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ON ANALYZING RANDOMIZED BLOCKS BY WEIGHTED RANKING

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ABSTRACT

The remarks introduced by Salama and Quade (1990), based on the method of weighted rankings introduced by Quade (1972, 1979) has been reviwed. Some lemmas proved so that the theorem introduced by Salama and Quade (1990) can be generalized from the case of two treatments and n blocks to the case of m treatments and n blocks. The case of three treatments and n blocks applied on an exponential case is introduced as an example.

1. INTRODUCTION

The standard non-parametric procedures for testing the hypothesis of no treatment effects in a complete blocks experiment depend entirely on the within-block rankings. If block effect are assumed additive, however, then between-block information may be recovered by weighting these rankings according their credibility with respect to treatment ordering.

Let X_{ij} be the observation of the j-th of m treatments in the i-th of n complete blocks, and consider the hypothesis of no treatments effects, specifically,

H_o: $X_{i1}, ..., X_{im}$ are interchangeable for each i.

Assume throughout:

(I) Independent blocks:

For i = 1, ..., n, the random vectors $X_i = (X_{i1}, ..., X_{im})$ (the blocks), are mutually independent.

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(IV) No Between-Block ties:

$$P\{D_i = D_{i'}\} = 0 \qquad \text{for } i \neq i'$$

This assures that there will be no ties in the ranking of the blocks. Let $0 \le b_1 \le ... \le b_n$, with $0 \neq b_n$, be a fixed set of block scores; and weight the i-th block proportionally to b_{Q_i}.

Write $(R_{i1}, ..., R_{im}) = R_i$, and consider $P(R_i = r \mid Q_i = k) = \dfrac{P(R_i = r, Q_i = k)}{P(Q_i = k)}$

where $P(Q_i = k) = \dfrac{1}{n}$.

So $P(Ri = r \mid Qi = k) = nP\left(R_i = r, D_{\cdot}^{k-1} < D_i < D_{\cdot}^{n-k}\right) = nP\left(R_i = r, D_{(k-1)} < D_i < D_{(k)}\right)$

where $D_{(j)}$ is the j-th order statistic from a sample of (n-1) values of D, (that is, all values except D_i).

Theorem 1.1

Let $g_{k-1\,k}$ be the joint density function of $D_{(k-1)}$ and $D_{(k)}$. Then

$$P(Ri = r \mid Qi = k) = n\int_0^{\infty}\int_0^{b} P\left(R_i = r, a < D_i < b\right) g_{k-1,k}(a,b)\,da\,db \qquad (1.2)$$

The proof can be found in Salama and Quade (1990).

2. GENERALIZATION

Consider the follows two lemmas:

Lemma 2.1 :

Consider m random variables $X_1, X_2, ..., X_m$, which we assume independent with density functions $f_i(X)$, $0 < x < \infty$, $i = 1, ..., m$. Let D_k be as defined in (1.1) and let $G(t) = P(D \le t)$. For a sample of size n, let $g_{(k)}(t)$ be the density function of $D_{(k)}$, $k = 1, ..., n$. Let $f(t)$ be *monotone increasing (decreasing)* function. Then:

$$I_k = \int_0^{\infty} f(t) g_{(k)}(t)\,dt \qquad (2.1)$$

$$I_{k+1} = \int_0^1 F(y)L_{k+1}(y)dy = \int_0^{y^*} F(y)L_{k+1}(y)dy + \int_{y^*}^1 F(y)L_{k+1}(y)dy$$

Since *F(y)* is *monotone increasing*, then

For $0 < y < y^*$; we have $F(y) < F(y^*)$, or $- F(y) > - F(y^*)$.

For $y^* < y < 1$, we have $F(y) > F(y^*)$. Then

$$\begin{aligned}
I_{k+1} - I_k &= \int_0^{y^*} F(y)[L_{k+1}(y) - L_k(y)]dy + \int_{y^*}^1 F(y)[L_{k+1}(y) - L_k(y)]dy \\
&= \int_{y^*}^1 F(y)[L_{k+1}(y) - L_k(y)]dy - \int_0^{y^*} F(y)[L_k(y) - L_{k+1}(y)]dy \\
&> F(y^*)\left[\int_{y^*}^1 [L_{k+1}(y) - L_k(y)]dy - \int_0^{y^*} [L_k(y) - L_{k+1}(y)]dy\right] \\
&= F(y^*)\int_0^1 [L_{k+1}(y) - L_k(y)]dy \\
&= 0
\end{aligned}$$

Hence I_k is *monotone increasing* in *k*. The case is similar when *f(t)* is *monotone decreasing*.

