Nonlinear Magneto-Heat Transfer in a Fluid-Particle Suspension Flowing in a Non-Darcian Channel with Heat Source and Buoyancy Effects: Numerical Study

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Abstract. We consider the steady, laminar nonlinear natural convection heat transfer of a particulate suspension in an electrically-conducting fluid through a two-dimensional channel containing a non-Darcian porous material in the presence of a transverse magnetic field. The transport equations for both fluid and particle phases are formulated using a two-phase continuum model and a heat source term is included which simulates either absorption or generation. A set of variables is implemented to reduce the ordinary differential equations for momentum and energy conservation (for both phases) from a two-dimensional coordinate system to a one-dimensional system. Finite element solutions are obtained for the dimensionless system under appropriate boundary conditions. A comprehensive
parametric study of the effects of heat source parameter (E), Prandtl number (Pr), Grashof number (Gr), momentum inverse Stokes number (Sk_m), Darcy number (Da), Forchheimer number (Fs), particle loading parameter (P_L), buoyancy parameter (B), Hartmann number (Ha) and temperature inverse Stokes number (Sk_T) on the dimensionless fluid phase velocity (U), dimensionless particle phase velocity (U_p), dimensionless fluid phase temperature (Φ) and the dimensionless temperature of particle phase (Φ_p) are presented graphically. Fluid phase velocities are found to be strongly reduce by magnetic field, Darcian drag and also Forchheimer drag; a lesser reduction is observed for the particle phase velocity field. Prandtl number is shown to depress both fluid temperature and particle phase temperature in the left hand side of the channel but to boost both temperatures at the right hand side of the channel (0.5 ≤ η ≤ 1). Inverse momentum Stokes number is seen to reduce fluid phase velocities and increase particle phase velocities. The influence of other thermophysical parameters is discussed in detail and computations compared with previous studies. The model finds applications in MHD plasma accelerators, astrophysical flows, geophysics, geothermics and industrial materials processing.

**Keywords**: Particle-Fluid Suspension; Hydromagnetics; Convection; Porous; Non-Darcy; Stokes number; Buoyancy; Numerical.

1. Introduction

Fluid-particle transport phenomena are encountered in many branches of engineering technology. In the presence of magnetic fields, flows become electrically-conducting and with heat transfer present are important in MHD (magnetohydrodynamic) accelerator technologies and hydromagnetic energy generators. The occurrence of solid particles such as soot or ash in plasma generators can cause particle suspensions in hydromagnetic flows\[1, 2\]. It is therefore important to study fluid-particulate convective hydromagnetic flows in order to evaluate the influence of the different phases on heat transfer processes. In geophysical and astrophysical flows, the plasma can also be conducting and with debris suspensions constitutes another type of two-phase hydromagnetic flow\[3\]. Several studies of two-phase particulate flows with and without heat transfer and with or without magnetic fields have been reported in the literature using a combination of experimental or numerical approaches. Studies have concerned both aerodynamic flows and liquid flows and dwelled on both laminar and turbulent flow regimes. Tien\[4\] presented an early study of heat transfer in a turbulent fluid-particle suspension conduit flow. Depew and Cramer\[5\] reviewed much of
the experimental work on convection heat transfer to horizontal gas-solid suspension flow processes. Sukomel et al.\textsuperscript{[6]} discussed dusty heat transfer in conduit particle-suspension flows. Other studies have been reported by Pechnegov \textit{et al.}\textsuperscript{[7]}, Mechaelidis\textsuperscript{[8]} and Apazidis\textsuperscript{[9, 10]}. Mwangi \textit{et al.}\textsuperscript{[11]} obtained experimentally convective heat transfer coefficients at the fluid-particle interface for shear flows of liquid food in a holding tube. The density difference between the fluid and particle phases was shown to strongly influence the residence time of the particles in the tube. Chamkha and Ramadan\textsuperscript{[12]} presented a theoretical study of flat-plate particle-fluid natural convection heat transfer. Chamkha and Adeeb\textsuperscript{[13]} analyzed numerically the oscillatory flow and transverse magnetic field effects on the fluid-particle suspension convection flow.

More recently Shakib-Manesh \textit{et al.}\textsuperscript{[14]} have used a direct numerical simulation method based on Lattice-Boltzmann techniques to analyze the mechanisms of momentum transfer and shear stress for liquid particle suspensions in two-dimensional Couette flow. They showed that the fluid phase field exhibits a non-linear velocity profile and that shear-thickening effects are related to the enhanced relative solid phase stress for rising shear rates. An excellent study of the dependence of viscosity of non-Brownian liquid-particle suspensions on the shape and concentration of suspended particles and on shear rate was presented recently by Raiskinmaki\textsuperscript{[15]}. It was shown that two possible mechanisms producing solid-phase momentum transfer are the effects of inertia in the flow and also the clustering of particles. Srivastava and Srivastava\textsuperscript{[16]} presented closed-form solutions for the peristaltic transport induced by sinusoidal waves of a fluid-particle suspension in Poiseuille flow. They computed exact solutions for limiting values of Reynolds number, Poiseuille parameter and wave number and used a Frobenius series solution technique for general solutions.

Very recently, Tiwari \textit{et al.}\textsuperscript{[17]} used a state-of-the-art multiphase numerical code to analyze the effects of Reynolds number and Dean number on two-phase particulate flows in helical curved conduits and U-bend systems. Particle inertia was shown to increase wall shear and a significant reduction in peak particle concentration was found for increasing Dean numbers.

