GENERALIZED BALANCED MODEL REDUCTION

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ABSTRACT. The balanced model reduction technique is generalized to produce reduced order models with poles clustered in the same region as the poles of the full order model. Generalized controllability and observability gramians are defined and used instead of the normal controllability and observability gramians. This method is also used to reduce the order of unstable systems.

1. INTRODUCTION
Despite all the nice properties of balanced-truncation reduced order models, there is no guarantee that the poles of the reduced order model will be clustered in the same region as the poles of the full order model. The location of the poles affect the time response characteristics such as overshoot, oscillations, settling time, etc... For the reduced order model to be a faithful representation of the full order model, the least expected is that the poles of the full and reduced order models lie in the same region. This will often ensure that the reduced order model has comparable response characteristics as the full order model. Moreover, most of the methods for model reduction are based on the assumption that the original system has constant parameters. However, it is shown in [8] that even if the reduced order model is stable and is a very accurate approximation of the full order model, the closed-loop system characteristics may not be acceptable and the stability of the closed-loop system may not even be preserved due to the effect of parameter variations. Typically in practical cases, parameter variations will result in movements of the system poles from their nominal values. If the system poles for all parameter variations are clustered in a certain region and the reduced order model has its poles in this region then it will be a faithful representation not only for the nominal model but also for all models resulted from parameter variations within that region.

Balanced model reduction and its properties will be reviewed in section II. In section III, a pole clustering theorem will be used to produce the generalized Lyapunov equation for the desired regions. This theorem will also be relaxed for controllable systems. In section IV, the modification of balanced model reduction will be developed and will be called generalized balanced model reduction. Finally, examples will be used to compare the balanced model reduction and the generalized model reduction.

2. BALANCED MODEL REDUCTION
Assume that the system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  \hspace{1cm} (1a)

\[ y(t) = Cx(t) \]  \hspace{1cm} (1b)

is asymptotically stable, controllable and observable where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), and \( C \in \mathbb{R}^{p \times n} \). The controllability gramian \( P \) and observability gramian \( Q \) are defined as:

\[ P = \int_0^\infty e^{At}BB^te^{A^t}dt \]  \hspace{1cm} (2)

\[ Q = \int_0^\infty e^{At}C^tCe^{A^t}dt \]  \hspace{1cm} (3)

and can be found as the unique positive definite solution of the following Lyapunov equations:
\[ AP + PA' + BB' = 0 \]
\[ A'Q + QA + C'C = 0 \]
where \((\cdot)'\) denotes the transpose. If the representation of the system of equation (1) is transformed to another representation using a non-singular transformation \(T\), then, the new state space representation of the system is \((\overline{A}, \overline{B}, \overline{C})\) where
\[
\overline{A} = T^{-1}AT, \quad \overline{B} = T^{-1}B, \quad \overline{C} = CT
\]
and the gramians will be transformed to
\[
\overline{P} = T^{-1}PT^{-1}, \quad \overline{Q} = T'QT, \quad \overline{PQ} = T^{-1}PQT
\]
Thus, the gramians \(P\) and \(Q\) depend on the state-space coordinates. However, the eigenvalues of their products \(PQ\) are invariant under state space transformations and are input/output invariant. The square root of the eigenvalues of \(PQ\) are called the Hankel singular values of the system \(G(s) = C(sI - A)^{-1}B\) and they are denoted by \(\sigma_i\) where \(\sigma_i = \lambda_i^{1/2}(PQ)\) and \(\lambda_i(PQ)\) is the \(i\)-th eigenvalue of \(PQ\). A realization \((\overline{A}, \overline{B}, \overline{C})\) of \(G(s)\) is said to be balanced, if \(\overline{P} = \overline{Q} = \Sigma\) where \(\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)\), \(\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0\). The states of a balanced representation are balanced between controllability and observability. Thus, they represent a convenient structure for model reduction since those states having week controllability and weak observability can be neglected without causing any imbalance in controllability or in observability properties of the remaining states [12]. The Hankel singular values provide the means of determining those states. Thus, the states corresponding to the smallest Hankel singular values can be neglected. If the realization \((\overline{A}, \overline{B}, \overline{C})\) of \(G(s)\) is balanced and matrices \(\overline{A}, \overline{B}, \overline{C}\) and \(\Sigma\) are partitioned compatibly as
\[
\overline{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}; \quad \overline{B} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}; \quad \overline{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}; \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}
\]
where the dimension of \(A_{11}\) and \(\Sigma_r\) are \(r \times r\), then a balanced-truncation reduced order model is defined as \(G_r(s) = C_r(sI - A_{11})^{-1}B_1\). The reduced order model \((A_{11}, B_1, C_1)\) is balanced and if \(\sigma_r > \sigma_{r+1}\), then it is asymptotically stable, controllable and observable [13]. Moreover, it enjoys the following \(L^\infty\)-norm error bound [3,4].
\[
\| G(jw) - G_r(jw) \|_\infty \leq 2(\sigma_{r+1} + \ldots + \sigma_r) = 2tr(\Sigma_r) \quad \text{for all } w
\]
where \(tr\) denotes the trace, the infinity norm is defined as \(\| X(s) \|_\infty = \sup_{w \geq 0} |\sigma(X(jw))| \) and \(\sigma(X)\) is the maximum singular value of \(X\).

