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On Common Fixed Point and Approximation Results of Gregus Type

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Abstract

Fixed point theorems of Ciric [3], Fisher and Sessa [4], Gregus [5], Jungck [10] and Mukherjee and Verma [17] are generalized to a locally convex space. As applications, common fixed point and invariant approximation results for subcompatible maps are obtained. Our results unify and generalize various known results to a more general class of noncommuting mappings.

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1. Introduction and preliminaries

In the sequel, \((E, \tau)\) will be a Hausdorff locally convex topological vector space. A family \(\{p_\alpha : \alpha \in I\}\) of seminorms defined on \(E\) is said to be an associated family of seminorms for \(\tau\) if the family \(\{\gamma U : \gamma > 0\}\), where \(U = \cap_{\alpha \in I} U_\alpha\) and \(U_\alpha = \{x : p_\alpha(x) < 1\}\), froms a base of neighborhoods of zero for \(\tau\). A family \(\{p_\alpha : \alpha \in I\}\) of seminorms defined on \(E\) is called an augmented associated family for \(\tau\) if \(\{p_\alpha : \alpha \in I\}\) is an associated family with property that the seminorm \(\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in I\}\) for any \(\alpha, \beta \in I\). The associated and augmented associated families of seminorms will be denoted by \(A(\tau)\) and \(A^*(\tau)\), respectively. It is well known that given a locally convex space \((E, \tau)\), there always exists a family \(\{p_\alpha : \alpha \in I\}\) of seminorms defined on \(E\) such that \(\{p_\alpha : \alpha \in I\} = A^*(\tau)\) (see[16, page 203]).
The following construction will be crucial. Suppose that $M$ is a $\tau$-bounded subset of $E$. For this set $M$ we can select a number $\lambda_0 > 0$ for each $\alpha \in I$ such that $M \subset \lambda_0 U_\alpha$, where $U_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly $B = \bigcap_\alpha \lambda_0 U_\alpha$ is $\tau$-bounded, $\tau$-closed absolutely convex and contains $M$. The linear span $E_B$ of $B$ in $E$ is $\bigcup_{n=1}^\infty nB$. The Minkowski functional of $B$ is a norm $\| \cdot \|_B$ on $E_B$. Thus $(E_B, \| \cdot \|_B)$ is a normed space with $B$ as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_0) = \|x\|_B$ for each $x \in E_B$ (for details see [16,25]).

Let $M$ be a subset of a locally convex space $(E, \tau)$. Let $I : M \to M$ be a mapping. A mapping $T : M \to M$ is called $I$-Lipschitz if there exists $k \geq 0$ such that $p_\alpha(Tx - Ty) \leq kp_\alpha(Ix - Iy)$ for any $x, y \in M$ and for all $p_\alpha \in A^*(\tau)$. If $k < 1$ (respectively, $k = 1$), then $T$ is called an $I$-contraction (respectively, $I$-nonexpansive). A point $x \in M$ is a common fixed point of $I$ and $T$ if $x = Ix = Tx$. The set of fixed points of $T$ is denoted by $F(T)$. The pair $\{I, T\}$ is called (1) commuting if $TIx = ITx$ for all $x \in M$, (2) $R$-weakly commuting if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rp_\alpha(Ix - Tx)$. If $R = 1$, then the maps are called weakly commuting [20]; (3) compatible [10,11,22] if for all $p_\alpha \in A^*(\tau)$, $\lim_n p_\alpha(TIx_n - ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some $t \in M$. Suppose that $M$ is $q$-starshaped with $q \in F(I)$ and is both $T$- and $I$-invariant. Then $T$ and $I$ are called (4) $R$-subcommuting on $M$ (see [21]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq \frac{B}{k} p_\alpha((1 - k)q + kTx) - Ix$ for each $k \in (0,1)$. If $R = 1$, then the maps are called 1-subcommuting [7]; (5) $R$-subweakly commuting on $M$ (see [8,9]) if for all $x \in M$ and for all $p_\alpha \in A^*(\tau)$, there exists a real number $R > 0$ such that $p_\alpha(ITx - TIx) \leq Rd_\alpha(Ix, [q, Tx])$, where $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$. It is well known that $R$-weakly commuting, $R$-subcommuting and $R$-subweakly commuting maps are compatible but not conversely in general (see [10-12]).

If $u \in E, M \subset E$, then we define the set $P_M(u)$ of best $M$-approximants to $u$ as $P_M(u) = \{y \in M : p_\alpha(y - u) = d_\alpha(u, M), \text{ for all } p_\alpha \in A^*(\tau)\}$, where $d_\alpha(u, M) = \inf(p_\alpha(x - u) : x \in M)$. A mapping $T : M \to E$ is called demiclosed at 0 if whenever $\{x_n\}$ converges weakly to $x$ and $\{Tx_n\}$ converges to 0, we have $Tx = 0$.

In [4], Fisher and Sessa obtained the following generalization of a theorem of Gregus [5].