In many areas of industrial transport phenomena, heat sources may also occur and these can have a significant effect on particle-fluid
suspension flows. Heat sinks are for example utilized commonly in enhancing the rate of heat dissipation in electronics systems such as multiple-chip circuit boards\cite{18}. Several researchers have considered heat generation/absorption effects in particle-fluid suspension heat transfer. Al-Subaie and Chamkha\cite{19} presented closed-form solutions for the effects of heat source on MHD free convection flows of a fluid-particulate suspension. Pop et al.\cite{20} studied numerically the effects of heat sinks and heat sources on mixed convection past an isothermal cylinder. Takhar and Ram\cite{21} reported on heat source, Hall current and wall-temperature fluctuation effects on MHD convection boundary layer flows. Chamkha\cite{22} studied the combined effects of radiative flux, gravity and heat sources on particle-fluid suspension heat transfer. Song et al.\cite{23} reported on heat source effects on electro-Marangoni droplet convection. Singh\cite{24} used a perturbation method to study the effects of strong wall suction, thermal diffusion, magnetic field and also heat source on natural hydromagnetic convection heat and mass transfer flows.

The studies reported above have all been confined to fluid-particle or fluid regimes. However the presence of porous media in geophysical systems and also chemical engineering applications is also of significant interest in transport modeling. Packed beds, solar porous wafer absorbers, metallic foams, ceramics, filtration systems and geomaterials are just several examples of porous materials which are encountered in engineering technologies. Particle capture by the porous matrix is important in deep-bed filtration, water flooding of oil reservoirs, wastewater treatment processes, etc. A lucid discussion of convection heat transfer flows in porous media for a variety of complex thermophysical flows has been presented by Ingham and Pop\cite{25}. In the context of fluid-particle suspension flows in porous media, applications include geothermal well hydraulics, corrosion in magneto-combustors etc. Balakrishnan and Pei\cite{26} studied experimentally the convective heat transfer between a packed bed porous medium of metallic oxides and a flowing gas-solid suspension. They computed convective Nusselt numbers in terms of a group of correlating parameters such as the Archimedes number, solid loading ratio, packing material shape ratio. It was shown that an increase in solids loading ratio substantially boosted the heat transfer rates.

More recently Santos and Bedrikovetsky\cite{27} have used a population-balance type model for studying the transport of particle suspensions in a
stochastic model of porous media, motivated by elucidating formation damage mechanisms in environmental technologies. Their model accurately simulates the particle flow reduction due to the restriction of larger particles in moving through small pores in the medium. Analytical solutions were presented for low particle concentration for the case of general particle and pore size distributions.

The effects of heat sources in convective heat transfer in porous media have also motivated some attention owing to important applications in radioactive waste disposal in geomaterials, “hot spots” in geothermal systems and also solar energy applications. Himasekhar and Baul\cite{28} studied the heat transfer in the vicinity of a heat source embedded in a rectangular saturated porous medium. Takhar et al.\cite{29} studied the hydromagnetic and heat source effects on convective boundary layer flow with Hall currents in a Darcian porous medium. Chamkha and Quadri \cite{30} obtained numerical solutions for the effects of heat source term on coupled hydromagnetic heat and mass past a conical geometry in a non-Darcian porous regime.

Very recently Bhargava et al.\cite{31} studied numerically the MHD particle-fluid suspension heat transfer in a non-Darcian porous channel considering inertial (Forchheimer) effects. In the present study we shall extend this work to consider the effects of heat absorption/generation on the fluid and particle phases velocities and temperature fields. We shall also consider numerically several special cases including non-porous flow, inviscid flow and non-magnetic heat transfer. Such a study has to the authors’ knowledge not appeared in the literature.

2. Mathematical Model

We study the laminar, steady, incompressible, hydromagnetic, fully-developed flow and heat transfer in a particle/fluid suspension in a vertical channel containing an isotropic, homogenous non-Darcian porous medium flowing against gravity. The x-direction is directed along the longitudinal axis of the plates and the y direction is normal to this. The channel plates are separated by a distance, s. A uniform transverse magnetic field, $B_0$, is applied normal to the flow direction, as shown in Fig. 1. The following assumptions are made in the analysis:

a) The number density of the particles is constant throughout the motion, and the particles are discrete and non-conducting.
b) The channel is infinitely long so that the flow is one-dimensional.

c) There is no electrical field and magnetic Reynolds number is taken as small enough to neglect induced magnetic field, Hall current, Joule heating and ion-slip effects.

d) The effects of thermal dispersion, anisotropic permeability and thermal stratification, variable porosity and Brinkman boundary friction effects in the porous medium are ignored.

e) The phases are modeled as two interacting continua and this interaction is confined to the interphase drag force (simulated using a Stokesian linear drag force model) and the interphase heat transfer, following Marble\cite{3}, Al-Subaie and Chamkha\cite{19} and Drew\cite{32}.

f) Particle phase volume fraction is assumed to be constant and finite to simplify numerical solutions.

g) Particle phase pressure gradient is neglected as the pressure is constant for uniform particle-phase volume fractions.

