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On the new integral transform "Tarig Transform" and systems of ordinary differential equations

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ABSTRACT
In this paper we introduce some properties and definition of the new transform, called Tarig transform. Farther, we use this transform to solve the linear systems of ordinary differential equations.

Introduction

In the literature there are numerous integral transforms and widely used in physics, astronomy as well as in engineering .The integral transform method is also an efficient method to solve the system of ordinary differential equations. Recently, Tarig M. ELzaki introduced a new transform and named as Tarig transform which is defined by the following formula.

$$T\left[f(t)\right]=F(u)=\frac{1}{u}\int_0^\infty f(t)e^{\frac{-t}{u^2}}dt \quad , \quad t\geq 0 \;\; , u\neq 0$$

And applied this new transform to the solution of system of ordinary differential equations. In this study, our purpose is to show the applicability of this interesting new transform and its efficiency in solving the linear system of ordinary differential equations.

Theorem 1:

If $T\left[f(t)\right]=F(u)$, then $T\left[f(at)\right]=\frac{1}{a}F(au)$

Proof:

We have $T\left[f(at)\right]=\int_0^\infty e^{\frac{-t}{u}}f(aut)dt$. Let $x=at$, then we get:

$$T\left[f(at)\right]=\frac{1}{a}\int_0^\infty e^{\frac{-x}{au}}f(ux)dx=\frac{1}{a}F(au)$$

Theorem 2:

If a,b are any constants and $f(t)$ and $g(t)$ are any functions, then:

$$T\left[af(t)+bg(t)\right]=aT\left(f(t)\right)+bT\left(g(t)\right)$$

Proof:

$T\left[f(t)\right]=\int_0^\infty f(ut)e^{\frac{-t}{u}}dt.$ And $T\left[g(t)\right]=\int_0^\infty g(ut)e^{\frac{-t}{u}}dt.$ then:

$$T\left[af(t)+bg(t)\right]=\int_0^\infty e^{\frac{-t}{u}}\left[af(ut)+bg(ut)\right]dt$$

Theorem 3:

If $T\left[f(t)\right]=F(u)$ then:

$$(i)\; T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0) \quad (ii)\; T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

$$(iii)\; T\left[f^{(n)}(t)\right]=\frac{F(u)}{u^{2n}}-\sum_{i=1}^{n}u^{2(i-n)-1}f^{(i-1)}(0)$$

Proof:

$(i)\; T\left[f'(t)\right]=\frac{1}{u}\int_0^\infty f'(t)e^{\frac{-t}{u^2}}dt$. Integrating by parts to find that:

$$T\left[f'(t)\right]=\frac{1}{u}\left\{-f(0)+\frac{1}{u}T\left[f(t)\right]\right\}$$

And $$T\left[f'(t)\right]=\frac{F(u)}{u^2}-\frac{1}{u}f(0)$$

(ii) By (i) $T\left[G'(t)\right]=\frac{T\left[G(t)\right]}{u^2}-\frac{1}{u}G(0)$. Let $G(t)=f'(t)$. then:

$$T\left[f''(t)\right]=\frac{T\left(f'(t)\right)}{u^2}-\frac{1}{u}f'(0)=\frac{1}{u^2}\left[\frac{F(u)}{u^2}-\frac{1}{u}f(0)\right]-\frac{1}{u}f'(0) \quad and$$

$$T\left[f''(t)\right]=\frac{F(u)}{u^4}-\frac{1}{u^3}f(0)-\frac{1}{u}f'(0)$$

The generalization to nth order derivatives in (iii) can be proved by using mathematical induction.

Theorem 4:

If $T\left[f(t)\right]=G(u)$ and $L\left[f(t)\right]=F(s)$ then:

$G(u)=\frac{F\left(\frac{1}{u^2}\right)}{u}$ where $F(s)$ is the Laplace transform of $f(t)$.

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Proof:

$T\left[f(t)\right]=\int_0^\infty f(ut)e^{\frac{-t}{u}}dt=G(u)$ Let $w=ut$, then we have:

$$G(u)=\int_0^\infty f(w)e^{\frac{-w}{u^2}}\frac{dw}{u}=\frac{F\left(\frac{1}{u^2}\right)}{u}$$

System of Ordinary Differential Equations

Tarig transform method is very effective for the solution of the response of a linear system governed by an ordinary differential equation to the initial data and | or to an external disturbance (or external input function). More precisely, we seek the solution of a linear system for its state at subsequent time $t>0$ due to the initial state at $t=0$ and / or to the disturbance applied for $t>0$.