There are several algorithms for computing the balancing transformation. One efficient algorithm is due to Laub et al. [10]. Saifonov and Chieng [15] developed a reliable algorithm for getting the reduced order model without computing a balancing transformation.

3. POLE CLUSTERING THEOREMS AND THE GENERALIZED GRAMIANS
In stability studies of linear time-invariant systems one is often concerned with the location of the poles of the system. Most famous are regions of pole clustering with respect to the left complex plane and the unit circle. The notation of relative stability introduces more regions. In general, one may ask: given a matrix \(A\) and an algebraic region \(S\) in the complex plane, find the necessary and sufficient conditions for the eigenvalues of \(A\) to lie in \(S\). Important results on pole clustering can be found in [11,2],[5-7],[9]. Here we are concerned with second order regions such as ellipses, parabolas, circles, etc. These regions are described by
\[
S = \{ (x, y) : \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy < 0 \}
\]
Let \(A \in C^{\infty}\), \(\lambda\) be an eigenvalue of \(A\) and \(\sigma(A) = \{ \lambda_1, \lambda_2, \ldots, \lambda_n \}\) denotes the spectrum of \(A\). The following theorem states the necessary and sufficient conditions for \(\sigma(A) \in S\).

**Theorem 1** [7],[9]: Let \(A \in C^{\infty}\) and consider \(S\) with \(\gamma_{00} + \gamma_{20} \geq 0\). For \(\sigma(A) \in S\), it is necessary
and sufficient that given any positive definite Hermitian matrix \( M \in C^{n \times n} \), there exists a unique positive definite Hermitian matrix \( H \in C^{n \times n} \) such that
\[
c_{00}H + c_{10}AH + c_{20}A^2H + c_{01}HA^* + c_{02}H(A^*)^2 + c_{11}AHA^* = -M
\]
(9)
where \( c_{00} = \gamma_{00}, \quad c_{10} = \overline{c}_{10} = \frac{1}{2}(\gamma_{10} + i\gamma_{01}), \quad i = \sqrt{-1} \quad c_{11} = \overline{c}_{10} = \frac{1}{2}(\gamma_{20} + \gamma_{02}), \quad c_{20} = \overline{c}_{20} = \frac{1}{4}(\gamma_{20} - \gamma_{02} + i\gamma_{11}) \)
and * denotes the complex-conjugate transpose.

The above theorem can be generalized by replacing the positive definite matrix \( M \) with a positive semi-definite matrix \( BB^* \) such that the system \((A,B)\) is controllable. This is done in theorem 2 below.

