Effective Moment of Inertia of Partially Cracked RC Beams

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Abstract. Effective moment of inertia of an RC beam is shown to be function of the service load-level. The beam is subjected to, the reinforcement ratio in the beam and the type of loading (uniform or point loads). The well known formulas available for evaluation of effective moment of inertia considered the effect of uniformly distributed load only. The test results show that the effective moment of inertia under point loads is significantly different from that obtained from the formulas. Refined models are presented here, which account for the effects of all the loading types and are compared with the test results and with other models available for this purpose. Applicability of the models as well as other models to beams of T-sections is also investigated.

1. Introduction

The limiting of flexural deflection is one of the major serviceability design requirements in reinforced concrete beams. The design of structural members employing high strength materials yields rather slender members which are stressed to higher levels. This situation calls for more refined models than the ones available to predict their deflections.

The use of effective moment of inertia is widely accepted for computation of deflections of partially cracked RC beams under service loads. Branson\textsuperscript{[1,2]} uses an interpolated value of the moment of inertia (MI), called effective moment of inertia, \(I_e\), between the well defined limits of the uncracked and fully cracked states. The procedure employs loading level in the form of moment ratio, \(R_{M'}\), of the cracking moment, \(M'_c\), to the applied service moment, \(M_s\), for interpolation purposes. However, recent studies\textsuperscript{[3,4]} show that \(I_e\) besides \(R_{M'}\) depends upon the type of loading and the
reinforcement ratio. The type of loading governs the extent of cracking along the reinforced concrete member, thereby in effect, rendering the beam 'nonprismatic'. A beam under a central point load tends to crack over a shorter length than the one under uniformly distributed load when subjected to the same $R_e$. On the other hand, the reinforcement ratio controls the rate at which cracks propagate towards the neutral axis, being slower for beams with higher steel percentages. IS-code\textsuperscript{3} formula for $I_s$ (which is based on the works by Rech\textsuperscript{6} and Brakel\textsuperscript{7}) indirectly employs in addition to $R_e$, the effect of reinforcement in evaluation of $I_s$.

2. A Review and Discussion of the Available Formulas

There are several formulas which express $I_s$ in function of the relevant parameters. The ACI code\textsuperscript{3} adopted the well known Branson equation for $I_s$, which is

$$I_s = I_{Ne} + \left[ I_s - I_{Ne} \right] \left( R_e \right)^m \leq I_s$$  \hspace{1cm} \text{(1)}$$

where, $m = 3$, and $I_{Ne}$ and $I_s$ are the moments of inertia of the gross and transformed cracked sections of RC beams, respectively. Among the relevant parameters affecting the value of $I_s$, Eq. (1) considers only the loading level which is reflected by the moment ratio, $R_e$.

In a recent study\textsuperscript{3}, two different approaches were used to incorporate the effect of the loading type in computation of $I_s$ of normally reinforced beams. The first approach employed the format of Eq. (1) with different values of the exponent, $m$, for each load-type. The proposed values were 2, 2.3, 1.8, and 1.3, for beams subjected to uniform, third-point, and midspan loads, respectively. The second approach incorporated the cracked length ratio, $R_L$, of the beam segment, $\Sigma_{cr}$, over which working moment exceeds $M_{cr}$, to the beam span, $L$, besides the moment ratio to account for the type of load and level of the loading. The effective moment of inertia, $I_{re}$, is expressed in terms of $R$ as

$$I_{re} = I_{Ne} - \left[ I_s - I_{Ne} \right] \left( R \right)^{m'}$$  \hspace{1cm} \text{(2)}$$

The exponent $m'$ was found to be equal to $R_e$.

In a subsequent study\textsuperscript{10}, the effect on $I_s$ of the reinforcement ratio, $\rho$, was also investigated. The study suggested modified values of the exponents $m$ and $m'$ to account for this effect. These values may be obtained from

$$m = a - 0.1 \rho \hspace{1cm} \text{(1 - a)}$$

$$m' = 8 \rho R_e \hspace{1cm} \text{(2 - a)}$$

For beams subjected to midspan point loads, the coefficient, $a$, was found to be $a = 3$.

Another format for computation of $I_s$ is available in the Indian Standard Code\textsuperscript{14} equation, which is valid for $I_s$ in the range $I_s < I_s < I_s$.
\[ I_c = \frac{I_{ec}}{1 + \frac{R_e}{Z_e} \left( 1 - \frac{b}{d} \right)} \]

where, \( I_{c} \) = levere area, \( Z_e \) = depth of the neutral axis, \( d \) = effective depth, \( b \) = breadth of web, and \( b \) = breadth of compression zone. For rectangular sections \( b = h \) is unity.