Example 1:

Consider the following system,

$$\begin{cases}\frac{dx(t)}{dt}+\frac{dy(t)}{dt}+x(t)+y(t)=1\\ \frac{dy(t)}{dt}=2x(t)+y(t)\end{cases} \qquad (1)$$

With the initial conditions:

$$x(0)=0\ ,\quad y(0)=1 \qquad (2)$$

Solution:-

Using Tarig transform into eq(1) we have:

$$\begin{cases}\frac{\bar{x}(u)}{u^2}-\frac{x(0)}{u}+\frac{\bar{y}(u)}{u^2}-\frac{y(0)}{u}+\bar{x}(u)+\bar{y}(u)=u\\ \frac{\bar{y}(u)}{u^2}-\frac{y(0)}{u}=2\bar{x}(u)+\bar{y}(u)\end{cases} \qquad (3)$$

Where $\bar{x}$, $\bar{y}$ are Tarig transform of x, y. Substituting eq(2) into eq(3) we get:

$$\begin{cases}\left(1+u^2\right)\bar{x}(u)+\left(1+u^2\right)\bar{y}(u)=u^3+u\\ \left(1-u^2\right)\bar{y}(u)-2u^2\bar{x}(u)=u\end{cases} \qquad (4)$$

And

$$\bar{y}(u)=\frac{2u^5+3u^3+u}{\left(1+u^2\right)^2}=2u-\frac{u}{1+u^2}\ \Rightarrow\ y(t)=F^{-1}\left[2u-\frac{u}{1+u^2}\right]=2-e^{-t}$$

Substituting $y(t)$ into eq (1) we get:

$$x(t)=\frac{1}{2}\left[\frac{dy(t)}{dt}-y(t)\right]=-1+e^{-t}$$

Example 2:

Consider the following non – homogenous differential system,

$$\begin{cases}\frac{dx(t)}{dt}-z(t)=-\cos t\\ \frac{dy(t)}{dt}-z(t)=-e^t\\ \frac{dz(t)}{dt}=x(t)-y(t)\end{cases} \qquad (5)$$

With the initial conditions:

$$x(0)=1\ ,\ y(0)=0\ ,\ z(0)=2 \qquad (6)$$

Solution:

Taking Tarig transform of eq(5)we have:

$$\begin{cases}\frac{\bar{x}(u)}{u^2}-\frac{x(0)}{u}-\bar{z}(u)=-\frac{u}{1+u^4}\\ \frac{\bar{y}(u)}{u^2}-\frac{y(0)}{u}-\bar{z}(u)=-\frac{u}{1-u^2}\\ \frac{\bar{z}(u)}{u^2}-\frac{z(0)}{u}=\bar{x}(u)-\bar{y}(u)\end{cases} \qquad (7)$$

Substituting eq (6) into eq(7) we get:

$$\begin{cases}\bar{x}(u)-u^2\bar{z}(u)=u-\frac{u^3}{1+u^4}\\ \bar{y}(u)-u^2\bar{z}(u)=\frac{-u^2}{1-u^2}\\ \bar{z}(u)-2u=u^2\bar{x}(u)-u^2\bar{y}(u)\end{cases} \qquad (8)$$

Or $\begin{cases}\bar{x}(u)-\bar{y}(u)=u-\frac{u^3}{1+u^4}+\frac{u^3}{1-u^2}\\ \bar{z}(u)-2u=u^2\bar{x}(u)-u^2\bar{y}(u)\end{cases}$ and,

$$z(u)=2u+u^3-\frac{u^5}{1+u^4}+\frac{u^5}{1-u^2}\quad and\quad z(u)=\frac{u}{1+u^4}+\frac{u}{1-u^2},\ Then:$$

$$z(t)=F^{-1}\left[\frac{u}{1+u^4}+\frac{u}{1-u^2}\right]=e^t+\cos t$$

Substituting $z(t)$ into eq (5) we have:

$$\frac{dx(t)}{dt}=z(t)-\cos t=e^t\ \Rightarrow\ x(t)=e^t+c$$

By using $x(0)=1$ we get $c=0$ and $x(t)=e^t$. Substituting $x(t)$ *and* $z(t)$ in to eq (5) yields,

$$y(t)=x(t)-\frac{dz}{dt}=\sin t$$

Example 3:

Consider the following second order system of differential equation,

$$\begin{cases}\frac{d^2x(t)}{dt^2}+y(t)=1\\ \frac{d^2y(t)}{dt^2}+x(t)=0\end{cases} \qquad (9)$$

With initial conditions:

$$x(0)=x'(0)\ =\ y(0)=y'(0)=0 \qquad (10)$$

Solution:

Applying Tarig transform to eq (9), we get:

$$\begin{cases}\frac{\bar{x}(u)}{u^4}-\frac{x(0)}{u^3}-\frac{x'(0)}{u}+\bar{y}(u)=u\\ \frac{\bar{y}(u)}{u^4}-\frac{y(0)}{u^3}-\frac{y'(0)}{u}+\bar{x}(u)=0\end{cases} \qquad (11)$$

Substituting eq(10) into eq(11) we have:

$$\begin{cases} \bar{x}(u) + u^4 \bar{y}(u) = u^5 \\ u^4 \bar{x}(u) + \bar{y}(u) = 0 \end{cases} \quad \text{Or}$$

$$\left(1-u^8\right)\bar{x}(u) = u^5 \Rightarrow \bar{x}(u) = \frac{u^5}{1-u^8} = \frac{u^5}{\left(1-u^2\right)\left(1+u^2\right)\left(1+u^4\right)}$$

$$\bar{x}(u) = \frac{u}{4(1-u^2)} + \frac{u}{4(1+u^2)} - \frac{u}{2(1+u^4)} \Rightarrow x(t) = F^{-1}\left[\frac{u}{4(1-u^2)} + \frac{u}{4(1+u^2)} - \frac{u}{2(1+u^4)}\right]$$

$$x(t) = \frac{1}{4}e^t + \frac{1}{4}e^{-t} - \frac{1}{2}\cos t$$

Substituting $x(t)$ in to eq(9) we get:

$$y(t) = 1 - \frac{d^2x(t)}{dt^2} = 1 - \frac{1}{4}e^t - \frac{1}{4}e^{-t} - \frac{1}{2}\cos t$$

Example 4:

Consider the following second order system of differential equation,

$$\begin{cases} 2\dfrac{d^2x(t)}{dt^2} + \dfrac{d^2y(t)}{dt^2} = 6 + 2e^{-t} \\ \dfrac{d^2x(t)}{dt^2} + \dfrac{dx(t)}{dt} - \dfrac{dy(t)}{dt} = 0 \end{cases} \quad (12)$$

With the initial conditions:

$$x(0) = 1, \ x'(0) = -1, \ y(0) = 0, \ y'(0) = 2 \quad (13)$$

Solution:

By using Tarig transform into eq (12) we get:

$$\begin{cases} \dfrac{2\bar{x}(u)}{u^4} - \dfrac{2x(0)}{u^3} - \dfrac{2x'(0)}{u} + \dfrac{\bar{y}(u)}{u^4} - \dfrac{y(0)}{u^3} - \dfrac{y'(0)}{u} = 6u + \dfrac{2u}{1+u^2} \\ \dfrac{\bar{x}(u)}{u^4} - \dfrac{x(0)}{u^3} - \dfrac{x'(0)}{u} + \dfrac{\bar{x}(u)}{u^2} - \dfrac{x(0)}{u} - \dfrac{\bar{y}(u)}{u^2} + \dfrac{y(0)}{u} = 0 \end{cases} \quad (14)$$

Substituting eq(13) in to eq(14) we have,

$$\begin{cases} 2\bar{x}(u) + \bar{y}(u) = 6u^5 + \dfrac{2u^5}{1+u^2} + 2u \\ \left(1+u^2\right)\bar{x}(u) - u^2\bar{y}(u) = u \end{cases} \quad \text{Or}$$

$$\left(1+3u^2\right)\bar{x}(u) = 6u^7 + \frac{2u^7}{1+u^2} + 2u^3 + u$$

$$\left(1+3u^2\right)\bar{x}(u) = \frac{6u^9 + 8u^7 + 2u^5 + 3u^3 + u}{\left(1+u^2\right)}$$

$$\bar{x}(u) = \frac{2u^5\left(3u^4 + 4u^2 + 1\right) + u\left(3u^2 + 1\right)}{\left(1+u^2\right)\left(1+3u^2\right)}$$

$$\bar{x}(u) = 2u^5 + \frac{u}{1+u^2}, \ Then: \ x(t) = F^{-1}\left[2u^5 + \frac{u}{1+u^2}\right] = t^2 + e^{-t}$$