h) The bulk effects of the porous medium are treated using a Darcian drag force and the inertial effects with a Forchheimer second order drag force.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Physical model and coordinate system.}
\end{figure}
Under these assumptions the governing flow equations for the fluid phase and particle phase momentum and heat conservation can be written as follows:

\[
\mu \frac{d^2u}{dy^2} = \frac{dp}{\partial x} + \rho_p N(u - u_p) + \rho g + \frac{\mu}{K} u + \frac{\rho_b}{K} u^2 + \sigma B^2 u \quad (1)
\]

\[
k \frac{d^2T}{dy^2} + \rho_p c_p N_T (T_p - T) + Q(T - T_o) = 0 \quad (2)
\]

\[
\mu_p \frac{d^2u_p}{dy^2} + \rho_p N(u - u_p) - \rho_p g = 0 \quad (3)
\]

\[
\rho_p c_p N_T (T_p - T) = 0 \quad (4)
\]

where all parameters have been defined in the nomenclature. The conservation of mass in both phases is also identically satisfied. Following Al-Subaie and Chamkha\[19\] we eliminate the longitudinal pressure gradient term from the fluid phase momentum equation (1), by re-defining the transport equations at a reference point within the channel. Using the classical Boussinesq approximation and defining, \( u = 0, T = T_o, \rho = \rho_0, \mu = \mu_0, \sigma = \sigma_0, u_p = u_{p0}, T_p = T_{p0}, \rho_p = \rho_{p0}, \mu_p = \mu_{p0} \), equation (1) then reduces to:

\[
\frac{\mu_0}{\rho_0} \frac{d^2u}{dy^2} + \beta * g(T - T_o) - \frac{\rho_{p0}}{\rho_0} N(u - u_p) - \frac{\rho_{p0}}{\rho_0} g = 0 \quad (5)
\]

The governing equations are now (2), (3), (4) and (5). The fourth term in (5) is a gravity term which leads directly from 4, as described by Drew\[32\] and Al-Subaie and Chamkha\[19\]. We prescribe the following boundary conditions at the channel walls:

**For the fluid phase:**

\[ u(0) = u(s) = 0; T(0) = T_1; T(s) = T_2 \quad (6a) \]
For the particle phase:

\[ u_p(0) = \omega \frac{du_p(0)}{dy} - \frac{g}{N} ; u_p(s) = -\omega \frac{du_p(s)}{dy} - \frac{g}{N} \]  

(6b)

where all terms are again defined in the nomenclature. The fluid velocity boundary conditions (6a) correspond to no-slip conditions for the fluid phase at the channel walls. The particle phase boundary conditions (6b) are generalized wall boundary conditions based on rarified gas dynamics\[^{19}\]. The model developed therefore constitutes a two-point nonlinear boundary value problem with nonlinearity due to the Forchheimer quadratic resistance in equation (5). To simplify numerical computations and achieve a parametric study independent of dimensions, we substitute a set of non-dimensional variables, defined as follows:

\[ y = s \eta, \quad u = \frac{\mu}{\rho_s} U, \quad u_p = \frac{\mu}{\rho_s} U_p, \quad T = [T_2 - T_0] \Phi + T_0, \]

\[ T_0 = \frac{[T_1 + T_2]}{2}, \quad T_p = [T_2 - T_0] \Phi + T_0, \quad Da = \frac{K}{s^2}, \quad Fs = \frac{b}{s}, \quad Pr = \frac{\mu c}{k}, \]

\[ \gamma = \frac{c_p}{c}, \quad Sk_T = \frac{\rho N_T s^2}{\mu}, \quad Sk_m = \frac{\rho N_s^2}{\mu}, \quad p_L = \frac{\rho_p}{\rho}, \quad Ha = \sqrt{\frac{\sigma B^2 s^2}{\mu}}, \]

\[ B = \frac{\rho^2 g s^3}{\mu^2}, \quad \Lambda = \frac{\mu_p}{k \mu}, \quad Gr = \frac{g \beta^* \rho^2 s^3 [T_2 - T_0]}{\mu^2}, \quad E = \frac{Q s^2}{\mu c} \]

(7)

where all quantities are defined in the nomenclature. The dimensionless conservation equations can then be presented as follows:

\[ \frac{d^2 U}{d\eta^2} + Gr \Phi - Sk_m p_L (U - U_p) - p_L B \]

\[-\frac{1}{Da} U - \frac{Fs}{Da} U^2 - Ha^2 U = 0 \]

\[ \frac{1}{Pr} \frac{d^2 \Phi}{d\eta^2} + p_L \gamma Sk_T (\Phi - \Phi) + E \Phi = 0 \]

(9)

\[ \Lambda \frac{d^2 U_p}{d\eta^2} + Sk_m (U - U_p) + B = 0 \]

(10)
\[ Sk_T (\Phi_p - \Phi) = 0 \quad (11) \]

The corresponding transformed boundary conditions now become:

**For the fluid phase:**

\[ U(0) = U(1) = 0; \Phi(0) = -1; \Phi(1) = 1 \quad (12a) \]

**For the particle phase:**

\[
\begin{align*}
U_p(0) &= \Omega \frac{dU_p(0)}{d\eta} - \frac{B}{Sk_m} \\
U_p(1) &= -\Omega \frac{dU_p(1)}{d\eta} - \frac{B}{Sk_m}
\end{align*}
\quad (12b, 12c)
\]

The parameter, \( \Omega \), *i.e.*, dimensionless particle-phase wall slip parameter, can be varied easily to study the effects of wall slip on the flow and heat transfer in the regime. With regard to the heat source parameter, \( E \), positive values imply heat generation and negative values, absorption.