**Theorem 1.1.** Let $T$ and $I$ be two weakly commuting mappings on a closed convex subset $C$ of a Banach space $X$ into itself satisfying the inequality,
\[\|Tx - Ty\| \leq a\|Ix - Iy\| + (1 - a) \max\{\|Tx - Ix\|, \|Ty - Iy\|\}, \quad (1.1)\]
for all \(x, y \in C\), where \(a \in (0, 1)\). If \(I\) is linear and nonexpansive on \(C\) and \(T(C) \subseteq I(C)\), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In 1988, Mukherjee and Verma [17] replaced linearity of \(I\) by affineness in Theorem 1.1. Subsequently, Jungck [12] obtained the following generalization of Theorem 1.1 and the result of Mukherjee and Verma [17].

**Theorem 1.2.** Let \(T\) and \(I\) be compatible self maps of a closed convex subset \(C\) of a Banach space \(X\). Suppose that \(I\) is continuous, linear and that \(T(C) \subseteq I(C)\). If \(T\) and \(I\) satisfy inequality (1.1), then \(T\) and \(I\) have a unique common fixed point in \(C\).

In this paper, we first prove that Theorems 1.1-1.2 can appreciably be extended to the setup of Hausdorff locally convex space. As applications, common fixed point and invariant approximation results for a new class of subcompatible maps are derived. Our results extend and unify the work of Al-Thagafi [1], Cirić [3], Fisher and Sessa [4], Gregus [5], Habisia [6], Hussain and Khan [7], Hussain et al. [8], Jungck [10], Jungck and Sessa [13], Khan and Hussain [14], Khan at el. [15], Mukherjee and Verma [17], Sahab, Khan and Sessa [18], Singh [23, 24] and many others.

2. Main Results

We begin with the definition of subcompatible mappings.

**Definition 2.1.** Let \(M\) be a \(q\)-starshaped subset of a normed space \(E\). For the selfmaps \(I\) and \(T\) of \(M\) with \(q \in F(I)\), we define \(S_q(I,T) := \cup\{S(I,T_k) : 0 \leq k \leq 1\}\) where \(T_k x = (1 - k)q + kTx\) and \(S(I,T_k) = \{\{x_n\} \subseteq M : \lim_n Ix_n = \lim_n T_k x_n = t \in M \Rightarrow \lim_n \|IT_k x_n - T_k I x_n\| = 0\}\). Now \(I\) and \(T\) are subcompatible if \(\lim_n \|IT x_n - TI x_n\| = 0\) for all sequences \(\{x_n\} \subseteq S_q(I,T)\).

We can extend this definition to locally convex space by replacing norm with a family of seminorms.

Clearly, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

**Example 2.2.** Let \(X = \mathbb{R}\) with usual norm and \(M = [1, \infty)\). Let \(I(x) = 2x - 1\) and \(T(x) = x^2\), for all \(x \in M\). Let \(q = 1\). Then \(M\) is \(q\)-starshaped with \(I q = q\). Note that \(I\) and \(T\) are compatible. For any sequence \(\{x_n\}\) in \(M\) with \(\lim_n x_n = 2\), we have, \(\lim_n Ix_n = \lim_n T_{1/2} x_n = 3 \in M\), \(\Rightarrow \lim_n \|T_{1/2} x_n - T_{1/2} I x_n\| = 0\). However, \(\lim_n \|IT x_n - TI x_n\| \neq 0\). Thus \(I\) and \(T\) are not subcompatible.
$q \in F(I)$ and $T(M) \subseteq I(M)$. If the pair $\{I, T\}$ is subcompatible and satisfies, for all $p_{\alpha} \in A^*(\tau)$, $x, y \in M$, and all $k \in (0, 1)$,

$$p_{\alpha}(Tx - Ty) \leq p_{\alpha}(Ix - Iy) + \frac{1 - k}{k} \max\{d_{p_{\alpha}}(Ix, [q, Tx]), d_{p_{\alpha}}(Iy, [q, Ty])\}, \quad (2.2)$$

then $I$ and $T$ have a common fixed point in $M$ provided one of the following conditions holds:

1. $M$ is $\tau$-compact and $T$ is continuous.
2. $M$ is weakly compact in $(E, \tau)$, $I$ is weakly continuous and $I - T$ is demiclosed at 0.

**Proof.** Define $T_n : M \to M$ by

$$T_n x = (1 - k_n)q + k_nTx$$

for some $q$ and all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then, for each $n$, $T_n(M) \subseteq I(M)$ as $M$ is convex, $I$ is linear, $Iq = q$ and $T(M) \subseteq I(M)$. Furthermore, since the pair $\{I, T\}$ is subcompatible and $I$ is linear with $Iq = q$ so, for any $\{x_m\} \subset M$ with $\lim_mIx_m = \lim_m T_n x_m = t \in M$, we have

$$\lim_m p_{\alpha}(T_nIx_m - IT_nx_m) = k_n \lim_m p_{\alpha}(TIx_m - ITx_m) = 0.$$ 

Thus the pair $\{I, T_n\}$ is compatible on $M$ for each $n$. Also, we obtain from (2.2),

$$p_{\alpha}(T_nx - T_ny) = k_n p_{\alpha}(Tx - Ty) \leq k_n \{p_{\alpha}(Ix - Iy) + \frac{1 - k_n}{k_n} \max\{p_{\alpha}(Ix - T_nx), p_{\alpha}(Iy - T_ny)\}\} = k_n p_{\alpha}(Ix - Iy) + (1 - k_n) \max\{p_{\alpha}(Ix - T_nx), p_{\alpha}(Iy - T_ny)\},$$

for each $x, y \in M$, $p_{\alpha} \in A^*(\tau)$ and $0 < k_n < 1$.

