A SHORT CONSTRUCTION OF MSM RINGS

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Abstract

Let us term the ring of a strict Morita context as an msm (Morita similar matrix) ring. We can always get an msm ring from an arbitrary Morita context. The aim here is to develop a very short and a direct technique to get such rings.

Let $K(A, B) = [A, B, M, N, \langle \cdot, \cdot \rangle_A, \langle \cdot, \cdot \rangle_B]$ be a Morita context (in short mc) in which $A$ and $B$ are associative rings with the multiplicative identities $1_A$ and $1_B$, respectively; $M$ and $N$ are $(B, A)$- and $(A, B)$-bimodules, respectively; and $\langle \cdot, \cdot \rangle_A : N \otimes_B M \rightarrow A$ and $\langle \cdot, \cdot \rangle_B : M \otimes_A N \rightarrow B$ are bimodule morphisms satisfying the associativity conditions:

(i) $m \langle n, m' \rangle_A = \langle m', n \rangle_B m,$
(ii) $\langle n, m \rangle_A n' = n \langle m, n' \rangle_B,$

where $m, m' \in M$ and $n, n' \in N$. If the mc maps $\langle \cdot, \cdot \rangle_A$ and $\langle \cdot, \cdot \rangle_B$ are onto, then they become isomorphisms and the mc is termed as a strict Morita context [1] or a pmc (projective Morita context) [2] and the matrix ring

\[ R = \begin{bmatrix} A & N \\ M & B \end{bmatrix} \]

2000 Mathematics Subject Classification: 16D90.
Keywords and phrases: Morita similar matrix rings.
Received February 19, 2008
is termed as a Morita similar matrix ring (in short an \textit{msm} ring). We present here a very short and a direct technique to construct such rings from any arbitrarily given mc. This is an improved method of the iteration as done in [2, Theorem 2.1].

Let us write \( R = R_2 \) and set

\[
R_3 = \begin{bmatrix}
A & N & A \\
M & B & M \\
A & N & A
\end{bmatrix},
\]

which is a ring under element wise addition and in multiplication we only consider that \( NM = \langle N, M \rangle_A \equiv A \) and \( MN = \langle M, N \rangle_B \equiv B \).

Let us have the datum

\[
K(R_2, R_3) = [R_2, M_{32}, N_{23}, R_3, \langle \cdot, \cdot \rangle_{R_2}, \langle \cdot, \cdot \rangle_{R_3}]
\]

in which \( M_{32} = \begin{bmatrix}
A & N \\
M & B \\
A & N
\end{bmatrix} \) is an \( (R_3, R_2) \)-bimodule; \( N_{23} = \begin{bmatrix}
A' & N \\
M' & B \\
A' & N
\end{bmatrix} \)

is an \( (R_2, R_3) \)-bimodule; and the maps \( \langle \cdot, \cdot \rangle_{R_2} \) and \( \langle \cdot, \cdot \rangle_{R_3} \) defined by the formulas

\[
\begin{align*}
\langle a_{11} n_{12} a_{13}, a_{11} n_{12} a_{13} \rangle_{R_2} & \equiv (a_{11} n_{12} a_{13}, a_{11} n_{12} a_{13}), \\
\langle a_{21} b_{22} m_{23}, a_{21} b_{22} m_{23} \rangle_{R_2} & \equiv (a_{21} b_{22} m_{23}, a_{21} b_{22} m_{23})
\end{align*}
\]

and

\[
\begin{align*}
\langle a_{11} n_{12}, a_{11} n_{12} a_{13} \rangle_{R_3} & \equiv (a_{11} n_{12}, a_{11} n_{12} a_{13}), \\
\langle a_{31} b_{32} m_{32}, a_{31} b_{32} m_{32} \rangle_{R_3} & \equiv (a_{31} b_{32} m_{32}, a_{31} b_{32} m_{32})
\end{align*}
\]

are mc maps. One can easily verify the existence of both associativity conditions for above maps and so \( K(R_2, R_3) \) is an mc.

\textbf{Theorem.} Every mc gives rise to an msm ring.

\textbf{Proof.} One can always get the mc \( K(R_2, R_3) \) from any arbitrarily given mc \( K(A, B) \). We will demonstrate that \( K(R_2, R_3) \) is strict.

In order to prove that \( \langle \cdot, \cdot \rangle_{R_2} \) is epic, assume that \( \begin{bmatrix} a & n \\ m & b \end{bmatrix} \in R_2 \). Set \( t \in N_{23} \otimes_{R_3} M_{32} \) in the form

\[
t = \begin{bmatrix}
a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & n
\end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0
\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix}.
\]

Then clearly,

\[
\langle \cdot, \cdot \rangle_{R_2} (t) = \begin{bmatrix} a & n \\ m & b \end{bmatrix}.
\]

Similarly, if we set

\[
t = \begin{bmatrix}
a_{11} & n_{12} \\ m_{21} & b_{22} \\ a_{31} & n_{32}
\end{bmatrix} \otimes \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix} a_{13} & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1
\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0
\end{bmatrix} \otimes \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_{32}
\end{bmatrix} \in M_{32} \otimes_{R_2} N_{23},
\]

then

\[
\langle \cdot, \cdot \rangle_{R_3} (t) = \begin{bmatrix}
a_{11} & n_{12} & a_{13} \\ m_{21} & b_{22} & m_{32} \\ a_{31} & n_{32} & a_{33}
\end{bmatrix},
\]

which is an arbitrary element in \( R_3 \).
Hence, both $mc$ maps of $K(R_2, R_3)$ are epimorphisms so $K(R_2, R_3)$ is strict. The $msm$ ring obtained from $K(R_2, R_3)$ is

$$R_5 = \begin{bmatrix} R_2 & N_{23} \\ M_{32} & R_3 \end{bmatrix} = \begin{bmatrix} A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \\ M & B & M & B & M \\ A & N & A & N & A \end{bmatrix}.$$ 

Let $K(A, B)$ be an $mc$. From this $mc$ we can form the sequence

$$\{A = R_1, R_2, R_3, R_5, \ldots, R_{2k+1}, \ldots\}.$$ 

Define a map $\alpha_{j(j+1)} : R_j \to R_{j+1}$ by:

$$\alpha_{j(j+1)}(r_j) = \begin{bmatrix} r_j & 0 \\ 0 & 0 \end{bmatrix}.$$ 

Clearly, $\alpha_{j(j+1)}$ is a well-defined homomorphism (not identity preserving). In fact, this is a (non-unital) embedding of $R_j$ into $R_{j+1}$. The transitivity of this process yields the increasing sequence

$$R_1 \leq R_2 \leq R_3 \leq R_4 \leq \cdots \leq R_n \leq \cdots.$$ 

The upper bound of this ascending sequence is

$$\overline{R} = \bigcup_{i=1}^{\infty} R_i.$$ 

It is important to note that each matrix in $\overline{R}$ has finite size.

In above sequence it is observed that the ring $R_4$ in general is not Morita similar to $R_2$. But as we have proved in Theorem that $R_2$ is Morita similar to $R_3$. Thus, $R_3$ is an $msm$ ring. Since the relation $Morita\ similar$ is transitive, it is clear that in the ascending sequence

$$R_5 \leq R_7 \leq \cdots \leq R_{2n+1} \leq \cdots,$$

every member is an $msm$ ring. Moreover, the context ring of $K(R_2, R_3)$ is $R_4$, and $R_2$ is Morita similar to $R_2$, so $R_4$ is an $msm$ ring. Finally, the Morita ring of $K(\overline{R}, \overline{R})$ is $\overline{R}$, and as $\overline{R}$, is Morita similar to itself, so $\overline{R}$ is an $msm$ ring. Hence we conclude that

**Corollary.** Let $K(A, B)$ be any $mc$. Then there is an ascending sequence of the $msm$ rings

$$R_4 \leq R_5 \leq \cdots \leq R_n \leq \cdots$$

including the upper bound $\overline{R}$ which is also an $msm$ ring.

**References**
