So, two confidence
intervals based on the parametric bootstrap methods are proposed: (i) percentile bootstrap method \(^9\), we refer it as Boot-p from now on, and (ii) bootstrap-t method \(^10\), we refer it as Boot-t from now on.

3. Estimation of \( R \) if \( \lambda \) is Known

In this section, we consider the estimation of \( R \) when \( \lambda \) is known. Without loss of generality, we can assume that \( \lambda = 1 \). Therefore, in this section it is assumed that \( X_1, X_2, ..., X_n \) is a random sample from \( TL(\alpha, 1) \) and \( Y_1, Y_2, ..., Y_m \) is a random sample from \( TL(\beta, 1) \) and based on the samples we want to estimate \( R \). First, we consider the MLE of \( R \) and its distributional properties.

3.1 MLE of \( R \)

Assuming \( \lambda \) is known, the MLE of \( R, \hat{R} = \hat{\alpha} / (\hat{\alpha} + \hat{\beta}) \) can be written as

\[
\hat{R} = \left[ 1 + \frac{m (\sum_i^n \ln(x_i) + \sum_i^n \ln(2 - x_i))}{n (\sum_j^m \ln(y_j) + \sum_j^m \ln(2 - y_j))} \right]^{-1}
\]

It is not difficult to notice that the random variable, say, \( u^* = -\ln(u(2 - u) \) follows an exponential distribution. Thus, \( \ln x_i - x_i \) follow a chi-square distribution with \( 2n \) degrees of freedom and we write \( 2\alpha \sum_i^n \ln (x_i(2 - x_i)) \sim \chi^2_{2n} \) and similarly \( 2\beta \ln \ln (y_j - j) \sim \chi^2_{2m} \). Based on this, we have

\[
\hat{R} = \left( 1 + \frac{\alpha}{\beta} F \right)^{-1}, \text{ or } \frac{\alpha}{\beta} \left( \frac{1 - \hat{R}}{\hat{R}} \right) \sim F,
\]

where \( F \) is a random variable having a \( F(2n; 2m) \) distribution with \( 2n \) and \( 2m \) degrees of freedom. The probability density function of \( F \) is

\[
f_F(r) = \frac{1}{B(n,m)} \left( \frac{n\alpha}{m\beta} \right)^n \frac{(\frac{1-r}{r})^{n-1}}{r^2 \left( 1 + \frac{n\alpha}{m\beta} \frac{1-r}{r} \right)^{n+m}}, \quad 0 < r < 1,
\]

(8)

where \( B(a, b) \) is the beta function. Here we mention some properties of \( \hat{R} \).

The calculations use the Gauss hypergeometric function. It is defined by

\[
\text{2F}_1(a, b, c, z) = F(a, b, c, z) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i i!} z^i, \quad |z| < 1,
\]
where \((a)_i = a(1 + a) \ldots (i + 1 + a)\). For further details of equation (8), as shown by Gradshteyn and Ryzhik\(^{[11]}\).

\[
E(\hat{R}) = \int_0^1 rf_{\hat{R}}(r) \, dr
\]

\[
= \frac{m}{m+n} \left( \frac{m \beta}{n \alpha} \right)^m \, _2F_1 \left( \begin{array}{c} m+n, m+1; m+n+1; 1 - \frac{m \beta}{n \alpha} \end{array} \right).
\]

The mean squared error (MSE) is defined as the expected value of the squared errors, that is:

\[
MSE(\hat{R}) = E(\hat{R} - R)^2
\]

\[
= \frac{m(m+1)}{(m+n)(m+n+1)} \left( \frac{m \beta}{n \alpha} \right)^m \, _2F_1 \left( \begin{array}{c} m+n, m+2; m+n+2; 1 - \frac{m \beta}{n \alpha} \end{array} \right)
- 2 \left( \frac{\alpha}{\alpha + \beta} \right) \left( \frac{m}{m+n} \right) \left( \frac{m \beta}{n \alpha} \right)^m \, _2F_1 \left( \begin{array}{c} m+n, m+1; m+n+1; 1 - \frac{m \beta}{n \alpha} \end{array} \right)
+ 1 \left( \frac{\alpha}{\alpha + \beta} \right)^2
\]