Lemma 2.2:

Consider *m* random variables $X_1, X_2, \ldots, X_m$, which we assume independent with density function $f_i(X)$, $0 < x < \infty$, $i = 1, \ldots, m$. Let D_k be as defined in (1.1) and let $G(t) = P(D \le t)$. For a sample of size *n*, let $g_{(k)}(t)$ be the density function of $D_{(k)}$, $k = 2, \ldots, n+1$. Let *f(t)* be *monotone increasing (decreasing)* function. Then:

$$P_k = \int_0^{\infty} f(t)[g_{(k)}(t) - g_{(k-1)}(t)]dt \tag{2.4}$$

is also *monotone increasing (decreasing)* in *k*; that is

$$P_{k+1} - P_k > 0 \;\forall k \qquad \text{or} \qquad P_{k+1} - P_k < 0 \;\forall k.$$

then: $L(y) = 0$ gives the roots 0, y_1^*, y_2^* and 1, such that $0 < y_1^* < y_2^* < 1$ and $\int_0^1 L(y)dy = 0$;

$$P_{k+1} - P_k = \int_0^{y_1^*} F(y)L(y)dy + \int_{y_1^*}^{y_2^*} F(y)L(y)dy + \int_{y_2^*}^{1} F(y)L(y)dy$$

$$\geq F(0)\int_0^{y_1^*} L(y)dy + F(y_2^*)\int_{y_2^*}^{y_1^*} L(y)dy + F(y_2^*)\int_{y_2^*}^{1} L(y)dy$$

$$\geq F(0)\int_0^{1} L(y)dy$$

$$\geq 0$$

Since $F(y)$ is *monotone increasing*, then :

For $0 < y < y_2^*$, we have $F(y) < F(y_2^*)$ or $-F(y) > -F(y_2^*)$

For $y_2^* < y < 1$; we have $F(y) > F(y_2^*)$

Hence P_k is *monotone increasing* in k. The case is similar when $f(t)$ is *monotone increasing*.

3. APPLICATION ON EXPONENTIAL DISTRIBUTION

Here, we will give an example on the exponential distribution for the case of three treatments.

Lemma 3.1:

Let $g_m(t)$ be the m-th order statistics corresponding to the p.d.f. $g(t)$. Then

$$\int_0^\infty e^{-at} g_m(t)dt = E\left(X_{(n-m)}^{a}\right)$$

Theorem 3.1

Let (x_{11}, x_{21}, x_{31}), ..., $(x_{1\ell}, x_{2\ell}, x_{3\ell})$, ..., (x_{1n}, x_{2n}, x_{3n}) be the observations corresponding to a design with three treatments and n blocks. Assume that X_1, X_2 and X_3 are independent, with density functions

$$-\frac{n\lambda i}{\lambda}\int_{0}^{\infty}\left(e^{-\lambda_j t_2}+e^{-\lambda_k t_2}-e-\left(\lambda_j+\lambda_k\right)t_2\right)g_{(m)}(t_2)dt_2 \tag{3.1}$$

$$=\frac{n\lambda_i}{\lambda}\int_{0}^{\infty}\left(e^{-\lambda_j t}+e^{-\lambda_k t}-e^{-(\lambda_j+\lambda_k)t}\right)\left(g_{(m-1)}(t)-g_{(m)}(t)\right)dt$$

Then

$$P_{m,i,1}=\frac{n\lambda_i}{\lambda}\left[\int_{0}^{\infty}\left(e^{-\lambda_j t}\left(g_{(m)}(t)-g_{(m)}(t)\right)\right)dt+\int_{0}^{\infty}\left(e^{-\lambda_k t}\right)\left(g_{(m-1)}(t)-g_{(m)}(t)\right)dt\right.$$

$$-\int_{0}^{\infty}\left(e^{-(\lambda_j+\lambda_k)t}g_{(m-1)}(t)-g_{(m)}(t)\right)dt \tag{3.2}$$

From lemma (3.1) we have

$$\int_{0}^{\infty}e^{-\lambda_j t}g_{(m)}(t)dt=E\left(X^{\lambda_j}(n-m)\right),$$

$$\int_{0}^{\infty}e^{-\lambda_j t}g_{(m-1)}(t)dt=E\left(X^{\lambda_i}(n-m+1)\right),$$

$$\int_{0}^{\infty}e^{-\lambda_j t}g_{(m-1)}(t)dt=E\left(X^{\lambda_i}(n-m+1)\right),$$

Then.

