On the ELzaki Transform and Ordinary Differential Equation with Variable Coefficients

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Abstract

The ELzaki transform, whose fundamental properties are presented in this paper, is little known and not widely used. Here the ELzaki transform is used to solve ordinary differential equation with variable coefficients without resorting to a new frequency domain.

Keyword: Elzaki transform - differential equations

Introduction

A new integral transform, called the ELzaki transform, is defined for functions of exponential order. We consider functions in the set A, defined by

\[ A = \left\{ f(t) : \exists M, \ k_1 \ and \ k_2 > 0: \left| f(t) \right| < Me^{jk_2}, \ if \ t \in (-1)^j \times [0, \infty) \right\} \] (1)

For a given function in the set A, the constant M must be finite, while \( k_1 \) and \( k_2 \) may be infinite. The variable \( v \) in the ELzaki transform is used to factor the variable \( t \) in the argument of the function \( f \). Specifically, for \( f(t) \) in A. The ELzaki transform is defined by:

\[ E[f(t)] = \int_0^\infty f(t) e^{-vt} dt = T(v), \ v \in (-k_1, k_2) \] (2)

The next theorem is very useful in the study of differential equations having non-constant coefficients.

Theorem I

If ELzaki transform of the function \( f(t) \) given by \( E[f(t)] = T(v) \), then:
\[(i) \ E[f'(t)]=v^2 \frac{d}{dv}\left[ \frac{T(v)}{v} - v f(0) \right] - v \left[ \frac{T(v)}{v} - v f(0) \right] \]

\[(ii) \ E[t^2 f'(t)]=v^4 \frac{d^2}{dv^2}\left[ \frac{T(v)}{v} - v f(0) \right] \]

\[(iii) E[t f^*(t)]=v^2 \frac{d}{dv}\left[ \frac{T(v)}{v^2} - f(0) - v f'(0) \right] - v \left[ \frac{T(v)}{v^2} - f(0) - v f'(0) \right] \]

\[(iv) \ E[t^2 f^*(t)]=v^4 \frac{d^2}{dv^2}\left[ \frac{T(v)}{v^2} - f(0) - v f'(0) \right] \]

**Proof**

To prove (i) we use the following formula

\[E[tf(t)]=v^2 \frac{d}{dv}T(v) - vT(v)\]

We have

\[
\frac{d}{dv}T(v) = T'(v) = \frac{d}{dv} \int_0^\infty ve^{-\frac{v}{v}} f(t) dt
\]

\[
= \int_0^\infty \frac{d}{dv} ve^{-\frac{v}{v}} f(t) dt = \int_0^\infty \frac{1}{v} e^{-\frac{v}{v}} (tf(t)) dt
\]

\[
+ \int_0^\infty e^{-\frac{v}{v}} f(t) dt = \frac{1}{v^2} E(tf(t)) + \frac{1}{v} E(f(t))
\]

Then we have

\[E(tf(t)) = v^2 \frac{d}{dv}T(v) - vT(v)\]

Now we put \(f(t) = f'(t)\) we have

\[E(tf'(t)) = v^2 \frac{d}{dv}\left[ E(f'(t)) \right] - v \ E[f'(t)] \]

\[= v^2 \frac{d}{dv}\left[ \frac{T(v)}{v} - vf(0) \right] - v \left[ \frac{T(v)}{v} - vf(0) \right] \]

The proof of (ii), (iii) and (iv) are similar to the Proof of (i).
Now we apply the above theorem to find ELzaki transform for some differential equations:

**Example I**
Solve the differential equation:

\[ y'' + ty' - y = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad t > 0 \quad (3) \]

By using ELzaki transform into equation (3) and the theorem I, we have

\[
E(y) - y(0) - vy'(0) + v^2 \frac{d}{dv} \left[ E(v) - vy(0) \right] - v \left[ E(y) - vy(0) \right] - E(y) = 0
\]

Using the initial conditions we get

\[
E'(y) + \left( \frac{1}{v} - \frac{3}{v^2} \right) E(y) = 1
\]

This is a linear differential equation for unknown function \( E \), have the solution in the form

\[
E(y) = v^3 + C \cdot v^3 \cdot e^{2v^2} \quad \text{and} \quad C = 0,
\]

then:

\[
E(y) = v^3.
\]

By using the inverse ELzaki transform we obtain the solution in the form of

\[ y = t \]

**Example II**
Consider the non constant coefficient differential equation in the form of

\[ ty'' + (1 - 2t)y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 2 \]

By using ELzaki transform and apply the initial condition we have

\[
\frac{E'(y)}{E(y)} = \frac{2}{v} + \frac{2}{1 - 2v}
\]

Then we obtain the solution

\[
E(y) = \frac{Cv^2}{1 - 2v} \quad \text{where} \quad C \text{ is constant}
\]

\[ E(y) = T(v) \Rightarrow T(0) = 1 \]

Then the constant \( C = 1 \) and

\[
E(y) = \frac{v^2}{1 - 2v}
\]

By taking inverse ELzaki transform we have:

\[ y = e^{2t} \]
Example III
Consider the following equation:

\[ t^2 y' + 2ty = \sinh t, \quad y(0) = \frac{1}{2} \]  

(4)

Solution
By taking ELzaki transform of equation (4) we have,

\[ v^2 \frac{d^2}{dv^2} \left[ \frac{\bar{y}}{v} - v \bar{y}(0) \right] + 2v^2 \frac{d}{dv} \bar{y} - 2v^2 \bar{y} = \frac{v^3}{1-v^2} \]

Where that \( \bar{y} \equiv E[y(t)] \)

We can write the last equation in the form

\[ v^3 \bar{y}'' = \frac{v^3}{1-v^2} \text{ or } \bar{y}'' = \sum_{i=0}^{\infty} v^{2i} \]

The solution of this equation is

\[ \bar{y} = c_1 + c_2 v + \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)} \]  

(5)

Substituting the condition into Eq (5) we get:

\[ \bar{y} = \sum_{i=0}^{\infty} \frac{v^{2i+2}}{(2i+1)(2i+2)} \]  

(6)

By taking inverse ELzaki transform to equation (6) we obtain the solution as follow,

\[ y = \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i+2)!} = \frac{1}{2!} + \frac{t^2}{4!} + \frac{t^4}{6!} + \frac{t^6}{8!} + \ldots \]

Example IV
Let us consider the differential equation

\[ t^2 y'' + 4ty' + 2y = 12t^2 \]  

(7)

With the initial conditions:

\[ y(0) = y'(0) = 0 \]  

(8)

Now we apply ELzaki transform to equation (7) we obtain,
By simplifying above equation, we have \( \dddot{y} = 24 v^2 \)
The solution of this equation can be written in the form.
\[
\ddot{y} = 2 v^4 + C_1 v + C_2
\]  
(9)

By substituting the initial condition (8) into equation (9) we get,
\[
\ddot{y} = 2 v^4
\]  
(10)

By using inverse ELzaki transform for equation (10) we obtain the solution of equation (7)
\[
y = t^2
\]

**Conclusion**
Application of the ELzaki transform to Solution of ordinary differential equation with variable Coefficients has been demonstrated.

**References**