**Theorem 2:** Let \( A \in C^{n \times n} \) and consider \( S \) with \( \gamma_{02} + \gamma_{20} \geq 0 \). For \( \sigma(A) \in S \), it is necessary and sufficient that given any matrix, \( B \in C^{n \times m} \) with \((A,B)\) controllable, there exists a unique positive definite Hermitian matrix \( H \in C^{n \times n} \) such that
\[
c_{00}H + c_{10}AH + c_{20}A^2H + c_{01}HA^* + c_{02}H(A^*)^2 + c_{11}AHA^* = -BB^*
\]
(10)
where \( c_{00} = \gamma_{00}, \quad c_{01} = \overline{c}_{10} = \frac{1}{2}(\gamma_{10} + i\gamma_{01}), \quad c_{11} = \overline{c}_{10} = \frac{1}{2}(\gamma_{20} + \gamma_{02}), \quad c_{20} = \overline{c}_{20} = \frac{1}{4}(\gamma_{20} - \gamma_{02} + i\gamma_{11}) \)

Moreover, if (10) still has a positive definite solution \( H \) for \( B \), \( \lambda_i \not\in S \), then \( \lambda_i \) belongs to the boundary of \( S \).

**Proof**

**Sufficiency:** Let \( x = Re\lambda, \quad y = Im\lambda \) and \( A^*v = \overline{A}v \Rightarrow v^*A = \lambda v^* \). Multiplying (10) from left by \( v^* \), from right by \( v \) and after simplifications we get
\[
(\gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy)v^* Hv = -v^* BB^* v
\]
since \( v^* Hv > 0 \) and \( v^* BB^* v \geq 0 \), this implies \( \gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy \leq 0 \)
Since \((A,B)\) is controllable, then \( v^* B \neq 0 \). Thus \( v^* BB^* v \neq 0 \); which implies
\[
\gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy < 0 \Rightarrow \sigma(A) \in S
\]

**Necessity:** Let \( \lambda \in S \), we want to prove that for every \( B \) with \((A,B)\) controllable there exists a positive definite solution \( H \) of (10). Since \( c_{11} \) can not be negative, the proof will be done for \( c_{11} = 0 \) and \( c_{11} > 0 \). Let \( c_{11} = 0 \), equation (10) becomes
\[
K^* H + HK = -BB^*
\]
where \( K = \frac{1}{2}c_{00} + c_{10}A + c_{20}A^2 \)
If \( v \) is an eigenvector of \( A \), then it is also an eigenvector of \( K \). Hence, \((K,B)\) is controllable implies \((K,B)\) is controllable. Now, let \( \lambda \) and \( \lambda \) be eigenvalues of \( K \) and \( A \) respectively, then
\[
\lambda = \frac{1}{2}c_{00} + c_{10}\lambda + c_{20}\lambda^2
\]
and let \( x = Re\lambda \) and \( y = Im\lambda \) and after simplification we get
\[
Re\lambda = \frac{1}{2}(\gamma_{00} + \gamma_{10}x + \gamma_{02}y^2 + \gamma_{11}xy)
\]
Clearly \( Re\lambda < 0 \), because \( \lambda \in S \). Hence, the Lyapunov equation has a unique positive definite solution \( H \). Now let \( c_{11} > 0 \), equation (10) is equivalent to \( n^2 \) linear equations, whose \( n^2 \times n^2 \) coefficient matrix has
\[
\gamma_{00} + \gamma_{10}x + \gamma_{20}x^2 + \gamma_{01}y + \gamma_{02}y^2 + \gamma_{11}xy\quad \text{where} \quad x = Re\lambda, \quad y = Im\lambda, \quad \text{and} \quad i,j = 1,2,\ldots,n
\]
as its eigenvalues. Since \( \lambda \in S \), (9) has a unique solution. From theorem 1, for any positive definite \( M \), their exist a positive definite solution \( H \) of (9). Let \( M_i = tBB^* + (1-t)M \) with \( 0 \leq t \leq 1 \)
Clearly, $M$ is positive definite for $0 \leq t < 1$ and positive semi-definite for $t = 1$. Equation (9) has unique solution $H$, and the eigenvalues of $H$, are real and vary continuously with $t$. Hence if we prove that $H$, never becomes singular, we complete the proof. Let

$$H = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}$$

we substitute $H = H$, in (9). Then every $(n,n)$ element of the matrix terms on the left of (9) is zero except that of $c_{11}AHA^*$. Hence we arrive at the contradiction to the fact - $M$ is negative semi-definite, since $c_{11}AHA^*$ is positive semi-definite, because $c_{11} > 0$.

The solution to equation (10) will be called the generalized controllability gramian. Similarly the generalized observability gramian is defined. The above theorem will be used to find Lyapunov-type matrix equations for sectors, circles, ellipses, parabolas and vertical strips in the complex plane. The following symmetric regions with respect to the $x$-axis are chosen because the poles of any real system are symmetric with respect to the $x$-axis.