### 3. Numerical Solutions

We have obtained computational solutions for the effects of all key parameters on the fluid phase and particle phase velocities and temperatures using the finite element method. This efficient and highly versatile numerical method has been implemented by the authors’ in many fluid dynamics and transport modeling scenarios. Extensive details are available in Bathe and Reddy. For computational purpose and without loss of generality, infinity (\( \infty \)) in the domain has been fixed as 1. The whole domain is divided into a set of 81 line elements of equal width, each element being two-noded. The variational form associated with equations (25)-(28) over a typical two-noded linear element \( \left( \eta_e, \eta_{e+1} \right) \) is given by:

\[
\int_{\eta_e}^{\eta_{e+1}} \left\{ \frac{d^2U}{d\eta^2} + Gr\Phi - Sk_m p_L (U - U_p) - p_L B - \frac{1}{Da} U - \frac{F_S}{Da} U^2 - H a U \right\} d\eta = 0 \quad (13)
\]
\[ \eta_{e+1} \int \eta_e w_2 \left\{ \frac{1}{Pr} \frac{d^2 \Phi}{d \eta^2} + p_L \gamma S_k (\Phi_p - \Phi) + E \Phi \right\} d\eta = 0 \quad (14) \]

\[ \eta_{e+1} \int \eta_e w_3 \left\{ \Lambda \frac{d^2 U_p}{d \eta^2} + S_k (U - U_p) + B \right\} d\eta = 0 \quad (15) \]

\[ \eta_{e+1} \int \eta_e w_4 \left\{ S_k (\Phi_p - \Phi) \right\} d\eta = 0 \quad (16) \]

where \( w_1, w_2, w_3, \text{and} \ w_4 \) are arbitrary test functions and may be viewed as the variation in \( U, \Phi, U_p \) and \( \Phi_p \) respectively. The finite element model can be generated from equations (13)-(16) by introducing finite element approximations of the form:

\[ U = \sum_{j=1}^{2} U_j \psi_j, \ \Phi = \sum_{j=1}^{2} \Phi_j \psi_j, \ U_p = \sum_{j=1}^{2} U_{pj} \psi_j, \ \Phi_p = \sum_{j=1}^{2} \Phi_{pj} \psi_j \quad (17) \]

with \( w_i = w_2 = w_3 = w_4 = \psi_i \quad (i = 1, 2) \). \( \psi_i \) are the “shape functions” for a typical element \((\eta_e, \eta_{e+1})\) which are defined as follows:

\[ \psi_1(e) = \frac{\eta_{e+1} - \eta}{\eta_{e+1} - \eta_e}, \ \psi_2(e) = \frac{\eta - \eta_e}{\eta_{e+1} - \eta_e} \quad \eta_e \leq \eta \leq \eta_{e+1} \quad (18) \]

The finite element model of the differential equations thus formed can be summarized in matrix-vector form as:

\[
\begin{bmatrix}
K^{11} & K^{12} & K^{13} & K^{14} \\
K^{21} & K^{22} & K^{23} & K^{24} \\
K^{31} & K^{32} & K^{33} & K^{34} \\
K^{41} & K^{42} & K^{43} & K^{44}
\end{bmatrix}
\begin{bmatrix}
U \\
\Phi \\
U_p \\
\Phi_p
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4
\end{bmatrix}
\quad (19)
\]

where \( [K^{mn}] \) and \( [b^m] \) \((m, n = 1, 2, 3, 4)\) are the matrices of order \(2 \times 2\) and \(2 \times 1\) respectively. All these matrices may be defined as follows.
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\[
K_{ij}^{11} = \eta_e + 1 \int \frac{d\psi_j}{d\eta} \frac{d\psi_i}{d\eta} d\eta - \frac{1}{Da} \int \psi_j \psi_i d\eta - \frac{F_s}{U_i} \int \psi_j \psi_i d\eta - \frac{F_s}{U_j} \int \psi_j \psi_i d\eta - \frac{H_k^2}{\eta_e} \int \psi_j \psi_i d\eta,
\]

\[
K_{ij}^{12} = Gr \int \psi_j \psi_i d\eta, \quad K_{ij}^{13} = Sk_m P_L \int \psi_j \psi_i d\eta, \quad K_{ij}^{14} = 0,
\]

\[
K_{ij}^{21} = 0,
\]

\[
K_{ij}^{22} = -\frac{1}{Pr} \eta_e + 1 \int \frac{d\psi_j}{d\eta} \frac{d\psi_i}{d\eta} d\eta + P_L \gamma Sk_T \int \psi_j \psi_i d\eta + E \int \psi_j \psi_i d\eta,
\]

\[
K_{ij}^{23} = 0,
\]

\[
K_{ij}^{24} = P_L \gamma Sk_T \int \psi_j \psi_i d\eta,
\]

\[
K_{ij}^{31} = Sk_m \int \psi_j \psi_i d\eta, \quad K_{ij}^{32} = 0,
\]

\[
K_{ij}^{33} = -\Lambda \int \frac{d\psi_j}{d\eta} \frac{d\psi_i}{d\eta} d\eta - Sk_m \int \psi_j \psi_i d\eta \int \psi_j \psi_i d\eta - Sk_m \int \psi_j \psi_i d\eta - \Lambda \int \psi_j \psi_i d\eta - \Lambda \int \psi_j \psi_i d\eta - \Lambda \int \psi_j \psi_i d\eta - \Lambda \int \psi_j \psi_i d\eta,
\]

\[
K_{ij}^{34} = 0,
\]

\[
K_{ij}^{41} = 0, \quad K_{ij}^{42} = -Sk_T \int \psi_j \psi_i d\eta, \quad K_{ij}^{43} = 0,
\]

\[
K_{ij}^{44} = 0.
\]
\[
K_{ij}^{44} = Sk_T \int_{\eta_e}^{\eta_{e+1}} \psi_i \psi_j \, d\eta
\]  \hfill (23)

\[
b_i^1 = -PrB \left( \psi_i \frac{d\psi}{d\eta} \right)^n_{e+1} \eta_e, \quad b_i^2 = -\frac{1}{\Pr} \left( \psi_i \frac{d\Phi}{d\eta} \right)^n_{e+1} \eta_e, \quad b_i^3 = B - \Lambda \left( \psi_i \frac{dU_f}{d\eta} \right)^n_{e+1} \eta_e
\]

