World Applied Sciences Journal 24 (7): 944-948, 2013
ISSN 1818-4952
© IDOSI Publications, 2013
DOI: 10.5829/idosi.wasj.2013.24.07.1041

Homotopy Perturbation Method and Elzaki Transform for Solving System of Nonlinear Partial Differential Equations

[1]Tarig. M. Elzaki and [2]J. Biazar

[1]Mathematics Department, Faculty of Sciences and Arts-Alkamil, King Abdulaziz University, Jeddah-Saudi Arabia. P.O. Box 110 Alkamil 21931
[2]Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O. Box 41335-1914, .P.C.4193822697, Guilan, Rasht, Iran

Submitted: Jul 21, 2013; **Accepted:** Aug 30, 2013; **Published:** Aug 29, 2013

Abstract: In this paper a new method called Elzaki transform homotopy perturbation method (ETHPM) is described to obtain the exact solution of nonlinear systems of partial differential equations. The proposed method was derived by combining Elzaki transform and homotopy perturbation method. This method was found to be more efficient and easy to solve linear and nonlinear differential equations. The presented examples in this paper have wide application in physics and engineering.

Key word: Elzaki transform · Homotopy perturbation method · Nonlinear systems of partial differential equations

INTRODUCTION

The exact solutions of nonlinear partial differential equations are difficult, therefore many various approximate methods have recently been developed such as homotopy perturbation method [1-5], Adomian's decomposition method [6], differential transform method [6-11] and projected differential transform method [8, 12] to solve linear and nonlinear differential equations. The homotopy perturbation method has the merits of simplicity and easy execution. Homotopy theory becomes a powerful mathematical tool; also homotopy theory can overcome the difficulties arising in calculation of Adomian's polynomials in Adomian's decomposition method.

Motivated and inspired by the on-going research in these areas, we consider a new method, which is called Elzaki transform [13-16] homotopy perturbation method (ETHPM). This method is suggested by combining the homotopy perturbation method and Elzaki transform, the suggested (ETHPM) provides the solution in a rapid convergent series which may lead the solution in a closed form. The fact that the proposed technique solves nonlinear problems without using so-called Adomian's polynomials is a clear advantage of this algorithm over the decomposition method.

In this paper we present some basic definitions of Elzaki transform and homotopy perturbation, also we present a reliable combination of homotopy perturbation method and Elzaki transform to obtain the solution of system of nonlinear partial differential equations.

Elzaki Transform: The basic definition of Elzaki transform is defined as follows, Elzaki transform of the function $f(t)$ is

$$E\left[f(t)\right]=v\int_0^\infty f(t)\,e^{-\frac{t}{v}}dt\,,\qquad t>0 \tag{1}$$

Where v is the factor of variable t.

Tarig M. Elzaki and Sailh M. Elzaki in [13-17], showed Elzaki transform [18-21], was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations.

To obtain Elzaki transform of partial derivative we use integration by parts and then we have:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{1}{v}T(x,v)-vf(x,0)$$

$$E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f(x,0)}{\partial t}$$

Corresponding Author: Tarig. M. Elzaki, Department of Mathematics, Faculty of Sciences and Arts-Alkamil, King Abdulaziz University, Jeddah-Saudi Arabia. P.O. Box 110 Alkamil 21931.

Proof: To obtain transforms of partial derivatives we use integration by parts as follows:

$$\mathrm{E}\left[\frac{\partial f}{\partial t}(x,t)\right]=\int_0^\infty v\frac{\partial f}{\partial t}e^{\frac{-t}{v}}dt=\lim_{p\to\infty}\int_0^p ve^{\frac{-t}{v}}\frac{\partial f}{\partial t}dt=$$

$$\lim_{p\to\infty}\left\{\left[ve^{\frac{-t}{v}}f(x,t)\right]_0^p-\int_0^p e^{\frac{-t}{v}}f(x,t)dt\right\}=\frac{T(x,v)}{v}-vf(x,0)$$

We assume that f is piecewise continuous and it is of exponential order.

To find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$ Le $\frac{\partial f}{\partial t}=g$, *then* we have:

$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\mathrm{E}\left[\frac{\partial g(x,t)}{\partial t}\right]=\mathrm{E}\frac{[g(x,t)]}{v}-vg(x,0)$ and

$\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f}{\partial t}(x,0)$

We can easily extend this result to the nth partial derivative by using mathematical induction.

Homotopy Perturbation Method: Let X and Y be the topological spaces. If f and g are continuous maps of the space X into Y, it is said that f is homotopic to g, if there is continuous map $F:X\times[0,1]\to Y$ such that $F(x,0)=f(x)$ and $F(x,1)=g(x)$, for each $x\in X$, then the map is called homotopy between f and g.

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(U)=0 \tag{2}$$

Where L is any differential operator, we define a convex homotopy $H(U,p)$by

$$H(U,p)=(1-p)F(U)+pL(U) \tag{3}$$

Where $F(U)$ is a functional operator with known solution V_0 which can be obtained easily. It is clear that, for

$$H(U,p)=0 \tag{4}$$

We have: $H(U,0)=F(U)$, $H(U,1)=L(U)$

In topology this show that $H(U,P)$ continuously traces an implicitly defined carves from a starting point $H(V_0,0)$ to a solution function $H(f,1)$. The HPM uses the embed ling parameter p as a small parameter and write the solution as a power series

$$U=U_0+pU_1+p^2U_2+p^3U_3+..... \tag{5}$$

If $p\to1$, then (5) corresponds to (3) and becomes the approximate solution of the form,

$$f=\lim_{p\to1}U=\sum_{i=0}^{\infty}U_i \tag{6}$$

We assume that (6) has a unique solution. The comparisons of like powers of p give solutions of various orders, for more details see [1-4].

Applications: In this section we apply the homotopy perturbation Elzaki transform method for solving system of nonlinear partial differential equations.

Example 2.1: Consider the following system of nonlinear partial differential equations

$$\begin{cases}U_t+VU_x+U=1\\V_t+UV_x-V=-1\end{cases} \tag{7}$$

With the initial conditions

$$U(x,0)=e^x\ ,\quad V(x,0)=e^{-x}$$

Taking Elzaki transform of equations (7) subject to the initial conditions, we have:

$$\begin{cases}E[U(x,t)]=v^2e^x-vE[VU_x+U-1]\\E[V(x,t)]=v^2e^{-x}-vE[V_xU-V+1]\end{cases} \tag{8}$$

The inverse Elzaki transform implies that:

$$\begin{cases}U(x,t)=e^x-E^{-1}\{vE[VU_x+U-1]\}\\V(x,t)=e^{-x}-E^{-1}\{vE[UV_x-V+1]\}\end{cases} \tag{9}$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases}\sum_{n=0}^{\infty}p^nU_n(x,t)=e^x-p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^nH_n(U)\right]\right]\right\}\\\sum_{n=0}^{\infty}p^nV_n(x,t)=e^{-x}-p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty}p^nH_n(V)\right]\right]\right\}\end{cases} \tag{10}$$

Where $H_n(U), H_n(V)$ are He's polynomials that represents the nonlinear terms.
Or

$$H_n(U):\ p\left[VU_x + U - 1\right] = 0$$
$$H_n(V):\ p\left[UV_x - V + 1\right] = 0$$

Where $U = U_0 + pU_1 + p^2U_2 + \ldots,$
$V = V_0 + pV_1 + p^2V_2 + \ldots$

The first few components of He's polynomials, are given by

$$\begin{cases} H_0(U) = V_0U_{0x} + U_0 - 1 \\ H_0(V) = U_0V_{0x} - V_0 + 1 \\ H_1(U) = V_0U_{1x} + V_1U_{0x} + U_1 \\ H_1(V) = U_0V_{1x} + U_1V_{0x} + V_1 \\ \quad . \\ \quad . \\ \quad . \end{cases}$$

Comparing the coefficients of the same powers of *p*, we get:

$$p^0: U_0(x,t) = e^x\ ,\quad V_0(x,t) = e^{-x},\ H_0(U) = e^x,\ H_0(V) = -e^{-x}$$
$$p^1: U_1(x,t) = -E^{-1}\left\{vE\left[H_0(U)\right]\right\} = -te^x\ ,\ H_1(U) = -te^x$$
$$p^1: V_1(x,t) = -E^{-1}\left\{vE\left[H_0(V)\right]\right\} = te^{-x}\ ,\ H_1(V) = -te^{-x}$$
$$p^2: U_2(x,t) = -E^{-1}\left\{vE\left[H_1(U)\right]\right\} = \frac{t^2}{2}e^x$$
$$p^2: V_2(x,t) = -E^{-1}\left\{vE\left[H_1(V)\right]\right\} = \frac{t^2}{2}e^{-x}$$
$$U_3(x,t) = -\frac{t^3}{3!}e^x\quad ,\quad U_3(x,t) = \frac{t^3}{3!}e^{-x}$$
. . .
. . .
. . .

Therefore the solutions $U(x,t)\ , V(x,t)$ are given by:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + \ldots. = e^{x-t}$$
$$V(x,t) = V_0(x,t) + V_1(x,t) + V_2(x,t) + \ldots. = e^{t-x}$$

Example 2.2: Consider the following system of nonlinear partial differential equations

$$\begin{cases} U_t + V_xW_y - V_yW_x = -U \\ V_t + W_xU_y + W_yU_x = V \\ W_t + U_xV_y + U_yV_x = W \end{cases} \tag{11}$$

With the initial conditions

$$U(x,y,0) = e^{x+y},$$
$$V(x,y,0) = e^{x-y},$$
$$W(x,y,0) = e^{-x+y}$$

Taking Elzaki transform of equations (11) subject to the initial conditions and then take the inverse Elzaki transform we have:

$$\begin{cases} U(x,y,t) = e^{x+y} + E^{-1}\left\{vE\left[V_yW_x - V_xW_y - U\right]\right\} \\ V(x,y,t) = e^{x-y} + E^{-1}\left\{vE\left[V - W_xU_y - W_yU_x\right]\right\} \\ W(x,y,t) = e^{-x+y} + E^{-1}\left\{vE\left[W - U_xV_y - U_yV_x\right]\right\} \end{cases} \tag{12}$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^nU_n(x,y,t) = e^{x+y} + p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^nH_n(U)\right]\right]\right\} \\ \sum_{n=0}^{\infty} p^nV_n(x,y,t) = e^{x-y} + p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^nH_n(V)\right]\right]\right\} \\ \sum_{n=0}^{\infty} p^nW_n(x,y,t) = e^{-x+y} + p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^nH_n(W)\right]\right]\right\} \end{cases} \tag{13}$$

Where $H_n(U), H_n(V), H_n(W)$ are He's polynomials that represents the nonlinear terms.
Or

$$H_n(U):\ p\left[V_yW_x - V_xW_y - U\right] = 0$$
$$H_n(V):\ p\left[V - W_xU_y - W_yU_x\right] = 0$$
$$H_n(W):\ p\left[W - U_xV_y - U_yV_x\right] = 0$$

Where $U = U_0 + pU_1 + p^2U_2 + \ldots,$
$V = V_0 + pV_1 + p^2V_2 + \ldots,$
$W = W_0 + pW_1 + p^2W_2 + \ldots$

Comparing the coefficients of the same powers of p, we get:

$$p^0 : U_0(x,y,t) = e^{x+y},\ V_0(x,y,t) = e^{x-y},\ W_0(x,y,t) = e^{-x+y}$$
$$H_0(U) = -e^{x+y},\ H_0(V) = e^{x-y},\ H_0(W) = e^{-x+y}$$
$$p^1 : U_1(x,y,t) = E^{-1}\{vE[H_0(U)]\} = -te^{x+y},\ H_1(U) = te^{x+y}$$
$$p^1 : V_1(x,y,t) = E^{-1}\{vE[H_0(V)]\} = te^{x-y},\ H_1(V) = te^{x-y}$$
$$p^1 : W_1(x,y,t) = E^{-1}\{vE[H_0(W)]\} = te^{-x+y},\ H_1(W) = te^{-x+y}$$
$$p^2 : U_2(x,y,t) = E^{-1}\{vE[H_1(U)]\} = \frac{t^2}{2}e^{x+y}$$
$$p^2 : V_2(x,y,t) = E^{-1}\{vE[H_1(V)]\} = \frac{t^2}{2}e^{x-y}$$
$$p^2 : W_2(x,y,t) = E^{-1}\{vE[H_1(W)]\} = \frac{t^2}{2}e^{-x+y}$$
. . .
. . .

Then the solutions $U(x,y,t), V(x,y,t), W(x,y,t)$ are given by:

$$U(x,y,t) = U_0(x,y,t) + U_1(x,y,t) + U_2(x,y,t) + ... = e^{x+y-t}$$
$$V(x,y,t) = V_0(x,y,t) + V_1(x,y,t) + V_2(x,y,t) + ... = e^{x-y+t}$$
$$W(x,y,t) = W_0(x,y,t) + W_1(x,y,t) + W_2(x,y,t) + ... = e^{-x+y+t}$$

Example 2.3: Consider the following Coupled Burger's system

$$\begin{cases} U_t - U_{xx} - 2UU_x + (UV)_x = 0 \\ V_t - V_{xx} - 2VV_x + (UV)_x = 0 \end{cases} \quad (14)$$

With the initial conditions

$$U(x,0) = \sin x\ ,\quad V(x,0) = \sin x$$

Using the same method in example 1 to obtain:

$$\begin{cases} U(x,t) = \sin x + E^{-1}\{vE[U_{xx} + 2UU_x - UV_x - U_xV]\} \\ V(x,t) = \sin x + E^{-1}\{vE[V_{xx} + 2VV_x - UV_x - U_xV]\} \end{cases} \quad (15)$$

Now applying the homotopy perturbation method, we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n U_n(x,t) = \sin x + p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right\} \\ \sum_{n=0}^{\infty} p^n V_n(x,t) = \sin x + p\left\{E^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n H_n(V)\right]\right]\right\} \end{cases} \quad (16)$$

Where $H_n(U), H_n(V)$ are He's polynomials that represents the nonlinear terms.
Or

$$H_n(U):\ p[U_{xx} + 2UU_x - UV_x - U_xV] = 0$$
$$H_n(V):\ p[V_{xx} + 2VV_x - UV_x - U_xV] = 0$$

Where $U = U_0 + pU_1 + p^2U_2 + ...,\ V = V_0 + pV_1 + p^2V_2 + ...$

The first few components of He's polynomials, are given by

$$\begin{cases} H_0(U) = U_{0xx} + 2U_0U_{0x} - U_0V_{0x} - U_{0x}V_0 \\ H_0(V) = V_{0xx} + 2V_0V_{0x} - U_0V_{0x} - U_{0x}V_0 \\ H_1(U) = U_{1xx} + 2U_0U_{1x} + 2U_1U_{0x} - U_0V_{1x} - U_1V_{0x} - U_{0x}V_1 - U_{1x}V_0 \\ H_1(V) = V_{1xx} + 2V_0V_{1x} + 2V_1V_{0x} - U_0V_{1x} - U_1V_{0x} - U_{0x}V_1 - U_{1x}V_0 \\ . \\ . \\ . \end{cases}$$

Comparing the coefficients of the same powers of p, we get:

$$p^0 : U_0(x,t) = \sin x\ ,\quad V_0(x,t) = \sin x,\ H_0(U) = -\sin x\ ,\ H_0(V) = -\sin x$$
$$p^1 : U_1(x,t) = E^{-1}\{vE[H_0(U)]\} = -t\sin x\ ,\ H_1(U) = t\sin x$$
$$p^1 : V_1(x,t) = E^{-1}\{vE[H_0(V)]\} = -t\sin x\ ,\ H_1(V) = t\sin x$$
$$p^2 : U_2(x,t) = E^{-1}\{vE[H_1(U)]\} = \frac{t^2}{2}\sin x$$
$$p^2 : V_2(x,t) = E^{-1}\{vE[H_1(V)]\} = \frac{t^2}{2}\sin x$$
$$U_3(x,t) = -\frac{t^3}{3!}\sin x\quad ,\quad U_3(x,t) = -\frac{t^3}{3!}\sin x$$
. . .
. . .

Therefore the solutions $U(x,t), V(x,t)$ are given by:

$$U(x,t) = U_0(x,t) + U_1(x,t) + U_2(x,t) + ... = e^{-t}\sin x$$
$$V(x,t) = V_0(x,t) + V_1(x,t) + V_2(x,t) + ... = e^{-t}\sin x$$

CONCLUSION

In this paper, we combined homotopy perturbation method and Elzaki transform for solving nonlinear system of nonlinear partial differential equations. The proposed method is successfully implemented by using the initial

conditions only. Finally, we conclude that Elzaki transform homotopy perturbation method considered as a nice refinement in existing numerical techniques.

REFERENCES

1. Sweilam, N.H. and M.M. Khader, 2009. Exact Solutions of some Capled nonlinear partial differential equations using the homotopy perturbation method. Computers and Mathematics with Applications. 58: 2134-2141.
2. Sharma, P.R. and Giriraj Methi, 2010. Applications of Homotopy Perturbation method to Partial differential equations. Asian Journal of Mathematics and Statistics 4(3): 140-150.
3. Jafari, M.A. and A. Aminataei, 2010. Improved Homotopy Perturbation Method. International Mathematical Forum, 5(32): 1567-1579.
4. Jagdev Singh, Devendra and Sushila, 2011. Homotopy Perturbation Sumudu Transform Method for Nonlinear Equations. Adv. Theor. Appl. Mech., 4(4):165-175.
5. Tarig M. Elzaki and Eman M.A. Hilal, 2012. Homotopy Perturbation and Elzaki Transform for Solving Nonlinear Partial Differential Equations, Mathematical Theory and Modeling, 2(3): 33-42.
6. Abdel-Hassan, I.H., 2004. Differential transformation technique for solving higher-order initial value problem. Applied Math. Comput, 154: 299-311.
7. Ayaz, F., 2004. Solution of the system of differential equations by differential transforms method.Applied math. Comput., 147: 54767.
8. Bongsoo Jang, 2009. Solving linear and nonlinear initial value problems by the projected differential transform method. Ulsan National Institute of Science and Technology (UNIST), Banyeon-ri-100, Ulsan 689-798 korea. Compu. Phys. Communication.
9. Hchen, C. and S.H. Ho, 1999. Solving Partial differential by two dimension differential transform method, APPL. Math. Comput., 106: 171-179.
10. Fatma Ayaz, 2004. Solution of the system of differential equations by differential transforms method. Applied. Math. Comput., 147: 547-567.
11. Kanglgil, F. and F.A. Yaz, 2008. Solitary wave Solution for kdv and M kdv equations by differential transform method, chaos solutions and fractions do1:10.1016/j. Chaos.02.009.
12. Tarig M. Elzaki and Eman M.A. Hilal, 2012. Solution of Linear and Nonlinear Partial Differential Equations Using Mixture of Elzaki Transform and the Projected Differential Transform Method, Mathematical Theory and Modeling, 2(4): 50-59.
13. Tarig M. Elzaki, 2011. The New Integral Transform "Elzaki Transform" Global Journal of Pure and Applied Mathematics, 1: 57-64.
14. Tarig M. Elzaki and Salih M. Elzaki, 2011. Application of New Transform "Elzaki Transform" to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, 1: 65-70.
15. Tarig M. Elzaki and Salih M. Elzaki, 2011. On the Connections between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, 6(1): 1-11.
16. Tarig M. Elzaki and Salih M. Elzaki, 2011. On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. 6(1): 13-18.
17. Tarig M. Elzaki, 2009. Existence and Uniqueness of Solutions for Composite Type Equation, Journal of Science and Technology. pp: 214-219.
18. Kilicman, A. and H.E. Gadain, 2009. An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math., 30(3): 214-223.
19. Zhang, J., 2007. A Sumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3): 303-313.
20. Hassan Eltayeb and Adem Kilicman, 2010. A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, 4(22): 1089-1098.
21. Hassan ELtayeh and Adem Kilicman, 2010. On Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, 4(3): 123-132.

Appendix:

Elzaki transform of some Functions

$f(t)$	$E[f(t)]=T(u)$
1	v^2
t	v^3
t^n	$n!\ v^{n+2}$
e^{at}	$\frac{v^2}{1-av}$
$sinat$	$\frac{av^3}{1+a^2v^2}$
$cosat$	$\frac{v^2}{1+a^2v^2}$