**Pole clustering inside a sector**

Consider the sector shown in figure 1a, and described by

$$S = \{(x,y): -y + nx + b < 0\}$$

The condition $\lambda_{02} + \lambda_{20} \geq 0$ is satisfied because $\lambda_{02} + \lambda_{20} = 0 + 0 = 0$. Thus

$$c_{00} = b, \quad c_{10} = c_{01} = \frac{1}{2}(m - i) \quad \text{and} \quad c_{02} = c_{20} = c_{11} = 0 \quad \text{and} \quad (9) \quad \text{becomes}$$

$$2bH + (m - i)AH + (m + i)HA^* = -M$$

(11)

**Pole clustering inside a circle**

Consider the circular region with radius $r$ and center at $-\alpha$ shown in figure 1b and described by

$$S = \{(x, y): (x + \alpha)^2 + y^2 - r^2 < 0\}$$

The condition $\lambda_{02} + \lambda_{20} = 1 + 1 = 2 \geq 0$ is satisfied. Thus

$$c_{00} = \alpha^2 - r^2, \quad c_{10} = c_{01} = \alpha, \quad c_{11} = 1, \quad \text{and} \quad c_{02} = c_{20} = 0 \quad \text{and} \quad (9) \quad \text{becomes}$$

$$\alpha(AH + HA^*) + AHA^* + (\alpha^2 - r^2)H = -M$$

(12)

**Pole clustering inside an ellipse**

Consider the region inside the ellipse shown in figure 1c and described by

$$S = \{(x, y): \left(\frac{x + \alpha}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - 1 < 0\}$$

The condition $\gamma_{02} + \gamma_{20} = 1 + \left(\frac{a}{b}\right)^2 \geq 0$ is satisfied. Let $c = \left(\frac{a}{b}\right)$ then

$$c_{00} = \alpha^2 - a^2, \quad c_{01} = c_{10} = \alpha, \quad c_{02} = c_{20} = \frac{1}{4}(1 - c) \quad \text{and} \quad c_{11} = \frac{1}{2}(1 + c) \quad \text{and} \quad (9) \quad \text{becomes}$$

$$(\alpha^2 - a^2)H + \alpha(2AH + HA^*) + \frac{1}{4}(1 - c)[A^2H + H(A^*)^2] + \frac{1}{2}(1 + c)AHA^* = -M$$

(13)

**Pole clustering inside a hyperbola**

Consider the hyperbola shown in figure 1d, and described by

$$S = \{(x, y): \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 < 0\}$$

The condition $\gamma_{02} + \gamma_{20} \geq 0$ is not always satisfied because

$$\gamma_{02} + \gamma_{20} = 1 - \left(\frac{a}{b}\right)^2 \geq 0 \Rightarrow -1 \leq \frac{a}{b} \leq 1$$

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This means that the asymptotes of the hyperbola should have slopes between -1 and 1. Let 
\( c = \left( \frac{a}{b} \right) \), then

\[
c_{00} = -a^2, \quad c_{01} = c_{10} = 0, \quad c_{02} = c_{20} = \frac{1}{4}(1 + c), \quad \text{and} \quad c_{11} = \frac{i}{2}(1 - c) \quad \text{and (9) becomes}
\]

\[
-a^2 H + \frac{1}{4}(1 + c)[A^2 H + H(A^\ast)^2] + \frac{1}{2}(1 - c)AHA^\ast = -M
\] (14)

**Pole clustering inside a parabola**

Consider the region inside the parabola shown in figure 1e and described by

\[ S = \{(x, y) : y^2 + k(x + \alpha) < 0\} \]

The condition \( \gamma_{02} + \gamma_{20} = 1 + 0 = 1 \) is satisfied. Thus

\[
c_{00} = \alpha k, \quad c_{01} = c_{10} = \frac{1}{2} k, \quad c_{02} = c_{20} = -\frac{1}{4}, \quad \text{and} \quad c_{11} = \frac{1}{4} \quad \text{and (9) becomes}
\]

\[
\alpha k H + \frac{1}{2} k(AH + H A^\ast) - \frac{1}{4}[A^2 H + H(A^\ast)^2] + \frac{1}{2} AHA^\ast = -M
\] (15)