Equation (3) accounts for the effect of the reinforcement ratio, \( R_e \), in incorporating the factor of the ratio, \( x/d \) ratio, which is function of \( \rho \). Other oversimplified models for computation of \( I_c \) are also available in the literature.

In view of the above discussion, the present work is aimed at accomplishing the following objectives:

1. Comparison of the \( I_c \) values predicted by various available models with the test results of beams subjected to midspan, third-point, and uniform loads.
2. Generalization of the widely used Eq. (1) to account for the effect of reinforcement ratio on \( I_c \) of RC beams subjected to various types of loading.
3. Modification of Eq. (3) which considers the effects of loading level and reinforcement ratio to account for the effect of loading type.

### 3. Summary and Analysis of Test Results

Details of the test beams and their material properties, instrumentation, and loading patterns were reported earlier. A summary of relevant information is presented in Table 1. Typical plots of the load-deflection curves of the experimental beams in the range beyond cracking are shown in Fig. 1. Tables 2 and 3 present the experimental and computed values of \( I_c \) of the test beams for various moment ratios, \( R_{cw} \), for beams carrying point loads at midspan and uniform loads, respectively.

<table>
<thead>
<tr>
<th>Beam</th>
<th>l, mm</th>
<th>b, mm</th>
<th>h, mm</th>
<th>d, mm</th>
<th>( R_{cw} )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1-U</td>
<td>38.2</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>1.3</td>
<td>[1], [10]</td>
</tr>
<tr>
<td>B2-C</td>
<td>38.2</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>1.3</td>
<td>[2], [13]</td>
</tr>
<tr>
<td>B3-T</td>
<td>38.2</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>1.3</td>
<td>[1], [10]</td>
</tr>
<tr>
<td>B4-C</td>
<td>25.5</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>2.0</td>
<td>[3], [14]</td>
</tr>
<tr>
<td>B5-C</td>
<td>3.4</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>2.0</td>
<td>[20]</td>
</tr>
<tr>
<td>B6-T</td>
<td>31.4</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>0.8</td>
<td>[10]</td>
</tr>
<tr>
<td>B7-L</td>
<td>33.8</td>
<td>200</td>
<td>200</td>
<td>155</td>
<td>0.8</td>
<td>[4]</td>
</tr>
</tbody>
</table>

1. U, C, and T in beam designations indicate uniform, midspan, and third-point loads, respectively.
2. Beam B4-C-1 was failed under midspan load.

Figure 2 shows the variation of the effective moment ratio, \( R_{cw} = I/I_{ec} \), with the moment ratio, \( R_{cw} \), determined from test results and predicted by Eq. (3). It can be seen that Eq. (3) does not distinguish between differently loaded beams, while the experimental results show a noticeable difference. The plot of cracked length
### Table 2: Comparison of computed and experimental values of effective moment of inertia for beam B(1) [15].

<table>
<thead>
<tr>
<th>Rs</th>
<th>Rs</th>
<th>Ix × 10⁶ m⁴</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.605</td>
<td>0.397</td>
<td>12.81</td>
<td>0.80</td>
<td>0.514</td>
<td>0.95</td>
</tr>
<tr>
<td>0.567</td>
<td>0.433</td>
<td>96.74</td>
<td>0.84</td>
<td>0.499</td>
<td>0.98</td>
</tr>
<tr>
<td>0.539</td>
<td>0.461</td>
<td>92.86</td>
<td>0.86</td>
<td>0.445</td>
<td>1.01</td>
</tr>
<tr>
<td>0.495</td>
<td>0.505</td>
<td>84.63</td>
<td>0.86</td>
<td>0.461</td>
<td>1.01</td>
</tr>
<tr>
<td>0.458</td>
<td>0.542</td>
<td>79.34</td>
<td>0.86</td>
<td>0.507</td>
<td>1.00</td>
</tr>
<tr>
<td>0.435</td>
<td>0.565</td>
<td>75.98</td>
<td>0.86</td>
<td>0.726</td>
<td>1.00</td>
</tr>
<tr>
<td>0.402</td>
<td>0.598</td>
<td>71.76</td>
<td>0.87</td>
<td>0.752</td>
<td>1.00</td>
</tr>
<tr>
<td>0.368</td>
<td>0.641</td>
<td>67.75</td>
<td>0.87</td>
<td>0.773</td>
<td>0.99</td>
</tr>
<tr>
<td>0.318</td>
<td>0.682</td>
<td>64.03</td>
<td>0.88</td>
<td>0.797</td>
<td>0.98</td>
</tr>
<tr>
<td>0.294</td>
<td>0.706</td>
<td>60.37</td>
<td>0.92</td>
<td>0.833</td>
<td>1.00</td>
</tr>
</tbody>
</table>

**Note:** Details of test beams B(1) at B: 
- b = 200 mm, d = 240 mm, As = 506 mm² (2.14), d = 100 mm, m = 0.8%. 
- Ix = 5.85 × 10⁶ mm⁴, Iy = 294.2 × 10⁶ mm⁴, h = 80 mm. 
Type of load: Point load at midpoint.

### Table 3: Comparison of computed and experimental values of effective moment of inertia for beams B(U) and B(1) [15].

<table>
<thead>
<tr>
<th>Rs</th>
<th>Rs</th>
<th>Ix × 10⁶ m⁴</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.735</td>
<td>0.51</td>
<td>73.85</td>
<td>1.03</td>
<td>0.97</td>
<td>1.07</td>
</tr>
<tr>
<td>0.676</td>
<td>0.57</td>
<td>67.74</td>
<td>1.00</td>
<td>0.70</td>
<td>1.04</td>
</tr>
<tr>
<td>0.675</td>
<td>0.60</td>
<td>62.65</td>
<td>0.97</td>
<td>0.72</td>
<td>1.02</td>
</tr>
<tr>
<td>0.578</td>
<td>0.55</td>
<td>58.98</td>
<td>0.96</td>
<td>0.75</td>
<td>1.00</td>
</tr>
<tr>
<td>0.506</td>
<td>0.67</td>
<td>54.34</td>
<td>0.93</td>
<td>0.77</td>
<td>0.97</td>
</tr>
<tr>
<td>0.491</td>
<td>0.74</td>
<td>50.35</td>
<td>0.93</td>
<td>0.81</td>
<td>0.96</td>
</tr>
<tr>
<td>0.405</td>
<td>0.77</td>
<td>47.74</td>
<td>0.93</td>
<td>0.83</td>
<td>0.96</td>
</tr>
<tr>
<td>0.368</td>
<td>0.79</td>
<td>45.87</td>
<td>0.95</td>
<td>0.84</td>
<td>0.96</td>
</tr>
<tr>
<td>0.338</td>
<td>0.81</td>
<td>44.37</td>
<td>0.94</td>
<td>0.85</td>
<td>0.96</td>
</tr>
<tr>
<td>0.312</td>
<td>0.83</td>
<td>43.62</td>
<td>0.94</td>
<td>0.86</td>
<td>0.96</td>
</tr>
<tr>
<td>0.281</td>
<td>0.84</td>
<td>42.83</td>
<td>0.94</td>
<td>0.86</td>
<td>0.96</td>
</tr>
</tbody>
</table>

**Note:** Details of test beams B(U) and B(1): 
- b = 200 mm, h = 200 mm, As = 42 mm² (2.14), d = 155 mm, m = 1.3%. 
- Ix = 3.34 × 10⁶ mm⁴, Iy = 133.35 × 10⁶ mm⁴, h = 80 mm. 
Type of load: Uniform load.

ratio, Rs, as determined from the fundamentals of structural analysis, versus moment ratio, Rs, produced in Fig. 3 clearly suggests that the difference is due to the variation in the extent of cracked length under the different loadings. The figure reveals the fact that, at the same moment ratio, Rs, the three differently loaded beams have different cracked lengths which definitely affect the averaging concepts employed in the computation of Ix. 

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Fig. 1. Typical load-deflection curves of the experimental beams.²⁻⁴.
Fig. 2. Variation of $R_1$ with $R_M$. 
Figure 4 compares the experimental values of $R_f$ with the computed ones using Eq. (3). The figure suggests that Eq. (3) needs to be modified to account for the effect of
type of loading and to overcome the inconsistency in the prediction error, which is higher for lightly cracked beams.

![Graph showing experimental values of $R_i$ vs. the corresponding theoretical values computed by using Eq. (7).](image_url)
4. Proposed Models

Equation (1) is modified to account for the effect of the reinforcement ratio on the \( I_e \) values for RC beams under a general state of loading. The test results, which were reported earlier\(^{[8,9]} \) on the beams summarized in Table 1, are utilized. The case of beams under central point loads was investigated in Ref. [4]. A relation in the form of Eq. (1-a) is obtained by linear fitting for beams under third-point loads, and by extrapolation for beams under uniform load. The extrapolation is based on the reported value of \( m \) at \( \rho = 1 \% \) from Ref. [3]. The values of the coefficient, \( a \), to be used in Eq. (1-a) are summarized in Table 4.

<table>
<thead>
<tr>
<th>Loading type</th>
<th>( a )</th>
<th>( b )</th>
<th>( \rho )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform load</td>
<td>3.8</td>
<td>1.15</td>
<td>-1.45</td>
</tr>
<tr>
<td>Third-point load</td>
<td>3.5</td>
<td>1.10</td>
<td>-1.50</td>
</tr>
<tr>
<td>Midpoint load</td>
<td>3.0</td>
<td>1.07</td>
<td>-1.55</td>
</tr>
</tbody>
</table>

In order to incorporate the effect of type of loading in Eq. (3) while maintaining its form, the ratio of effective \( M_I / R_c \) is modeled as:

\[
R = \frac{1}{a + b \cdot X_H}
\]  

(4)

where, \( X_H = R_d (Z/d) (1 - X/d) b / \beta \), \( a \) & \( b \) are coefficients to be determined by regression analysis of the experimental test data.

The experimental values of \( R_I \) for the differently loaded rectangular beams summarized in Table 1, are then plotted against the corresponding \( X_H \) values as shown in Fig. 5. The linear regression coefficients \( a \) and \( b \) for the three loading types are presented in Table 4. Thus, \( I_e \) may be expressed as

\[
I_e = \frac{I_{zl}}{a + b \cdot R_d \cdot \frac{Z}{d} \left(1 - \frac{S}{d}\right) R_c}
\]

(5)

Computed values of ratio of effective \( M_I / R_c \), from Eq. (5), at various moment ratios, are compared with the corresponding experimental values of \( R_I \) in Fig. 6. The plot shows that Eq. (5) adequately accounts for the effect of loading type in the computation of \( I_e \).

5. Comparison of the Various Models

Accuracy of the effective \( M_I / R_c \) models presented in this work is checked against typical test results from Ref. [3, 4 and 10] on beams with rectangular and T-sections. The effect on \( I_e \) of the parameters like the level and type of loading, and the reinforcement ratio is compared.
Fig. 5. Variation of $R^{-1}$ with $X_0$ for different types of load.
Fig. 6. Experimental values of $R_i$ vs. the corresponding theoretical values by using Eq. (15).
5.1 Rectangular Sections

Table 2 presents the experimental values of a loaded rectangular beam under a central point load and compares them with values obtained from ACI-Code formula, IS-Code formula, Eq. (1), Eq. (2) and the proposed Eq. (5), which is a modified version of the IS-Code formula. Table 3 does the same for the average values obtained for two identical uniformly loaded beams. The following observations are made on the information presented in these tables:

1. The IS-Code formula, Eq. (3), underestimates \( I \) of both the centrally and the uniformly loaded beams.

2. As mentioned previously, the ACI-Code formulai was derived on the basis of experimental results of uniformly loaded beams. This explains the good agreement with the test results in Table 3.

3. Equations (1) and (2) show a superior ability to predict \( I \) under all circumstances. Equation (5) is also fairly accurate as compared to its original version in the form of Eq. (3).

5.2 T-Sections

Table 5 reports the results of two T-beams carrying a midspan and third-point loads. Pertinent information of the test beams is summarized in the footnote of the table. The values of \( R_e \) using Eq. (1), (2), and (5) show good agreement with the test results. On the other hand, the IS-Code formula is observed to be the least accurate.

<table>
<thead>
<tr>
<th>Beam designation</th>
<th>( R_e )</th>
<th>( I )</th>
<th>( I / \text{mm}^4 )</th>
<th>( I / \text{mm}^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B15-3M</td>
<td>0.600</td>
<td>0.400</td>
<td>91.85</td>
<td>91.85</td>
</tr>
<tr>
<td>midspan load</td>
<td>0.429</td>
<td>0.571</td>
<td>75.00</td>
<td>75.00</td>
</tr>
<tr>
<td></td>
<td>0.333</td>
<td>0.607</td>
<td>71.82</td>
<td>71.82</td>
</tr>
<tr>
<td></td>
<td>0.273</td>
<td>0.727</td>
<td>67.54</td>
<td>67.54</td>
</tr>
<tr>
<td></td>
<td>0.231</td>
<td>0.769</td>
<td>66.21</td>
<td>66.21</td>
</tr>
<tr>
<td>B16-3T</td>
<td>0.571</td>
<td>0.610</td>
<td>91.08</td>
<td>91.08</td>
</tr>
<tr>
<td>third-point load</td>
<td>0.444</td>
<td>0.704</td>
<td>77.62</td>
<td>77.62</td>
</tr>
<tr>
<td></td>
<td>0.364</td>
<td>0.757</td>
<td>69.68</td>
<td>69.68</td>
</tr>
<tr>
<td></td>
<td>0.308</td>
<td>0.795</td>
<td>67.41</td>
<td>67.41</td>
</tr>
<tr>
<td></td>
<td>0.267</td>
<td>0.822</td>
<td>65.21</td>
<td>65.21</td>
</tr>
</tbody>
</table>

Note: Details of test beams B15-3M and B16-3T are:

- \( d = 350 \text{ mm}, b = 150 \text{ mm}, h = 200 \text{ mm}, \alpha = 60 \text{ mm}, \beta = 105 \text{ mm}, A_s = 500 \text{ mm}^2 \) and 100 mm, respectively.
- \( I = 6.5 \times 10^5 \text{ mm}^4, J = 8.5 \times 10^5 \text{ mm}^4, z_t = 1.7 \text{ mm} \).

6. Conclusion

The study points to the fact that effective moment of inertia of an RC beam depends upon the service load level, the reinforcement ratio and the load type applied to the beam. Various models, currently in use, only account for either the load level or the load level and reinforcement ratio. This study presents refined models which,
in addition to the above mentioned factors, consider the load type. The models are compared with the test results.

The p-and-load-type-dependent value of exponent m given by Eq. (1-a) enables Eq. (1) to account for both the reinforcement ratio and the loading type with an excellent accuracy.

In case of rectangular beams carrying midspan load, the effective moment of inertia, \( I_e \), by the IS-Code formula, at various moment ratios, varies from the experimental values by \(-44.6\% \) to \(+16.7\%\), while \( I_e \) obtained by using the modified version of the formula does so by \(-4\% \) to \(+8\%). These variations in case of T-beams carrying midspan load range from \(-31\% \) to \(-5\% \) and from \(-16\% \) to \(-2\%\), respectively. However, more test data is necessary to strengthen the above conclusions.

References


Notation

- \( s \): Numerical coefficient
- \( A_t \): Area of reinforcing steel
- \( b \): Width of compression zone
- \( b_r \): Breadth of web of a flanged section
- \( d \): Effective depth of beam section
- \( f_c \): Compressive strength of concrete
- \( h \): Height of beam section
- \( h_k \): Depth of flange of a flanged section
- \( I_m \): Moment of inertia of the uncracked cracked section
- \( I_{cr} \): Effective moment of inertia of the partially cracked beam
- \( I_e \): Experimental value of \( I_f \)
- \( I_{cc} \): Moment of inertia of the gross concrete section
$L$: Beam span
$L_a$: Cracked length (beam segment on which the working moment exceeds $M_p$)
$m$: Exponent for use in Eq. (1)
$m'$: Exponent for use in Eq. (2)
$M_p$: Maximum service load-moment acting on the beam
$M_e$: Cracking moment of beam
$R_f$: Ratio of effective moment of inertia, $I/I_e$
$R_e$: Ratio of cracked length, $L_a/L$
$R_{pe}$: Ratio of moment, $M/M_p$
$X$: Depth of neutral axis
$Z$: Lever arm
$\rho$: Reinforcement ratio
$\alpha, \beta$: Numerical coefficients.
عزم التصور الذاتي التفاعل للمعوارض الحرسانية المسلحة المشتقة حديثًا

راجع نجد زيد

فيما، عمدة المدينة، وكالة 영ئحة، جامعة الملك سعود، الرياض، المملكة العربية السعودية

المستفض: أثبتت الدراسة أعلاه أن عزم التصور الذاتي التفاعل للمعوارض الحرسانية المؤشرة على كلي من مستوى عمل العناصر ومتنا من حدود المراد، وكذلك على نوي الأغلفة من حيث توزيعها أو ترتيبها على العرض. حيث أثبتت النتائج أن عزم التصور الذاتي التفاعل للمعوارض الحرسانية المسلحة المؤشرة بأعمال مرکزة تختلف بدرجة كبيرة مع البالغة للعوارض المرتفع بأعمال مراعاة، وتم تم على أساسها تطور الإنتاج والفوضى للعوارض الذاتي التفاعل. أما إذا فحص هذه الدراسة ومحدودة حسب عزم التصور الذاتي التفاعل، فإنها تقدم نماذج أعمق أثرًا في الأداء. حيث وظائف الأغلفة التي يمكن أن تبرر هذه المعوارض الحرسانية. كما تثبت نتائجها مع النتائج التجريبية من النتائج الأخرى المعروفة هذا العرس. أن أيضًا بعض مثابرة استخدام نماذج المراعية بالنسبة للمعوارض الحرسانية ذات طيف على شكل حرف 2.