\[
b_i^4 = 0
\]  \hfill (24)

where \( \overline{U} = \sum_{i=1}^{2} U_i \psi_i \). Each element matrix is of the order 8×8. As the entire solution domain is divided into a set of 81 line elements, consequently, post-assembly of all the elements equations leads to an effective matrix of order 328×328. This system of equations is nonlinear therefore an iterative scheme is utilized to solve it. The system is linearized by incorporating the function \( \overline{U} \), which is assumed to be known. After applying the given boundary conditions, the system of equation has been solved using Gauss elimination method by maintaining an accuracy of 0.0005. The finite element solutions have been compared for the purely fluid case (\( Da \to \infty \) and \( Fs = 0 \)) with the non-porous analytical solutions of Al-Subaie and Chamkha\(^{[19]} \), and also with a finite difference solution; in both cases an excellent agreement up to three decimal places was achieved. Comparisons have been excluded here for lack of space.

4. Results and Discussion

Representative results for the fluid phase and particle phase hydrodynamics and temperature distributions have been obtained. The flow regime is dictated by 12 parameters, namely the heat source parameter (E), Prandtl number (Pr), Grashof free convection number (Gr), Hartmann hydromagnetic number (Ha), inverse Stokes number (Sk\(_m\)), Darcy number (Da), Forcheimmer number (Fs), particle loading parameter (P\(_L\)), buoyancy parameter (B), temperature inverse Stokes number (Sk\(_T\)), viscosity ratio (\( \Lambda \)), specific heat ratio (\( \gamma \)) and wall slip parameter (\( \Omega \)). From equation (16) the \( \Phi \) value (dimensionless
temperature of fluid phase) and the \( \Phi_p \) (dimensionless temperature of particle phase) will be equivalent.

We have prescribed the following default values for the thermophysical parameters: \( \text{Pr} = 1, \) \( \text{Gr} = 1, \) \( \text{Ha} = 1, \) \( \text{Sk}_m = 1, \) \( \text{Da} = 1, \) \( \text{Fs} = 1, \) \( \text{P}_L = 1, \) \( \text{B} = 1, \) \( \text{Sk}_T \) = 1, \( \Lambda = 1, \) \( \gamma = 1, \) \( \Omega = 1 \) and \( E = 1. \) These correspond to free convective non-Darcian hydromagnetic heat transfer with heat generation in a porous channel. We shall discuss in turn the individual and relative influence of the majority of these parameters on the fluid and particle phase velocity and temperature distributions. For brevity we have omitted the variation of the flow variables with the \( \Lambda, \gamma \) and \( \Omega \) parameters.
Fig. 6. $U$ versus $\eta$ for $Ha$ values.

Fig. 7. $\rho U$ versus $\eta$ for $Ha$ values.

Fig. 8. $U$ versus $\eta$ for $Sk_m$ values.

Fig. 9. $\rho U$ versus $\eta$ for $Sk_m$ values.

Fig. 10. $U$ versus $\eta$ for $Da$ values.

Fig. 11. $\rho U$ versus $\eta$ for $Da$ values.
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Fig. 12. $U$ versus $\eta$ for various $Fs$ values.

Fig. 13. $U_p$ versus $\eta$ for various $F_s$ Values.

Fig. 14. $U$ versus $\eta$ for various $P_L$ values.

Fig. 15. $\Phi, \Phi_p$ versus $\eta$ for $P_L$.

Fig. 16. $U_p$ versus $\eta$ for $P_L$.

Fig. 17. $U$ versus $\eta$ for $B$.
Figure 2 illustrates the distribution of both particle phase temperature and fluid phase temperature ($\Phi$, $\Phi_p$) with heat source parameter, $E$. For heat absorption effects, $E < 0$ and we observe that in the left hand side of the channel ($0 < \eta < 0.5$), both temperatures are increased with negative $E$ values ($-5$, $-10$) and reduced for positive $E$ values. The case $E = 0$ clearly corresponds to the absence of a heat source. In the second half of the channel, ($0.5 < \eta < 1.0$), a reversal in effects is observed so that as $E$ becomes more positive, temperatures are boosted. The peak values of $\Phi$ and $\Phi_p$ (both equal to 1) occur for $E = 10$ at the right hand side channel wall in accordance with the boundary condition prescribed there. We note that particle phase temperature boundary conditions are not specified in the model and these are in fact driven by the fluid phase temperature field. A smooth development in profiles for all $E$ values is observed indicating that efficient numerical solutions have been obtained. We have compared computations favourably with earlier studies\[19\] but these are not reported here for brevity.

Figure 3 shows the variation of particle phase temperature and fluid phase temperature ($\Phi$, $\Phi_p$) with Prandtl number (Pr) for different $E$ values. Pr defines the ratio of momentum diffusivity to thermal diffusivity. Larger Pr values imply lower thermal conductivities which amount to smaller thermal diffusion rates. Smaller Pr fluids have higher thermal conductivities so that heat can diffuse faster. Physically Pr = 3, 5, 7 can represent electrically-conducting solutions at high temperatures. High values of Pr are also physically accurate for denser particulate-fluid suspensions such as corrosive suspensions, debris-laden flows, contaminated oils... etc, as described by Incropera and Dewitt\[35\].
computations indicate that with negative E (= –1) i.e., for heat absorption cases, a rise in Pr from 3, to 5 to 7, slightly increases the temperatures in the saturated porous regime, which become increasingly more positive in the right half of the channel (0.5 < η < 1.0); however temperatures decrease slightly in the left half of the channel (0 < η < 0.5) with a rise in Pr. For the case of heat generation (E = 1), the converse is apparent. In the left half of the channel, (0 < η < 0.5), temperatures (Φ, Φ_p) now decrease and the change is much more dramatic over the same range of Pr values than for the heat absorption case. In the right half of the channel, temperatures however increase markedly as Pr rises from 3, through 5 to 7, for E = 1. These trends for positive E values are verified in Fig. 4, where we have also plotted temperatures for Pr = 0.1 (liquid metal suspensions) and Pr = 2, 10.

