Response of first order systems

Outline:

- Definition of first order systems
- The general form of transfer function of first order systems
- Response of first order systems to some common forcing functions (predict and understand how it responds to an input)

Time behavior of a system is important. When you design a system, the time behavior may well be the most important aspect of its behavior (How quickly a system responds is important)

What is a first order system?

It is a system whose dynamic behavior is described by a first order differential equation.

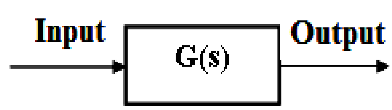
Synonyms for first order systems are first order lag and single exponential stage.

Transfer function

The transfer function is defined as the ratio of the output and the input in the Laplace domain.

It describes the dynamic characteristics of the system.

$$G(s) = \frac{output}{input}$$



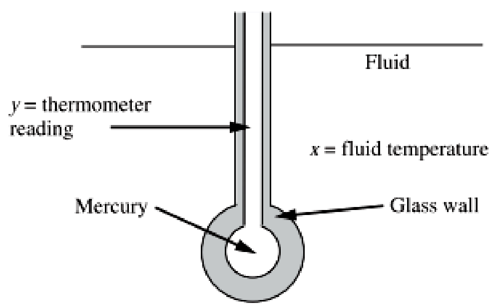
General rules to develop a transfer function

1. Make unsteady state balance (mass, heat or momentum)
2. Make steady state balance

3. Subtract the steady state equation from the unsteady state equation (why?)
4. Transform the resulting equation into the Laplace domain
5. Rearrange the equation to get the ratio of the (out/in) in one side and the other parameters in the other side (the resulting is the transfer function)

Example on first order systems

A mercury thermometer: consider the mercury thermometer shown in the figure



Cross view of thermometer

Basic assumptions:

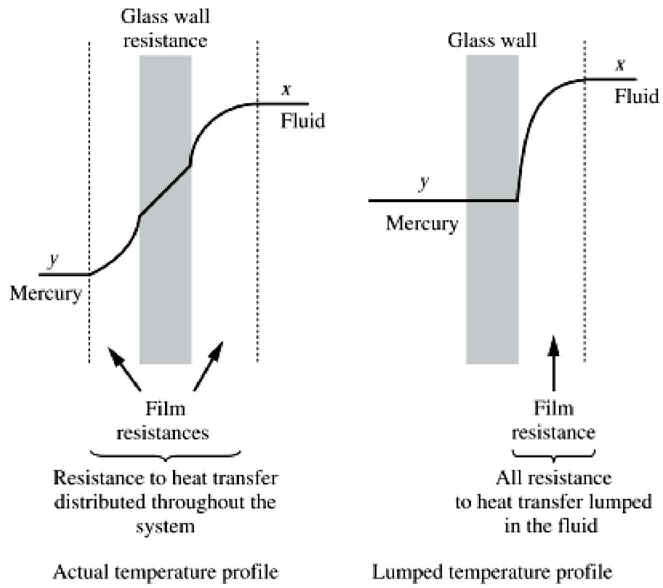
1. All the resistance to heat transfer resides in the film surrounding the bulb (conduction resistance is neglected).
2. All the thermal capacity is in the mercury.
3. The mercury assumes a uniform temperature throughout.
4. The glass wall containing the mercury does not expand or contract during the transient response.
5. Constant properties.

To develop the transfer function of the thermometer we will follow the steps mentioned earlier;

1. Unsteady state heat balance

Input – Output = Rate of change

$$hA(x - y) - 0 = mc_p \frac{dy}{dt}$$



Temperature profile in the thermometer

Note: output from the thermometer = 0.0 (mercury is expanded when heated)

2. Steady state balance

$$hA(x_s - y_s) = 0$$

3. Unsteady state balance – steady state balance

$$hA[(x - x_s) - (y - y_s)] = mc_p \frac{d(y - y_s)}{dt}$$

Note: $\frac{d(y-y_s)}{dt} = \frac{dy}{dt}$

Write the above equation in terms of the deviation variables

$$hA[X - Y] = mc_p \frac{dY}{dt}$$

$$[X - Y] = \frac{mc_p}{hA} \frac{dY}{dt}$$

4. Laplace transform

$$[X(s) - Y(s)] = \frac{mc_p}{hA} sY(s)$$

5. Output/Input

$$\frac{Y(s)}{X(s)} = \frac{1}{\frac{mc_p}{hA} s + 1}$$

The parameter $\frac{mc_p}{hA}$ has the units of time and is called the time constant of the system and is denoted by τ

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

Note:

- The time constant is a measure to how fast be the system response. The smaller is the time constant, the more responsive is the system.
- τ also called "dead time" or "dynamic lag"

Standard form of first order transfer functions

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

- The important characteristics of the standard form are as follows:
- The denominator must be of the form $\tau s + 1$
- The coefficient of the s term in the denominator is the system time constant τ
- The numerator is the steady-state gain K.

Example 1: A first order system has a transfer function $\frac{Y(s)}{X(s)} = \frac{2}{s+\frac{1}{3}}$. Identify the time constant and the steady state gain.

Properties of transfer functions:

- Superposition is applicable

$$G(s) = \frac{Y(s)}{X(s)}$$

If $X(s) = a_1 X_1(s) + a_2 X_2(s)$

$$Y(s) = G(s)\,X(s)$$

$$Y(s) = a_1 G(s) X_1(s) + a_2 G(s) X_2(s)$$

$$= a_1 Y_1(s) + a_2 G(s) Y_2(s)$$

i.e. The output of multi-input is the sum of the output to the individual inputs.

Response of first order systems to some common forcing functions

1. Step response $(X(t) = A\,u(t); Y(t) = ??)$

Required:

- Sketch of the input and the corresponding output
- General equation of the output
- The ultimate and maximum value of the output and their corresponding time.

To get the response to any input follow the following steps (scheme)

$$[Input]\,X(t) \xrightarrow{\mathcal{L}T} X(s) \xrightarrow{TF} Y(s) \xrightarrow{\mathcal{L}^{-1}} Y(t)[Output]$$

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

$$X(s) = \frac{A}{s}$$

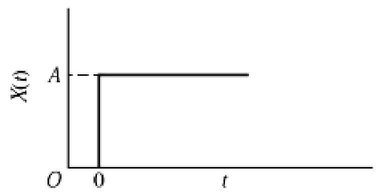
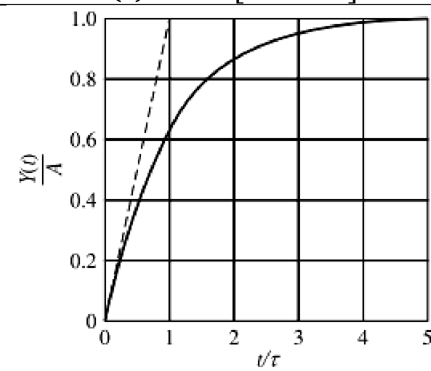
$$Y(s) = \frac{A}{s}\frac{K}{\tau s + 1} = \frac{C_1}{s} + \frac{C_2}{\tau s + 1}$$

Solving by partial fractions

$$Y(s) = \frac{KA}{s} - \frac{KA}{s + \frac{1}{\tau}}$$

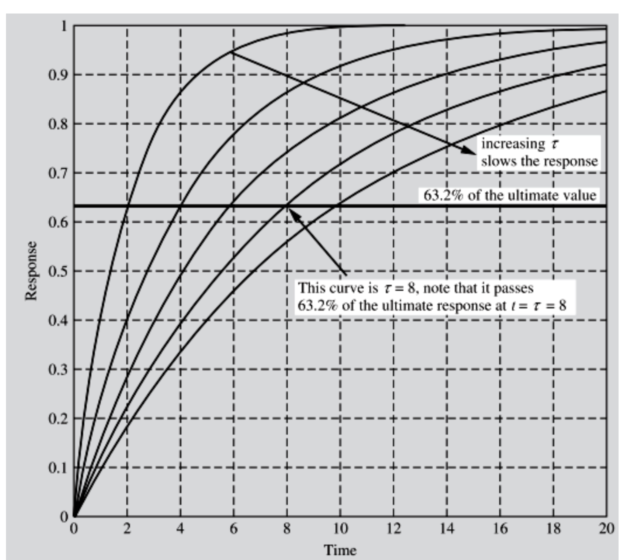
$$Y(t) = KA[1 - e^{-\frac{t}{\tau}}]$$

The above equation is the general form of first order system response to step change.

Input X(t)	Output Y(t)
$X(t) = A\,u(t)$	$Y(t) = KA[1 - e^{-\frac{t}{\tau}}]$
	 Ultimate value = KA Maximum value = KA

<u>Characteristics of step response</u>

A. The value of the output reaches 63.2% of its ultimate value after $t = \tau$
B. If the initial rate of change is maintained the response will be completed after $t = \tau$
C. The response is completed after $t = 5\tau$
D. The speed of the response of a first-order system is determined by the time constant for the system. as τ increases, it takes longer for the system to respond to the step disturbance.



Effect of the time constant on the step response of a first order system

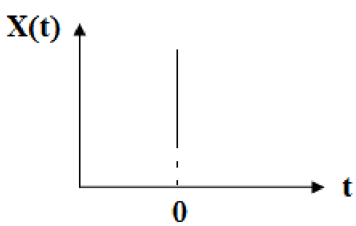
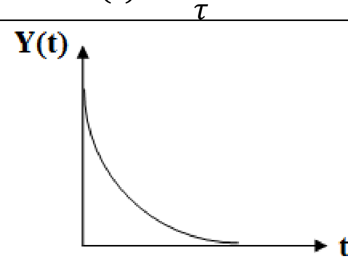
2. Impulse response $(X(t) = A\,\delta(t); Y(t) = ??)$

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

$$X(s) = A$$

$$Y(s) = \frac{KA}{\tau s + 1} = \frac{KA/\tau}{s + \frac{1}{\tau}}$$

$$Y(t) = \frac{KA}{\tau}\, e^{-\frac{t}{\tau}}$$

Input X(t)	Output Y(t)
$X(t) = A\,\delta(t)$	$Y(t) = \frac{KA}{\tau}\, e^{-\frac{t}{\tau}}$
	 The response rises immediately and then decays exponentially. Ultimate value = 0 Maximum value = $\frac{KA}{\tau}$ at t = 0

3. Pulse response

$$X(t) = \begin{cases} H & 0 < t < T \\ 0 & T < t \end{cases}$$

Write the function X(t) in terms of the unit step function

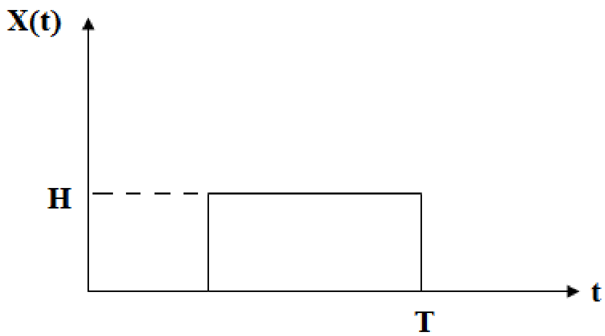
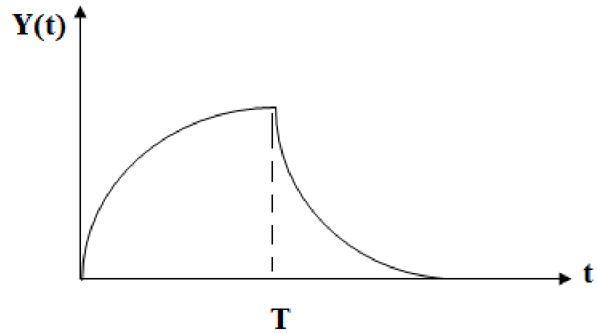
$$X(t) = H\,u(t) - Hu(t - T)$$

$$X(s) = \frac{H}{s} - \frac{H}{s} e^{-Ts}$$

$$Y(s) = \left(\frac{H}{s} - \frac{H}{s} e^{-Ts}\right)\frac{K}{\tau s + 1}$$

$$Y(s) = \frac{H}{s}\frac{K}{\tau s + 1} - \frac{H}{s}\frac{K}{\tau s + 1} e^{-Ts}$$

$$Y(t) = KH\left[1 - e^{-\frac{t}{\tau}}\right]u(t) - KH\left[1 - e^{-\frac{(t-T)}{\tau}}\right]u(t-T)$$

Input X(t)	Output Y(t)
$X(t) = A\,\delta(t)$	$Y(t) = \frac{KA}{\tau}\,e^{-\frac{t}{\tau}}$
	 Ultimate value = 0 Maximum value = $KH\left[1 - e^{-\frac{T}{\tau}}\right]$ at t =T

4. **Ramp response** $(X(t) = At\,u(t); Y(t) = ??)$ where A is the slope of the ramp function

$$\frac{Y(s)}{X(s)} = \frac{K}{\tau s + 1}$$

$$X(s) = \frac{A}{s^2}$$

$$Y(s) = \frac{KA}{s^2(\tau s + 1)} = \frac{\frac{KA}{\tau}}{s^2\left(s + \frac{1}{\tau}\right)} = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3}{\left(s + \frac{1}{\tau}\right)}$$

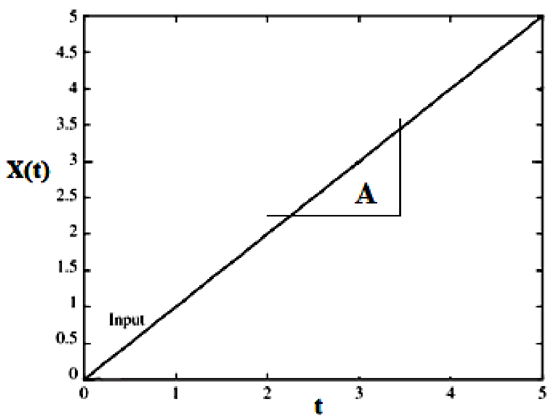
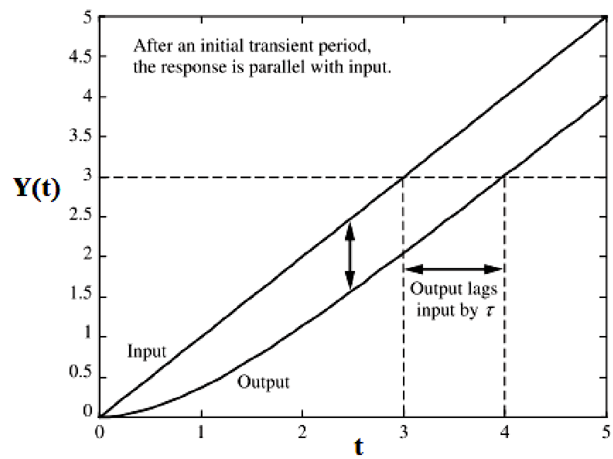
Solving by partial fractions

- $C_1 = KA$
- $C_2 = -KA\tau$
- $C_3 = KA\tau$

$$Y(s) = \frac{KA}{s^2} - \frac{KA\tau}{s} + \frac{KA\tau}{\left(s + \frac{1}{\tau}\right)}$$

$$Y(t) = KAt - KA\tau + KA\tau\, e^{-\frac{t}{\tau}}$$

$$Y(t) = KAt - KA\tau\left(1 - e^{-\frac{t}{\tau}}\right)$$

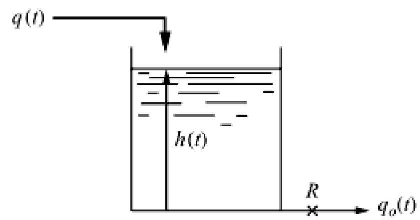
Input X(t)	Output Y(t)
$X(t) = At\, u(t)$	$Y(t) = KAt - KA\tau\left(1 - e^{-\frac{t}{\tau}}\right)$
	 Ultimate value = ∞ Maximum value = ∞

Physical examples of first order systems

The objective of this part is to develop the transfer function of some first order systems and to confirm that the time constant depends on the system parameters.

1. Liquid level

A tank of uniform cross sectional area A, the inlet flow is q and the outlet is q_o. The liquid level in the tank is h and the tank has a linear flow resistance at the outlet (e.g. valve). $\left(q_o = \frac{h}{R}\right)$



Liquid level system

Required: the transfer function $\frac{H(s)}{Q(s)}$ and $\frac{Q_o(s)}{Q(s)}$ (note: this system is called single input multi-output SIMO)

Basic assumptions:

a) Constant density b) Linear resistance c) Constant cross sectional area

i. Unsteady state mass balance

$$\rho q - \rho q_o = \frac{d(\rho V)}{dt} = \frac{d(\rho A h)}{dt}$$

$$q - q_o = A\frac{dh}{dt}$$

ii. Steady state mass balance

$$q_s - q_{o_s} = 0$$

iii. Subtract the steady state equation from the unsteady state one

$$Q - Q_o = A\frac{dH}{dt}$$

Where:

$Q = q - q_s$	$Q_o = q_o - q_{o_s}$	$H = h - h_s$

The above parameters are called deviation variables.

iv. Taking the transform of the resulting equation

$$Q(s) - Q_o(s) = AsH(s)$$

$$\mathrm{Q_o} = \frac{\mathrm{H}}{\mathrm{R}}$$

$$Q(s) - \frac{\mathrm{H(s)}}{\mathrm{R}} = AsH(s)$$

$$\frac{\mathrm{H(s)}}{Q(s)} = \frac{R}{ARs + 1}$$

$$\frac{\mathrm{H(s)}}{Q(s)} = \frac{R}{\tau s + 1}$$

$$\tau = AR$$

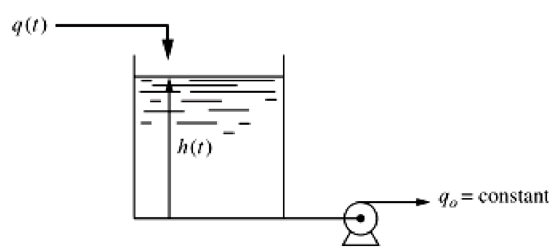
Note: tanks having small cross sectional area are more responsive.

$$\frac{\mathrm{Q_o(s)}}{Q(s)} = \frac{1}{\tau s + 1}$$

N.B: the steady state gain depends on the input and the output in the last transfer function it is dimensionless because the input and the output have the same units.

The steady state gain is a conversion factor that relates the input and the output at steady state.

2. Liquid level with constant flow outlet



Liquid level with constant flow outlet

In this example the resistance R is replace by a constant flow pump $\left(q_o(t) = q_{o,s}\right)$

Required: the transfer function $\frac{\mathrm{H(s)}}{Q(s)}$

Basic assumptions:

d) Constant density e) Constant flow outlet f) Constant cross sectional

$$\left(q_o(t) = q_{o,s}\right) \qquad\qquad \text{area}$$

i. Unsteady state mass balance

$$q - q_o = A\frac{dh}{dt}$$

ii. Steady state mass balance

$$q_s - q_{o_s} = 0$$

iii. Subtract the steady state equation from the unsteady state one

$$Q = A\frac{dH}{dt}$$

Where: $q_o - q_{o_s} = 0$

iv. Taking the transform of the resulting equation

$$Q(s) = AsH(s)$$

$$\frac{\mathrm{H(s)}}{Q(s)} = \frac{1}{As}$$

$$\mathrm{H(s)} = \frac{Q(s)}{As}$$

Note: Inverse of the above equation yields:

$$\mathrm{H(t)} = \frac{1}{A}\int_0^t Q(t)dt$$

Ex: Find the response of this system to a unit step change in input

$$Q(t) = 1\,u(t)$$

$$Q(s) = \frac{1}{s}$$

$$H(s) = \frac{1}{As^2}$$

$$H(t) = \frac{1}{A}t$$

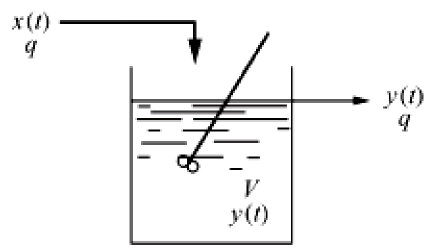
- The response is a ramp function that grows with time without limit. Such a system that grows without limit for a sustained change in input is said to have **non-regulation**.
- Systems that have a limited change in output for a sustained change in input are said to have regulation.

Important note:

The transfer function for the liquid-level system with constant outlet flow given can be considered as a special case of Eq. $\frac{H(s)}{Q(s)} = \frac{R}{ARs+1}$ as $R \to \infty$

$$lim_{R\to\infty}\left(\frac{R}{ARs+1}\right) = \frac{1}{As}$$

3. Mixing process



Mixing process

Description:

Consider the mixing process shown in the above figure in which a stream of solution containing dissolved salt flows at a constant volumetric flow rate (q) into a tank of constant holdup volume V. The concentration of the salt in the entering stream x (mass of salt/volume) varies with time. It is desired to determine the transfer function relating the outlet concentration y to the inlet concentration x $\left(\frac{Y(s)}{X(s)}\right)$.

Basic assumptions:

- Constant density
- Perfect mixing (outlet concentration equal the concentration inside the tank
- Constant holdup

i. Unsteady state mass balance

$$qx - qy = \frac{d(Vy)}{dt} = V\frac{dy}{dt}$$

ii. Steady state mass balance

$$qx_s - qy_s = 0$$

iii. Subtract the steady state equation from the unsteady state one

$$qX(t) - qY(t) = V\frac{dY}{dt}$$

iv. Taking the transform of the resulting equation

$$qX(s) - qY(s) = VsY(s)$$

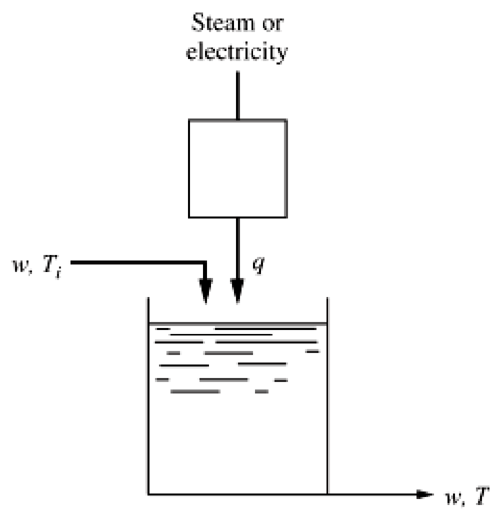
$$\frac{Y(s)}{X(s)} = \frac{1}{\frac{V}{q}s + 1}$$

$$\tau = \frac{V}{q}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{\tau s + 1}$$

Note: the steady state gain is dimensionless.

4. Heating Process



Heating process

A stream at temperature T_i is fed to the tank. Heat is added to the tank by means of an electric heater. The tank is well mixed, and the temperature of the exiting stream is T. The flow rate to the tank is constant at w lb/h.

Required: the transfer function relating the change in the inlet temperature and the change in q to the change in the outlet temperature

i. Unsteady state mass balance

$$wc_p(T_i - T_{ref}) + q - wc_p(T - T_{ref}) = \frac{d\,\rho V c_p(T - T_{ref})}{dt} = \rho V c_p \frac{dT}{dt}$$

ii. Steady state mass balance

$$wc_p(T_{is} - T_{ref}) + q_s - wc_p(T_s - T_{ref}) = 0$$

iii. Subtract the steady state equation from the unsteady state one

$$wc_p(\bar{T}_i - \bar{T}) + Q = \rho V c_p \frac{d\bar{T}}{dt}$$

Where:

$\bar{T}_i = T_i - T_{i,s}$	$\bar{T} = T - T_s$	$Q = q - q_s$

This system has two inputs (change in inlet temperature and change in q)

We will fix one input and account for the effect of the other one:

First: Assume q is constant:

$$\therefore wc_p(\bar{T}_i - \bar{T}) = \rho V c_p \frac{d\bar{T}}{dt}$$

iv. Taking the transform of the resulting equation

$$w[\bar{T}_i(s) - \bar{T}(s)] = \rho V s \bar{T}(s)$$

$$\bar{T}_i(s) = \bar{T}(s)\left[\frac{\rho V}{w} s + 1\right]$$

$$\frac{\bar{T}(s)}{\bar{T}_i(s)} = \frac{1}{\frac{\rho V}{w} s + 1}$$

$$\tau = \frac{\rho V}{w}$$

Effect of q: Assume T_i is constant:

$$wc_p\bar{T} + Q = \rho V c_p \frac{d\bar{T}}{dt}$$

Taking the transform of the resulting equation

$$\frac{Q(s)}{wc_p} = \bar{T}(s)\left[\frac{\rho V}{w} s + 1\right]$$

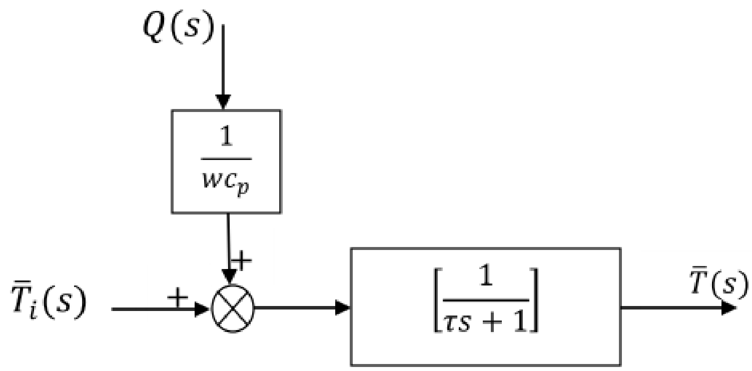
$$\frac{\bar{T}(s)}{Q(s)} = \frac{\frac{1}{wc_p}}{\frac{\rho V}{w} s + 1}$$

$$\tau = \frac{\rho V}{w}$$

This heating process can be represented by the following equation:

$$\left(\bar{T}_i(s) + \frac{Q(s)}{wc_p}\right)\left[\frac{1}{\tau s + 1}\right] = \bar{T}(s)$$

The block diagram for this process is:



Important note:

Regardless the type of input, the time constant is the same (the denominator of the transfer function is constant for all inputs, but the numerator depends on the relation between the input and the output).

Linearization

What is linearization?

Linearization is approximation of nonlinear equation in a linear form.

What is a linear system?

Linear system means that the dynamic behavior of the system is described by a linear differential equation.

What is the importance of linearization?

Most physical systems of practical importance are nonlinear. Characterization of a dynamic system by a transfer function can be done only for linear systems (those described by linear differential equations)

Techniques for linearization (Taylor series expansion)

Assume a nonlinear function $y = cx^2$

By means of a Taylor series expansion, the function may be expanded around the steady-state value x_s.