**Pole clustering inside a horizontal strip**

Consider the region inside the horizontal strip shown in figure 1f, and described by

\[ S = \{(x, y) : -w^2 + y^2 < 0\} \]

The condition \( \lambda_{02} + \lambda_{20} = 1 + 0 = 1 \) is satisfied. Thus

\[
c_{00} = -w^2, \quad c_{01} = c_{10} = 0, \quad c_{02} = c_{20} = -\frac{1}{4}, \quad \text{and} \quad c_{11} = \frac{1}{2} \quad \text{and (9) becomes}
\]

\[
-w^2 H - \frac{1}{4}[A^2 H + H(A^\ast)^2] + \frac{1}{2} AHA^\ast = -M
\] (16)

**4. GENERALIZED BALANCED MODEL REDUCTION**

Examining the balanced model reduction, we note the following. The poles of a stable continuous system are clustered in the left half plane and the system satisfies the standard continuous-time Lyapunov equation. By using the continuous-time Lyapunov equation for model reduction, the reduced order model will be stable, i.e. the poles of the reduced order model are clustered in the left half plane. Similarly the poles of a stable discrete-time system are clustered inside the unit circle and the system satisfies the standard discrete-time Lyapunov equation. If the discrete-time Lyapunov equation is used for model reduction, the reduced order model will be stable, i.e. the poles of the reduced order model will be clustered inside the unit circle. Hence, if the poles of the system are clustered in some region and if a Lyapunov type equation that is satisfied by the system can be found, then this Lyapunov equation may be used, instead of continuous-time or discrete-time Lyapunov equations, to produce a reduced order model with poles clustered in the same region. We will restrict the regions to second order regions \( S \) which are described by equation (8). Second order regions are chosen because they are simple to describe and they represent most of the regions of interest. The algorithm, for obtaining reduced order models with poles clustered in \( S \), is similar to the balanced model reduction algorithm. The difference is in replacing the controllability or observability gramian with the generalized controllability or observability gramian (10). The following algorithm gives parallel steps to the balanced model reduction algorithm given in [10].

**Step 1** Solve for the generalized controllability gramian and the standard observability gramian

\[
c_{00}P + c_{10}AP + c_{20}A^2P + c_{01}PA^\ast + c_{02}P(A^\ast)^2 + c_{11}APA^\ast = -BB^\ast
\] (17)

\[
A^\ast Q + QA = -C^\ast C
\] (18)

Or solve for the standard controllability gramian and the generalized observability gramian
\[ AP + PA^* = -BB^* \]  
\[ c_{00}Q + c_{10}A^*Q + c_{20}(A^*)^2Q + c_{01}QA + c_{02}QA^2 + c_{11}A^*PA = -C^*C \]  

Step 2 Compute the Cholesky factors of the gramians  
\[ P = L_cL_c^*; \quad Q = L_oL_o^* \]  
where \( L_c \) and \( L_o \) denotes the lower triangular Cholesky factors of \( P \) and \( Q \) respectively.

Step 3 Compute the singular value decompositions of the product of the Cholesky factors  
\[ L_c^*L_c = U \Sigma V^* \]

Step 4 The balanced transformation matrix \( T \), its inverse and the balanced system are given by  
\[ T = L_cV^{\frac{1}{2}}; \quad T^{-1} = \Sigma^{\frac{1}{2}}U^*L_o^* \]
\[ A_b = T^{-1}AT; \quad B_b = T^{-1}B; \quad C_b = CT \]

Step 5 Partition the balanced system \((A_b, B_b, C_b)\) as  
\[ A_b = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_b = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C_b = \begin{pmatrix} C_1 & C_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix} \]

and the reduced order model is \((A_{11}, B_1, C_1)\).

In this algorithm, one standard Lyapunov equation is used to guarantee the stability of the reduced model and one generalized Lyapunov equation is used to guarantee clustering of the poles of the reduced order model in \( S \). In step 1 of the algorithm, we use either the generalized controllability gramian or observability gramian. There is no known optimal way to choose either one of them. Nevertheless, they will be compared by simulation. Also in some cases such as the reduction of unstable systems, the two generalized gramians should be used. Otherwise, \( P \) or \( Q \) will not be positive definite.