Substituting $x(t)$ into eq (12) yields:

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} + \frac{dx}{dt} = 2 + 2t \quad \Rightarrow \quad y(t) = 2t + t^2 + c$$

By using, $y(0) = 0$ we get, $c = 0$, and $y(t) = 2t + t^2$

Conclusion

Tarig transform provides powerful method for analyzing derivatives. It is heavily used to solve differential equations and system of ordinary differential equations.

References

[1] Tarig M. Elzaki, The New Integral Transform "Elzaki Transform" Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 57-64.

[2] Tarig M. Elzaki & Salih M. Elzaki, Application of New Transform "Elzaki Transform" to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 65-70.

[3] Tarig M. Elzaki & Salih M. Elzaki, On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1(2011),pp. 1-11.

[4] Tarig M. Elzaki & Salih M. Elzaki, On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6, Number 1(2011),pp. 13-18.

[5] Tarig M. Elzaki, Adem Kilicman, Hassan Eltayeb. On Existence and Uniqueness of Generalized Solutions for a Mixed-Type Differential Equation, Journal of Mathematics Research, Vol. 2, No. 4 (2010) pp. 88-92.

[6] Tarig M. Elzaki, Existence and Uniqueness of Solutions for Composite Type Equation, Journal of Science and Technology, (2009). pp. 214-219.

[7] Lokenath Debnath and D. Bhatta. Integral transform and their Application second Edition, Chapman & Hall /CRC (2006).

[8] A.Kilicman and H.E.Gadain. An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math.30 (3) (2009), pp.214-223.

[9] J. Zhang, Asumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3) (2007), pp – 303-313.

[10] Hassan Eltayeb and Adem kilicman, A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4,2010, no.22,1089-1098.

[11] Kilicman A. & H. ELtayeb. A note on Integral transform and Partial Differential Equation, Applied Mathematical Sciences, 4(3) (2010), PP.109-118.

[12] Hassan ELtayeh and Adem kilicman, on Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, Vol, 4, 2010, no.3, 123-132.

Appendix Tarig Transform of Some Functions

S.N0.	$f\left(t\right)$	$F\left(u\right)$
1	1	u
2	t	u^3
3	e^{at}	$\frac{u}{1-au^2}$
4	t^n	$n!\,u^{2n+1}$
5	t^a	$\Gamma(a+1)u^{2a+1}$
6	$\sin at$	$\frac{au^3}{1+a^2u^4}$
7	$\cos at$	$\frac{u}{1+a^2u^4}$
8	$\sinh at$	$\frac{au^3}{1-a^2u^4}$
9	$\cosh at$	$\frac{u}{1-a^2u^4}$
10	$H\left(t-a\right)$	$ue^{\frac{-a}{u^2}}$
11	$\delta\left(t-a\right)$	$\frac{1}{u}e^{\frac{-a}{u^2}}$
12	te^{at}	$\frac{u^3}{\left(1-au^2\right)^2}$
13	$e^{at}\sin bt$	$\frac{bu^3}{\left(1-au^2\right)^2+b^2u^4}$
14	$\int_0^t f(\omega)d\omega$	$u^2F\left(u\right)$
15	$\left(f*g\right)(t)$	$u\,M\left(u\right)N\left(u\right)$
16	$e^{at}\cos bt$	$\frac{u\left(1-au^2\right)}{\left(1-au^2\right)^2+b^2u^4}$
17	$e^{at}\cosh bt$	$\frac{u\left(1-au^2\right)}{\left(1-au^2\right)-b^2u^4}$
18	$e^{at}\sin bt$	$\frac{bu^3}{\left(1-au^2\right)^2-b^2u^4}$
19	$t\sin at$	$\frac{2au^5}{\left(1+a^2u^4\right)^2}$
20	$t\cos at$	$\frac{u^3\left(1-a^2u^4\right)}{\left(1+a^2u^4\right)^2}$
21	$t\sinh at$	$\frac{2au^5}{\left(1-a^2u^4\right)^2}$
22	$t\cosh at$	$\frac{u^3\left(1+a^2u^4\right)}{\left(1-a^2u^4\right)^2}$