1. $M$ being $\tau$-compact is $\tau$-bounded and $\tau$-complete. Thus by Theorem 2.6, for each $n \geq 1$, there exists an $x_n \in M$ such that $x_n = Ix_n = T_n x_n$. Now the $\tau$-compactness of $M$ ensures that $\{x_n\}$ has a convergent subsequence $\{x_j\}$ which converges to a point $x_0 \in M$. Since $x_j = T_j x_j = k_j T_j x_j + (1 - k_j)$ and $T$ is continuous, so we have, as $j \to \infty$, $Tx_0 = x_0$. The continuity of $I$ implies that

$$Ix_0 = I(\lim_j x_j) = \lim_j I(x_j) = \lim_j x_j = x_0.$$
(ii) Weakly compact sets in \((E,\tau)\) are \(\tau\)-bounded and \(\tau\)-complete so again by
Theorem 2.6, \(T_n\) and \(I\) have a common fixed point \(x_n\) in \(M\) for each \(n\). The
set \(M\) is weakly compact so there is a subsequence \(\{x_{j}\}\) of \(\{x_n\}\) converging
weakly to some \(y \in M\). The map \(I\) being weakly continuous gives that \(Iy = y\).
Now
\[
x_{j} = I(x_{j}) = T_{j}(x_{j}) = k_{j} Tx_{j} + (1 - k_{j})q
\]
implies that \(Ix_{j} - Tx_{j} = (1 - k_{j})[q - Tx_{j}] \to 0\) as \(j \to \infty\). The demiclosedness
of \(I - T\) at \(0\) implies that \((I - T)(y) = 0\). Hence \(Iy = Ty = y\).

An application of Theorem 2.7 establishes the following result in best
approximation theory.

**Theorem 2.8.** Let \(T\) and \(I\) be selfmaps of Hausdorff locally convex space
\((E,\tau)\) and \(M\) a subset of \(E\) such that \(T(\partial M) \subseteq M\), where \(\partial M\) denotes
boundary of \(M\) and \(u \in F(T) \cap F(I)\). Suppose that \(P_{M}(u)\) is nonempty
convex containing \(q\), \(q \in F(I), I\) is nonexpansive and linear on \(P_{M}(u)\) and
\(I(P_{M}(u)) = P_{M}(u)\). If the pair \(\{I, T\}\) is subcompatible on \(P_{M}(u)\) and satisfies,
for all \(x \in P_{M}(u) \cup \{u\}\), \(p_{a} \in A^{*}(\tau)\) and \(k \in (0, 1)\),
\[
p_{a}(Tx - Ty)
\]
\[
\leq \begin{cases} 
    p_{a}(Ix - Iu) & \text{if } y = u, \\
    p_{a}(Ix - Iy) + \frac{1 - k}{k} \max\{d_{p_{a}}(Ix, [q, Tx]), d_{p_{a}}(Iy, [q, Ty])\} & \text{if } y \in P_{M}(u),
\end{cases}
\]
then \(P_{M}(u) \cap F(I) \cap F(T) \neq \emptyset\), provided one of the following conditions is
satisfied:
(i) \(P_{M}(u)\) is \(\tau\)-compact and \(T\) is continuous on \(P_{M}(u)\).
(ii) \(P_{M}(u)\) is weakly compact in \((E, \tau)\), \(I\) is weakly continuous and \(I - T\) is
demiclosed at \(0\).

**Proof.** Let \(y \in P_{M}(u)\). Then as in the proof of Theorem 2.6 of [15](see also
[9,12]) \(Ty \in P_{M}(u)\) which implies that \(T\) maps \(P_{M}(u)\) into itself and the
conclusion follows from Theorem 2.7.

**Remark 2.9.** (i) 1-subcommuting maps are subcompatible, consequently,
Theorem 2.2-Theorem 3.3 due to Hussain and Khan [7] and Theorem 2.3 of
Khan and Hussain [14] are improved and extended.
(ii) Commuting maps are subcompatible so Theorems 2.7-2.8 are proper
generalization of the main results of Brosowski [2], Habiniak [6], Sahab et al. [18],
Sahney et al. [19], Singh [23,24], Tarafdar [25], Theorems 6-7 due to Jungck
and Sessa [13] and Theorem 2.6 due to Khan et al.[15].
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