The generalized balanced model reduction will produce a generically controllable and observable reduced order models because the weakly controllable and weakly observable states are neglected. The reduced order models will also be generically asymptotically stable because if we partition the observability Gramian (18) as in (25), we get 
\[ A_{11}^*\Sigma_{11} + \Sigma_{11}A_{11} = -C_1^*C_1 \]

Since the reduced order model is generically observable, it is generically asymptotically stable from (26). The stability can also be shown using the controllability gramian (19). To determine if the poles of the reduced order model are clustered in \( S \), consider the generalized controllability gramian (17). Using the partitioning (25) and after simplifications, we get 
\[ c_{00}\Sigma_1 + c_{10}A_{11}\Sigma_1 + c_{20}A_{11}^*\Sigma_1 + c_{01}\Sigma_1A_{11} + c_{02}\Sigma_1(A_{11}^*)^2 + c_{11}A_{11}\Sigma_1A_{11}^* = -B_1^*B_1 - c_{20}A_{11}A_{21}\Sigma_1 - c_{02}\Sigma_1A_{21}^*A_{12} - c_{11}A_{11}\Sigma_2A_{12}^* \]

The reduced order model will have poles clustered in \( S \), if the right hand side of (27) is negative definite. The terms \(-B_1^*B_1\) and \(-c_{11}A_{11}\Sigma_2A_{12}^*\) are negative semi-definite since \( c_{11} = \frac{1}{2}(\lambda_{20} + \lambda_{20}) \geq 0 \). The other two terms, \(-c_{20}A_{11}A_{21}\Sigma_1\) and \(-c_{02}\Sigma_1A_{21}^*A_{12}\), are not known to be definite. But if \( c_{02} = c_{20} = 0 \), then (27) is satisfied and the reduced order model is proven to have poles clustered in \( S \). The restriction \( c_{02} = c_{20} = 0 \) will produce first order regions or circles. Finally, we point out that the reduced order model is not internally balanced, this is clear from (27).

5. EXAMPLES
In the following, the generalized model reduction technique is applied to three examples. In
the first one, a ninth order transfer function model is reduced to a third order model. The poles of the system are clustered around the line \( s = -1 \). Using balanced model reduction, the poles of the reduced order model are clustered around the line \( s = -0.3 \). The generalized model reduction is used to restrict the poles to be clustered in an ellipse with center at \( s = -1 \), and major axis with length 8 along y-axis and minor axis with length 0.2 along x-axis. In the second example, a sixth order system with three-inputs and three-outputs is used. The poles of the system are clustered in a sector with slope 0.2867. After reducing the order of the system to a third order using the standard balancing technique, two poles of the reduced order model became outside the sector. Then, the generalized balanced model reduction is applied to produce a reduced order model with poles clustered inside the sector. Example three is a reduction of sixth order model of fighter described in [14]. This model has two unstable poles. Therefore, the balanced model reduction can not be applied. However, the generalized balanced model reduction can be applied to reduce it. In this example, the generalized balancing is applied four times. In each time, the reduction is performed with respect to a different region.

Example 1
The transfer function of a closed-loop system is given by [11]:

\[
G(s) = \frac{\frac{-0.2466(s - 2.2591)}{(s + 0.9846)(s + 0.9998) + 15.9928\frac{(s + 1.0018^2 + 9.032)^3 + 3.95}{(s + 1.0125)^2 + 1.0255}}}{(s + 0.9846)(s + 0.9998) + 15.9928\frac{(s + 1.0018^2 + 9.032)^3 + 3.95}{(s + 1.0125)^2 + 1.0255}}
\]

It is obvious that all the poles have almost the same real part. When the balanced model reduction technique is used, the reduced order model is

\[
G_{br}(s) = \frac{-0.2466(s - 2.2591)}{(s + 0.2948)^2 + 0.7106}
\]

and its poles are located at \(-0.2948 \pm 0.8426i\) as seen in figure 2a. Clearly, the poles of the reduced order model are far away from the poles of the full order model. In order to get a reduced order model with poles located in the same region as the full order model, we choose an ellipse that contains all the poles. This ellipse has center at \( s = -1 \), major axis has length 8 along the y-axis and the minor axis has length 0.2 along the x-axis as shown in figure 2a. The generalized balanced model reduction is applied twice. First the generalized controllability gramian and the observability gramian are used. The reduced order model is:

\[
G_{gbr1}(s) = \frac{-0.3996(s - 2.2304)}{(s + 0.9758)^2 + 0.8789}
\]

When the controllability gramian and generalized observability gramian are used, the reduced order model becomes

\[
G_{gbr2}(s) = \frac{-0.4044(s - 2.2417)}{(s + 0.9992)^2 + 0.8315}
\]

Clearly, from figure 2a both reduced order models have their poles clustered inside the ellipse. Figure 2b shows a plot of the maximum singular value of the transfer function of the reduction error for each method versus frequency. The standard balanced model reduction has a smaller error than the generalized one. However, this is expected because we are restricting the left half plane to an ellipse. Hence, the price that we pay for pole restriction is this increase in the error frequency response. In [16], a similar statement was made. That is, when the region that encloses the poles is increased, the error response is decreased and vice versa. Therefore, as rule of thumb, it is better to make the region that encloses the poles as big as possible in order to make the error decreases. Figure 2c shows the step responses of the full order, the balanced and generalized balanced reduced order models. The step response of the balanced reduced model is closer to the step response of the full order model, but there is a big difference in the transient due to the oscillations in the balanced reduced order model. While there is a big difference in the steady state of the generalized balanced reduced model and the full order model, they have similar shapes and transients.

Example 2
Consider the following sixth order system with three-inputs and three-outputs given in [11]:

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\[
A = \begin{bmatrix}
-0.3110 & 1.0280 & 0 & 0 & 2.0000 & -6.3000 \\
-0.1100 & -1.2560 & 0 & -6.3000 & -1.0370 & 4.1000 \\
0 & 0 & -1.3000 & 0 & 0.0048 & 0 \\
0 & 0.0025 & 0 & -0.1760 & -0.0030 & 0 \\
0 & -0.0380 & -6.4000 & -3.7500 & -0.9750 & 0 \\
0.0250 & -0.0400 & 0 & 0 & -0.0091 & -1.8400
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0.0300 & 0 & 0 & 0.0520 \\
0.0039 & 0 & 0 & 0.0025 \\
0 & 0.0244 & 0 & 0.0370
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
0 & 0 & 1.0000 & 0 & 2.2400 & 0 \\
0 & 2.2400 & 0 & 1.0000 & 0 & 0 \\
2.2400 & 0 & 0 & 0 & 1.0000
\end{bmatrix}
\]

The poles of the system are: \(-0.1621, -1.4942, 0.4188i, -0.4761, -1.1985, -1.0328\).

They are clustered inside a sector with slopes \(\pm 0.2789\) as shown in figure 3a. The poles of the balanced-truncation reduced order model are \(-0.0921, -1.3578, 0.6375i\). They are not inside the sector, as can be seen in figure 3a. The generalized balanced model reduction will be used to produce a reduced order model with poles clustered inside this sector. But the generalized Lyapunov equation for a sector (11) gives a solution with complex coefficients. The balancing transformation \(T\) will also be complex. Hence, the transfer function of the reduced order model will be complex. This is not acceptable since there is no real system with complex coefficients. Thus another approach will be used. Consider the hyperbola (14) with the parameter \(a\) tending to zero and \(c = (0.2789)^2\). This is the degenerate case of a hyperbola. It is a two intersecting straight lines of slopes \(\pm 0.2789\) and passing through the origin. Now, we can use the generalized Lyapunov equation of the hyperbola to make the poles of the reduced order model cluster inside the sector. Since, there are two sectors one in the left half plane and the other in the right half plane, there is a possibility of getting poles of the reduced order model in the right half plane. However, this will not happen, because the other Lyapunov equation (18) or (19) will ensure that the poles of the reduced order model are clustered in the left half plane. Applying this method, the reduced order model using equations (17) and (18) is

\[
A_{gbl1} = \begin{bmatrix}
-0.7314 & -0.8794 & -0.6702 \\
-0.4729 & -0.8979 & -0.9020 \\
0.0893 & 0.0878 & -1.3588
\end{bmatrix}, \quad B_{gbl1} = \begin{bmatrix}
0.0370 & -0.0376 & -0.0378 \\
-0.0029 & 0.0673 & -0.0351 \\
-0.0344 & -0.0623 & 0.1048
\end{bmatrix}, \quad C_{gbl1} = \begin{bmatrix}
0.0566 & -0.1977 & 0.1064 \\
-1.2744 & -1.8762 & -0.6177 \\
2.7432 & 1.7545 & 1.2146
\end{bmatrix}
\]

and the reduced order model using equations (19) and (20) is

\[
A_{gbl2} = \begin{bmatrix}
-0.2688 & 0.3011 & -0.0969 \\
0.3999 & -1.3684 & 0.2084 \\
0.0541 & -0.6671 & -1.3591
\end{bmatrix}, \quad B_{gbl2} = \begin{bmatrix}
-0.1281 & 1.2766 & 1.1042 \\
0.6795 & 0.3179 & -3.4828 \\
2.1417 & 0.4086 & -0.3267
\end{bmatrix}, \quad C_{gbl2} = \begin{bmatrix}
-0.0136 & -0.0010 & 0.0197 \\
0.0171 & -0.0601 & 0.0316 \\
-0.0672 & 0.0085 & 0.0533
\end{bmatrix}
\]

Figure 3a shows that the poles of the two generalized balanced reduced order models are clustered inside the sector. Figure 3b shows the error frequency response for the two generalized balanced methods and balancing.

**Example 3**

An unstable sixth order model of a fighter is used in this example. A full description of this model is given in [14]. Its poles are \(-5.6757, 0.6898 \pm 0.2488i, -0.2578, -30.0, -30.0\). This model has two unstable poles. Thus the controllability and observability gramians do not exist and the solutions of the standard Lyapunov equations are indefinite. Thus, they can not be used in balanced model reduction. To produce a positive definite \(P\) and \(Q\), we should use the generalized controllability and observability gramians (17 and 20). They should be used with region \(S\) that includes all the poles of the model. In this example, four regions are chosen to examine the effect of \(S\) on model reduction. The first region denoted by \(S_1\) is the region inside the horizontal strip intersecting the imaginary axis at \(s=0.5\) shown in figure 4a. The second region \(S_2\) is the region inside the ellipse whose center is at \(s=-15\) and a major axis of length 40 along x-axis and a minor axis of length 1 along y-axis shown in figure 4b. The third region \(S_3\) is the region inside the parabola \(y^2 = -0.018(x-5)\) shown in figure 4c and the fourth region \(S_4\) is the region inside
the parabola $y^2 = \frac{(0.3)^2}{35} (x + 35)$ as shown in figure 4d. The error frequency response for each method is shown in figure 4e. It is clear from the graph, that the error frequency response depends highly on the chosen region. Region $S_3$ gives the minimum error over all the other three regions for this example. The poles of the reduced order models for every region are clustered inside the same region as shown in figures 4a, b, c and d.

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REFERENCES

Figure 1: Different examples of the region S
Figure 2a: Poles of $G(s)$, $G_{br}(s)$, $G_{brf}(s)$ and $G_{br2}(s)$
(x) poles of $G(s)$, (*) poles of $G_{br}(s)$
(o) poles of $G_{brf}(s)$, (+) poles of $G_{br2}(s)$

Figure 2b: Error Frequency Response

Figure 2c: Step response of $G(s)$, $G_{br}(s)$, $G_{brf}(s)$ and $G_{br2}(s)$

Figure 3a: Poles of the full and reduced order models
(x) poles of $A(s)$, (*) poles of $A_{br}(s)$
(o) poles of $A_{brf}(s)$, (+) poles of $A_{br2}(s)$

Figure 3b: Error Frequency Response
Figure 4a: Poles of the full order model (x) and the reduced order model (o) clustered inside a horizontal strip

Figure 4b: Poles of the full order model (x) and the reduced order model (o) clustered inside an ellipse

Figure 4c: Poles of the full order model (x) and the reduced order model (o) clustered inside a parabola

Figure 4d: Poles of the full order model (x) and the reduced order model (o) clustered inside a parabola

Figure 4e: Error frequency response