**Theorem 3.1** \(\hat{R}\) is asymptotically unbiased and consistent estimator for \(R\).

**Proof.** Since \(\frac{\alpha}{\beta} \left( \frac{1 - \hat{R}}{\hat{R}} \right) \sim F_{(2n,2m)}\), then this can be used to study the properties of \(\hat{R}\) and it could be shown:

\[
E(\hat{R}) = \delta(m) \left[ 1 + \frac{m+n+1}{n(m-2)}(1 - \delta(m))^2 \right], \text{where } \delta(m)
\]

\[
= \left( 1 + \frac{\beta}{\alpha} \left( \frac{m}{m-1} \right) \right)^{-1}
\]

Note that with fixed \(n\), we have

\[
\lim_{m \to \infty} E(\hat{R}) = R \left[ 1 + \frac{1}{n} (1 - R^2) \right].
\]

Hence, for \(n, m \to \infty\), \(E(\hat{R}) = R\). This means that \(\hat{R}\) is asymptotically unbiased. Also,

\[
Var(\hat{R}) = \frac{m+n-1}{n(m-2)} [\delta(m)]^2 [1 - \delta(m)]^2, \text{and}
\]
\[
\lim_{m \to \infty} \text{Var}(\hat{R}) = \frac{1}{n} R^2 [1 - R]^2.
\]

Hence, for \( n, m \to \infty \), \( \text{Var}(R') = 0 \). Therefore, \( \hat{R} \) is a consistent estimator.

The MSE of \( \hat{R} \) is given as:
\[
\text{MSE}(\hat{R}) = \text{Var}(\hat{R}) + (R - E(\hat{R}))^2
\]
\[
= \frac{m + n - 1}{n(m - 2)} \left[ \delta(m)^2 [1 - \delta(m)]^2 \right]
+ \left[ \frac{\alpha}{\alpha + \beta} - \delta(m) \left( 1 + \frac{m + n - 1}{n(m - 2)} [1 - \delta(m)]^2 \right) \right]^2
\]

The 100(1 - \( \gamma \)) % confidence interval of \( R \) can be obtained as
\[
\left[ \frac{1}{1 + F\left(2n,2m;\frac{1}{\gamma}\right) \times \left( \frac{1}{R} - 1 \right)}, \frac{1}{1 + F\left(2n,2m;\frac{1}{\gamma}\right) \times \left( \frac{1}{R} - 1 \right)} \right],
\]
where \( F\left(2n,2m;\frac{1}{\gamma}\right) \) and \( F\left(2n,2m;\frac{1}{\gamma}\right) \) are the lower and upper \( \gamma/2 \)th the percentile points of a \( F \) distribution with \( 2n \) and \( 2m \) degrees of freedom.

### 3.2 Bayes Estimator

We obtain the Bayes estimator for \( R \). We assume that \( \alpha \) and \( \beta \) are random variables each having gamma prior with some parameters. Therefore, we can write:
\[
\pi(\alpha) = \frac{\theta_1^{\gamma_1} \alpha^{\gamma_1 - 1} \exp(-\theta_1 \alpha)}{\Gamma(\gamma_1)}, \quad \theta_1, \gamma_1, \alpha > 0.
\]
\[
\pi(\beta) = \frac{\theta_1^{\gamma_1} \beta^{\gamma_1 - 1} \exp(-\theta_1 \beta)}{\Gamma(\gamma_1)}, \quad \theta_1, \gamma_1, \beta > 0.
\]

Then the posteriors of \( \alpha \) and \( \beta \) are, respectively, as follows:
\[
\pi(\alpha|x) \propto \alpha^{\delta_1 - 1} \exp(-\alpha \xi_1), \text{where} \quad \delta_1 = n + \gamma_1, \xi_1 = \theta_1
\]
\[
- \sum_{i=1}^{n} \ln(x_i(2 - x_i)),
\]
\[
\pi \left( \beta \mid y \right) \propto \beta^{\delta_2-1} \exp(-\beta \xi_2), \text{ where } \delta_2 = m + \gamma_2, \xi_2 = \theta_2
\]

\[-\sum_{j=1}^{m} \ln \left( y_j (2 - y_j) \right),\]

Since \(\alpha\) and \(\beta\) are independent, then the joint density function of \((\alpha, \beta)\) is given by

\[
\pi \left( \alpha, \beta \mid x, y \right) = \frac{\xi_1^{\delta_1} \xi_2^{\delta_2}}{\Gamma(\delta_1) \Gamma(\delta_2)} \alpha^{\delta_1-1} \beta^{\delta_2-1} \exp(-\alpha \xi_1 - \beta \xi_2)
\]

The posterior pdf of \(R\) is

\[
f_R(r) = K \frac{r^{\delta_1-1} (1 - r)^{\delta_2-1}}{(1 - r) \xi_2 + r \xi_2^{\delta_1 + \delta_2}}, \text{ for } 0 < r < 1
\]

and \(0\) otherwise, where \(K\) is a constant given by

\[
K = \frac{\Gamma(\delta_1 + \delta_2)}{\Gamma(\delta_1) \Gamma(\delta_2)} \xi_1^{\delta_1} \xi_2^{\delta_2}
\]

The Bayes estimate of \(R\) under squared error mean \(E(R \mid x, y)\) can be computed analytically. The Bayes estimate of \(R\), say \(\hat{R}_B\) is

\[
E \left( \left( R \mid x, y \right) \right) = \begin{cases} 
\frac{(\xi_1)^{\delta_1}}{\xi_2} \frac{(\delta_1)}{(\delta_1 + \delta_2)} 2F_1 \left( (\delta_1 + \delta_2, \delta_1 + 1); \delta_1 + \delta_2 + 1; 1 - \frac{\xi_1}{\xi_2} \right), & \text{if } \xi_1 \leq \xi_2; \\
\frac{(\xi_2)^{\delta_2}}{\xi_1} \frac{\delta_1}{(\delta_1 + \delta_2)} 2F_1 \left( (\delta_1 + \delta_2, \delta_2); \delta_1 + \delta_2 + 1; 1 - \frac{\xi_2}{\xi_1} \right), & \text{if } \xi_1 > \xi_2.
\end{cases}
\]

4. Simulation study

In this section, our main aim is to compare the Bayes estimators with the classical maximum likelihood estimators. Since the estimators are quite difficult to compute numerically, we compare the MLEs with the Bayes estimates in terms of biases and the mean squared errors for different sample sizes and for different values for \((\alpha = 2, 2.5, 3, 3.5, 4)\). The simulation is conducted by generating 5000 uniform \((0,1)\) random samples. The usual transformation technique has been used to get the corresponding Topp-Leone random samples. Table 1 shows the biases and MSEs of the MLEs, and Bayes estimators of \(R\). Table 2 represents the average confidence lengths and coverage percentages, reported in parentheses, based on estimated asymptotic distribution, Boot-p and Boot-t methods. From Table 1, we notice that the Bayes estimates have the
smallest mean square errors and the biases are significantly higher in most cases. It is observed that when \( m \) and \( n \) increase then MSEs of all the estimators decrease. In addition, Boot-p method performs well when compared to other methods for small samples as seen from Table 2.

Table 1. Biases and MSEs of the MLEs, and Bayes estimators of \( R \).
Table 2. The average confidence lengths and coverage percentages based on estimated asymptotic distribution, Boot-p and Boot-t methods.

<table>
<thead>
<tr>
<th>(n, m)</th>
<th>(α, β, λ)</th>
<th>True-MLE</th>
<th>Asy-MLE</th>
<th>Boot-p</th>
<th>Boot-t</th>
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<td>-5.5</td>
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References:


الاستدلال حول صلاحية الإجهاد- القوة لتوزيع توب و ليون

بندر الزهراني، علي الشمراني

قسم الإحصاء - كلية العلوم - جامعة الملك عبد العزيز
صر. ١٤١٩ – ٢٠٠٨ المملكة العربية السعودية

المستخلص: هدفنا في هذه الدراسة هو تقدير دالة الصلاحية لنموذج الإجهاد- القوة (R = P(Y < X) عندما تكون متغيرين عشوائيين يتبعان توزيع توب و ليون مع اختلاف معلم القياس. قمنا بدراسة مقدر الإمكان الأكبر (R) و أوجدنا الخصائص التقريبية له مع إيجاد فترات ثقة. وعلى افتراض أن معلمة الشكل معروفة سلفاً قمنا بدراسة مقدرات الإمكان الأكبر و بيز لـ R و أوجدنا أيضاً فترات الثقة. وكمثال توضيحي لنتائج استخدمنا أساليب المحاكاة.