$$P_{m,i,1}=\frac{n\lambda_i}{\lambda}\left[\left(E\left(X^{\lambda_j}{}_{(n-m+1)}\right)-E\left(X^{\lambda_j}{}_{(n-m)}\right)\right)+\left(E\left(X^{\lambda_k}{}_{(n-m+1)}\right)-E\left(X^{\lambda_k}{}_{(n-m)}\right)\right)\right.$$

$$\left.-\left(E\left(X^{\lambda_j+\lambda_k}{}_{(n-m+1)}\right)-E\left(X^{\lambda_j+\lambda_k}{}_{(n-m)}\right)\right)\right]=d_{(n-m)} \tag{3.3}$$

From lemma (2.2)

and for $\lambda_i > \lambda_j > \lambda_k$. $P_{m,i,1}$ is monotone increasing in "m"

Hence $P_{1,i,1} \leq P_{2,i,1} \leq \ldots \leq P_{n,i,1}$.

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$$f_{x_1}(x_1)=\lambda_1 e^{-\lambda_1 x_1}, f_{x_2}(x_2)=\lambda_2 e^{-\lambda_2 x_2} \text{ and } f_{x_3}(x_3)=\lambda_3 e^{-\lambda_3 x_3}$$

respectively $0 < x_1, x_2, x_3 < \infty$.

Let $P_{m,i,\gamma} = P\{R(x_i)=\gamma \mid R(D_i)=m\}$, where $D_i = \max x_i - \min x_i$. If $\lambda_1 > \lambda_2 > \lambda_3$, then $\{P_{m,i,\gamma}\}$ is monotone increasing in "m"; that is $P_{1,i,\gamma} \le P_{2,i,\gamma} \le \ldots \le P_{n,i,\gamma}$. for $\gamma = 1, 2, 3$ and $i = 1, 2, 3$

We will give the proof for $\gamma = 1$. A similar way can be followed for $\gamma = 2$ and $\gamma = 3$

Proof:

we have

$$P_{i,1} = P\{R(x_i)=1, 0 \le D < t\} = \frac{\lambda_i}{\lambda}\left\{\left(2-e^{-\lambda_j t}-e^{-\lambda_k t}\right)+\left(e^{-(\lambda_j+\lambda_i)t}-1\right)\right)$$

So we can defined $P'_{i,1}$ and $P''_{i,1}$ to be

$$P'_{i,1} = P\{R(x_i)=1, t_1 \le D < t_2\} =$$

$$\frac{\lambda_i}{\lambda}\left\{\left(e^{-\lambda_j t_1}+e^{-\lambda_k t_1}-e^{-\lambda_k t_2}\right)+\left(e^{-(\lambda_j+\lambda_k)t_2}-e^{-(\lambda_j+\lambda_k)t_1}\right)\right\}$$

$$P''i,1 = P\{R(xi)=1, t \le D < \infty\} = \frac{\lambda_i}{\lambda}\left\{\left(e^{-\lambda_j t}+e^{-\lambda_k t}\right)-e^{-(\lambda_j+\lambda_k)t_1)}\right\}$$

for m = 2, ..., n, we have

$$P_{m,i,1} = P\{R(x_i)=1 \mid R(D_i)=m\}$$

$$= n\int_0^{\infty}\int_0^{t_2} P\{R(x_i)=1, t_1 \le D_i < t_2\}\, g_{(m-1,m)}(t_1,t_2)\,dt_1 dt_2$$

Where, $g_{(m-1,m)}(t_1, t_2)$ is the joint density function of $D_{(n-1)}$ and $D_{(n)}$

$$P_m,i,1 = \frac{n\lambda_i}{\lambda}\int_0^{\infty}\left(e^{-\lambda_j t_1}+e^{-\lambda_k t_1}-e^{-(\lambda_j+\lambda_k)t_1}\right)g_{(m-1)}(t_1)\,dt_1$$

Proof:

$$P_k = \int_0^t f(t)\left[g_{(k)}(t) - g_{(k-1)}(t)\right]dt$$

$$= \int_0^\infty f(t)\left[\frac{n!}{(k-1)!(n-k)!}\left[G(t)\right]^{k-1} g(t)\left[1-G(t)\right]^{n-k}\right.$$

$$\left. - \frac{n!}{(k-2)!(n-k+1)!}\left[G(t)\right]^{k-2} g(t)\left[1-G(t)\right]^{n-k+1}\right]dt$$