Figure 5 illustrates the influence of Grashof (free convection) parameter on the distribution of fluid phase velocity with transverse coordinate, η. As Gr increases from 0 through 100, 200, 300 and 500, reduces U values considerably for the regime 0 < η < 0.5, i.e., the left half of the channel. For higher Gr values, i.e., 300 and 500 which correspond to very strong buoyancy fluid phase velocity in fact becomes negative indicating flow reversal approximately over the range 0 < η < 0.3. Conversely in the second half of the channel, 0.5 < η < 1.0, the U values remain positive and are boosted substantially with buoyancy. The maximum fluid phase velocity in the channel corresponds to Gr = 500 at η ~ 0.75. For the forced convection case, Gr = 0, the velocity profile is parabolic and perfectly symmetrical about the channel centre line at η ~ 0.5. Increasing buoyancy effects, i.e., rising Gr, forces profiles downwards over the first half of the channel and upwards over the second half of the channel.

Figure 6 illustrate the effects of U for various Hartmann numbers, Ha on the fluid phase velocity, U distributions. As Ha increases from 0 (no magnetic field effects) to 3, 7 and 20, the velocity profiles are progressively decreased. In all cases the profiles are symmetrically parabolic about the channel centre-line, η = 0.5, so that for each profile the maximum velocity is at the centre of the channel. The maximum velocity computed is approximately 7 for Ha = 0, which decreases to about 0.25 for Ha = 20. Consequently as with single phase magnetohydrodynamic flows the magnetic field has a retarding effect on
the fluid phase velocity field as it produces a drag perpendicular to the direction of application (x-direction) which therefore acts in the direction of flow (longitudinal) and causes deceleration.

In Fig. 7 the influence of Hartmann number (Ha) on the particle phase velocity, $U_p$, is shown. Again an impeding effect is observed for a rise in Ha values from 0 to 3, 7, and 20. In all cases the velocity is negative and is made more so by a rise in Ha. However the separation of profiles is less distinct compared with the fluid velocity phase, as this is attributed to the indirect influence of magnetic field on the particle phase momentum development, via the coupling in the $Sk_m(U - U_p)$ term, with the fluid phase momentum equation. Profiles are all smooth and partially parabolic. Magnetic field therefore has a primary effect on the fluid phase momentum and a secondary effect on the particle phase momentum. The least peak value of $U_p$ is observed for Ha = 20 (maximum magnetic field strength case) and occurs at $\eta = 0.5$ and is equal to $-62$ approximately. For Ha = 0 (non-conducting case) the least peak value is about -58. In all cases the particle phase velocities are found to be negative indicating that particles flow in the opposite direction to the main fluid flow field. The maximum values for all values of Ha correspond to the wall where due to symmetry they are identical for each profile on the left wall and the right wall of the channel.

Figures 8 and 9 show the effects of momentum (hydrodynamic) inverse Stokes number, $Sk_m$, on U and $U_p$ profiles versus $\eta$. The parameter, $Sk_m$, signifies the hydrodynamic coupling between the fluid and the particle phases via the interphase momentum transfer coefficient (N). A rise in $Sk_m$ from 0.1 through 0.3, 0.7, 1, 2 and up to 5 implies a progressively greater transfer of momentum from the fluid phase to the particle phases. This causes a major decrease in U (fluid phase) velocity throughout the channel, as shown in Fig. 8. Profiles also become increasingly flatter as $Sk_m$ increases, i.e., they are more parabolic for lower $Sk_m$ values. Maximum values of U for any profile are located at the channel centre line ($\eta = 0.5$). For the profile $Sk_m = 1$ an equal momentum is achieved for both phases. U values remain zero at both channel walls, as prescribed by the wall boundary conditions. The maximum U velocity is about 8.5 for $Sk_m = 0.1$ and 2.5 for $Sk_m = 5$. In Fig. 9 it can be seen that particle phase velocities, i.e., $U_p$ values conversely to the U values in Fig. 8, increase with a rise in $Sk_m$, i.e., they become less negative as $Sk_m$
increases from 0.1 through 0.3, 0.7, 1, 2 to 5. The $U_p$ profiles are truncated parabolas and become less curved, \textit{i.e.}, flatter as $S_{km}$ increases. The peak centerline values rise from $-105$ at $\eta = 0.5$ (for $S_{km} = 0.1$ which implies 90\% of the momentum is retained by the fluid phase, \textit{i.e.}, only 10\% to the particulate phase) to about $-17$ for $S_{km} = 5$ (for which five times as much momentum is transferred to particle phase as to the fluid phase).

The influence of the porous drag parameters, Darcy number, $D_a$, and Forchheimer number, $F_s$, on $U$ and $U_p$ distributions are illustrated respectively in Fig. 10-13. The Darcian impedance, $-\frac{1}{D_a}U$, is inversely proportional to $D_a$ in the fluid phase momentum equation. $D_a$ is directly proportional to the hydraulic conductivity, \textit{i.e.}, permeability, $K$, of the porous medium. A rise in $D_a$ therefore implies that the porous matrix in the channel increases in permeability, \textit{i.e.}, becomes increasingly occupied by the fluid-particle suspension. In the limit as $D_a \rightarrow \infty$, the porous fibres vanish as does the Darcian impedance (and the Forchheimer drag force). A rise in $D_a$ is seen to clearly strongly increase the $U$ value. All $U$ profiles are again parabolic, a trend consistent with fully developed Newtonian channel flow in classical hydrodynamics\textsuperscript{[36]}. Hence maximum values always occur at the channel centre line and these rose for $D_a = 0.01$ (lowest permeability) from approximately 0.6 to 2.5 for $D_a = 0.1$, to 6 for $D_a = 1$ up to 10 for $D_a = 200$ (high permeability). Figure 11 shows that the particle phase velocity, $U_p$, also increases as Darcy number rises from 0.01, through 0.1 and 1 to 200. $U_p$ values hence become less negative with an increase in $D_a$ but the increase is significantly less pronounced than for the fluid phase velocity profiles. As with the magnetic field effects discussed earlier on $U$ and $U_p$ profiles, the lesser effect of Darcian resistance on particle phase velocities is due to the coupling of the $U$ and $U_p$ fields which indirectly permits the Darcy number to effect the particle phase velocity, $U_p$, by affecting directly the $U$ velocity field first.

Figures 12 and 13 depict the distribution of $U$ and $U_p$ versus $\eta$ with the quadratic porous parameter, \textit{i.e.}, Forchheimer number, $F_s$. For the case of $F_s = 0$ the regime is Darcian, \textit{i.e.}, no inertial drag is present. Generally such a regime corresponds to very low Reynolds number flows which are viscous-dominated. As $F_s$ is increased to 1, 5 and then 50
(strong inertial drag), the velocity $U$ is substantially reduced as shown in Fig. 11. Increasing $F_s$ from 1, through 5 to 50 implies an increase in inertial drag of five times and fifty times respectively since $D_a$ is unity in the computations (Forchheimer drag $= -\frac{F_s}{D_a} U^2$ in the fluid phase momentum equation (25). As with the case of magnetic field (Fig. 3), the parabolic profiles also become increasingly flatter with higher $F_s$ values. The maximum $U$ value for each profile is there seen at the centerline of the channel for and has a value of 10 for $F_s = 0$, 6.5 for $F_s = 1$, 3.75 for $F_s = 5$ and 1 for $F_s = 50$. Particle phase velocity, \textit{i.e.}, $U_p$ also decreases with a rise in $F_s$, \textit{i.e.}, values become more negative, as seen in Fig. 13, however the transition between profiles is much less pronounced than for the velocity profiles, again owing to the indirect influence of the Forchheimer drag via a coupling between $U$ and $U_p$ in the particle phase momentum equation. Forchheimer drag does not arise in this momentum equation as it does in the fluid phase momentum equation. Therefore Forchheimer quadratic drag induced principally a deceleration in the fluid and to a much lesser extent in the particle motions.

Figures 14-16 illustrate the influence of particle loading parameter, $P_L$, on the fluid phase velocity, fluid temperature, particle phase temperature and particle phase velocity. This parameter ($P_L = \frac{\rho_p}{\rho}$) represents the ratio of the densities of the particle and fluid phases. As $P_L$ rises the fluid phase velocity, $U$, is increased considerably, as seen in Fig. 14. All profiles are smooth parabolas and symmetric about $\eta \sim 0.5$, with the exception of $P_L = 0.1$ for which the density of the fluid is much greater than the particle phase. This latter profile is considerably flatter than for higher $P_L$ values. In Fig. 15 both fluid and particle phase temperatures, $\Phi$ and $\Phi_p$, decrease in the left hand half of the channel with a rise in $P_L$ from 0.1 through 1, 2, 3 and to 4; a reversal in this trend is apparent however for the second (right) half of the channel ($0.5 < \eta < 1$), where both temperatures are increased with a rise in $P_L$. Of course these computations are for the case of heat generation ($E = 1$). In Fig. 16 the particle-phase velocity, $U_p$, is seen to increase slightly with a rise in $P_L$ from 0.1 through 1, 2, 3 and 4. Hence particle phase velocities become less negative as $P_L$ increases. The changes are however much less dramatic than for the fluid phase velocity (Fig. 14).
The effect of gravitational buoyancy parameter, $B$, on the variation of $U$ and $U_p$ are shown in Fig. 17 and 18. The buoyancy parameter occurs as a positive body force in the fluid phase momentum equation (25) \( i.e., + p_L B \) and therefore aids in momentum development in the fluid i.e. accelerates the fluid. Hence a dramatic increase in fluid phase velocity, $U$, corresponds to an increase in $B$ from 0.2 through 0.3, 0.5, 0.7 and to 1.0 (Fig. 17). The fluid phase momentum is therefore strongly boosted by a rise in $B$ values. All profiles are again symmetric parabolas. On the other hand, the particle phase velocity, $U_p$, decreases markedly with a rise in $B$ value from 0.2 to 1.0 (Fig. 18). The $U_p$ value for $B = 0.2$ at the channel center-line is approximately $-8$; for $B = 1.0$ this value plummets to about $-62$. In the particle phase momentum equation, the gravitational buoyancy parameter occurs as a negative body force term, $-B$ and this in contrast to the fluid phase momentum equation, decelerates the flow with increasing $B$ values.

The variation of both fluid and particle phase temperatures, $\Phi$ and $\Phi_p$, with $\eta$ for various $Sk_T$, \( i.e. \) temperature inverse Stokes numbers are shown in Fig. 19. An increase in $Sk_T$ causes the thermal coupling, \( i.e., \) energy transfer between the fluid and particle phases to increase. We observe that as $Sk_T$ rises, the temperatures do indeed decrease considerably in the left half of the channel ($0 < \eta < 0.5$) but conversely in the second half (right) of the channel temperatures are then boosted considerably. In the study by Chamkha\cite{2} separate temperature distributions were obtained for either phase since in that flow scenario, a separate differential equation was used for the particle phase. In the present case, the two temperature fields are actually identical owing to the equation (16).

5. Conclusions

In this paper, we have presented a mathematical model for the magnetohydrodynamic flow and free convection heat transfer of a fluid-particle suspension in a non-Darcian porous channel. Using a set of non-dimensional variables the model has been made dimensionless and solutions obtained for the particle and fluid phase velocity and temperature fields for a wide range of the thermophysical parameters. Results have been presented graphically and discussed at length both with regard to modeling aspects and also physical implications. While no
experimental data was available for comparison of the general flow model, test computations have been compared with previous numerical studies and found to be favourable. The present model can be extended to three-dimensional flows and also to incorporate transient effects. These aspects will be addressed in the future.

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**Notations**

*Dimensional*

\[ \begin{align*}
\rho & = \text{density of fluid phase} \\
t & = \text{time} \\
\nabla & = \text{Del operator} \\
\mathbf{V} & = \text{fluid phase velocity vector} \\
\mathbf{P} & = \text{hydrodynamic pressure} \\
\mu & = \text{dynamic viscosity of fluid phase} \\
\rho_p & = \text{density of particle phase} \\
N & = \text{interphase momentum transfer coefficient} \\
\mathbf{V}_p & = \text{particle phase velocity vector} \\
g & = \text{gravitational acceleration} \\
\mathbf{B} & = \text{magnetic field vector} \\
K & = \text{permeability of porous medium} \\
\sigma & = \text{electrical conductivity of fluid phase} \\
b & = \text{Forcheimmer inertial resistance parameter} \\
c & = \text{specific heat of fluid phase at constant pressure (isobaric)} \\
T & = \text{temperature of fluid phase} \\
k & = \text{thermal conductivity of fluid phase} \\
c_p & = \text{specific heat of particle phase at constant pressure (isobaric)} \\
N_T & = \text{interphase heat transfer coefficient} \\
T_p & = \text{temperature of particle phase}
\end{align*} \]
\( \mu_p \) = dynamic viscosity of particle phase \\
\( x \) = direction parallel to channel \\
\( y \) = direction transverse to channel \\
\( u \) = velocity of fluid phase \\
\( u_p \) = velocity of particle phase \\
\( \beta^* \) = coefficient of volumetric expansion \\
\( s \) = separation of plates comprising channel \\
\( \omega \) = particle phase wall-slip coefficient \\
\( Q \) = heat source term \\

**Non-dimensional Parameters** \\
\( \eta \) = transformed \( y \) coordinate \\
\( U \) = dimensionless \( \eta \)-direction fluid phase velocity \\
\( U_p \) = dimensionless \( \eta \)-direction particle phase velocity \\
\( \Phi \) = dimensionless temperature of fluid phase \\
\( \Phi_p \) = dimensionless temperature of particle phase \\
\( Da \) = Darcy number \\
\( Fs \) = Forcheimmer number \\
\( Pr \) = Prandtl number \\
\( \gamma \) = specific heat ratio \\
\( Sk_T \) = inverse temperature Stokes number \\
\( Sk_m \) = inverse hydrodynamic (momentum) Stokes number \\
\( p_L \) = particle loading parameter \\
\( B \) = dimensionless buoyancy parameter \\
\( Ha \) = Hartmann hydromagnetic number \\
\( \Lambda \) = viscosity ratio \\
\( Gr \) = Grashof number \\
\( \Omega \) = dimensionless particle-phase wall slip parameter \\
\( E \) = dimensionless heat source term
References


انتقال الحرارة اللاخطي لسائل وعوائق تسير في قناة غير
دارسية مع وجود مصدر حرارة واعتبار الطفو والمغناطيسية:
دراسة عددية

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البديل الخصص: تمت دراسة انتقال الحرارة المستقر بالحمل الطبيعي
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الحرارة كي يتم إمتصاص الحرارة أو توليدها. تم استخدام
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المعادلات غير البعدية عديماً بأسلوب العناصر الصغيرة (FEM)
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الحرارة (E), ورقم برايتل (Pr), ورقم غراشوف (Gr), والرقم
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ورقم هارتمان (Ha), ورقم ستوكس العكسي لدرجة الحرارة
(SKT), وذلك على السرعة غير البعيدة للمائع U وسرعة العوائق \( U_p \) غير البعيدة ودرجة حرارة المائع غير البعيدة \( \phi \) ودرجة حرارة العوائق \( \phi_p \)، أيضاً غير البعيدة، وتم تمثيل ذلك على أشكال بيانية.

وجد أن سرعة المائع تقل كثيراً مع تخفيف الحقل المغناطيسي وقوة دارسي وقوة فورشيمر. بينما لوحظ أن التأثير على سرعة العوائق كان أقل. وجد أن رقم براندل يخفض كلاً من درجة حرارة المائع والعوائق في الجانب الأيسر من القناة، ولكن يزيد معها في الوسط والجانب الأيمن من القناة. وجد أن رقم ستوكس العكسي للحركة يخفض سرعة المائع ويزيد من سرعة العوائق. لقد تم أيضاً دراسة تأثير المتغيرات الفيزيوحرارية الأخرى ومقارنتها بالدراسات السابقة الأخرى.

يجد هذا الموضوع تطبيقات في تعجيل السريان المغناطيسي في البلازما والسريان في علم فيزياء الفضاء، وفيزياء الجيولوجية، وعلم حرارة الأرض، وعمليات المواد الصناعية.