Let $f(t)$ be a *monotone increasing* function. Let $y = G(t)$ then $dy = g(t)\ dt$, $[0,\infty) \Rightarrow [0,1)$ and $t = G^{-1}(y)$. Note that both y and $G^{-1}(y)$ are *monotone increasing* functions. Now;

$$P_k = \int_0^1 f\left(G^{-1}(y)\right)\left[\frac{n!}{(k-1)!(n-k)!}\, y^{k-1}\left[1-y\right]^{n-k}\right.$$

$$\left. - \frac{n!}{(k-2)!(n-k+1)!}\, y^{k-2}\left[1-y\right]^{n-k+1}\right]dt$$

$$= \int_0^1 F(y)\left[L_k(y) - L_{k-1}(y)\right]dy$$

Where $F(y) = f(G^{-1}(y))$ is also *a monotone increasing* function and $L_k(y)$ is defined as in (2.2). Now

$$P_{k+1} = \int_0^1 F(y)\left[L_{k+1}(y) - L_k(y)\right]dy$$

and

$$P_k = \int_0^1 F(y)\left[L_k(y) - L_{k-1}(y)\right]dy$$

Therefore:

$$P_{k+1} - P_k = \int_0^1 F(y)\left[L_{k+1}(y) - 2L_k(y) - L_{k-1}(y)\right]dy$$

Note that if

$L(y) = L_{k+1}(y) - 2L_k(y) - L_{k-1}(y).$

is also *monotone increasing (decreasing)* in k; that is

$$I_{k+1} - I_k > 0 \,\forall k \qquad \text{or} \qquad I_{k+1} - I_k < 0 \,\forall k \,.$$

Proof:

$$I_k = \int_0^{t} f(t) g_{(k)}(t)dt = \int_0^{\infty} f(t)\frac{n!}{(k-1)!1!(n-k)!}[G(t)]^{k-1} g(t)[1-G(t)]^{n-k} dt$$

Let $f(t)$ be a *monotone increasing* function. Let $y = G(t)$ then $dy = g(t)\ dt$, $[0,\infty) \Rightarrow [0,1)$ and $t = G^{-1}(y)$. Note that both y and $G^{-1}(y)$ are *monotone increasing* functions. Now;

$$I_k = \int_0^1 f\left(G^{-1}(y)\right)\frac{n!}{(k-1)!1!(n-k)!} y^{k-1} g(t)[1-y]^{n-k} dy = \int_0^1 F(y)L_k(y)dy$$

Where $F(y) = f(G^{-1}(y))$ is also a *monotone increasing* function and

$$L_k(y) = \frac{n!}{(k-1)!(n-k)!}\, y^{k-1}[1-y]^{n-k} \,. \tag{2.2}$$

Note that

$$\int_0^1 L_k(y)dy = \frac{n!}{(k-1)!(n-k)!}\int_0^1 y^{k-1}[1-y]^{n-k} dy$$
$$= \frac{n!}{(k-1)!(n-k)!}\,\beta(k, n-k+1) = 1 \tag{2.3}$$

Also; $L_k(0) = L_k(1) = 0$, L_k is unimodel and $\exists$ a y^* such thta $0 < y^* < 1$ with $L_k(y^*) = L_{k+1}(y^*)$.

Now

$$I_k = \int_0^1 F(y)L_k(y)dy = \int_0^{y^*} F(y)L_k(y)dy + \int_{y^*}^1 F(y)L_k(y)dy$$

and

(II) No within-blocks ties:

$$P(X_{ij} = X_{ij'}) = 0 \qquad for\ j \neq j'$$

The alternative under consideration can be fairly general, however, there may be additive treatment effects, as follows:

Unordered case:

$H_1(u)$: There exist quantities $\tau_1, \ldots, \tau_m$ (treatment effects) not all equal to zero, such that for i = 1, ..., n, $X_{i1} - \tau_1, \ldots, X_{im} - \tau_m$ are interchangeable.

Ordered case:

$H_1(0)$: The quantities $\tau_1, \ldots, \tau_m$ (treatment effects) satisfy $\tau_1 \leq \tau_2 \leq \ldots \leq \tau_m$ with $\tau_1 \neq \tau_m$.

(III) Additive block effects:

There exist quantities $\beta_1, \ldots, \beta_n$ (block effects) such that the random vectors $(X_{i1} - \beta_i, \ldots, X_{im} - \beta_i)$ are identically distributed.

By assumption III, comparisons of observations are possible between blocks as well as within, so procedures which use only within-block comparison waste information. A method of weighted within-block rankings, which makes use of assumption III, has been introduced by Quade (1972, 1979). The idea behind this method is that blocks in which the observations are more distinct are more likely to reflect any underlying true ordering of the treatment effects.

To determine the weight for the i-th block, for j = 1, ..., m let

$$D_i = \max_j \{X_{ij}\} - \min_j \{X_{ij}\} \tag{1.1}$$

that is D_i is the range of the block i. Let $Q_i = R(D_i)$, that is Q_i be the rank of D_i among $D_1, \ldots, D_n$.
Assume: