

The New Integral Transform "ELzaki Transform"

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Abstract

In this paper a new integral transform namely Elzaki transform was applied to solve linear ordinary differential equations with constant coefficients.

Keywords: Elzaki transform- Differential Equations.

Introduction

ELzaki Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Elzaki transform and its fundamental properties.

Elzaki transform was introduced by Tarig ELzaki to facilitate the process of solving ordinary and partial differential equations in the time domain.

Typically, Fourier, Laplace and Sumudu transforms are the convenient mathematical tools for solving differential equations,

Also ELzaki transform and some of its fundamental properties are used to solve differential equations.

A new transform called the ELzaki transform defined for function of exponential order we consider functions in the set A defined by:

$$A = \left\{ f(t) : \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{k_j}}, \text{ if } t \in (-1)^j X [0, \infty) \right\} \quad (1)$$

For a given function in the set A, the constant M must be finite number, k_1, k_2 may be finite or infinite.

The ELzaki transform denoted by the operator E (.) defined by the integral equations

$$E[f(t)] = T(v) = v \int_0^{\infty} f(t) e^{\frac{-t}{v}} dt, \quad t \geq 0, \quad k_1 \leq v \leq k_2 \quad (2)$$

The variable v in this transform is used to factor the variable t in the argument of the function f . This transform has deeper connection with the Laplace transform. We also present many different properties of this new transform and Sumudu transform, few properties expent.

The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

ELzaki Transform of the Some Functions

For any function $f(t)$, we assume that the integral equation (2) exist. The Sufficient Conditions for the existence of ELzaki transform are that $f(t)$ for $t \geq 0$ be piecewise continuous and of exponential order, Otherwise ELzaki transform may or may not exist

In this section we find ELzaki transform of simple functions.

(i) let $f(t) = 1$, then:

$$E(1) = v \int_0^{\infty} e^{-\frac{t}{v}} dt = v \left[-ve^{-\frac{t}{v}} \right]_0^{\infty} = v^2$$

(ii) let $f(t) = t$, then:

$$E(t) = v \int_0^{\infty} t e^{-\frac{t}{v}} dt,$$

Integrating by parts to find that: $E(t) = v^3$

In the general case if $n > 0$ is integer number, then.

$$E(t^n) = n! v^{n+2}$$

$$(iii) E[e^{at}] = v \int_0^{\infty} e^{-\frac{t}{v}} e^{at} dt = \frac{v^2}{1-av}$$

This result will be useful, to find ELzaki transform of:

$$E[\sin at] = \frac{av^3}{1+a^2v^2}, \quad E[\cos at] = \frac{v^2}{1+a^2v^2}$$

$$E[\sinh at] = \frac{av^3}{1-a^2v^2}, \quad E[\cosh at] = \frac{v^2}{1-a^2v^2}$$

Theorem:

Let $T(v)$ is the ELzaki transform of $[E(f(t)) = T(v)]$. then:

$$(i) \quad E[f'(t)] = \frac{T(v)}{v} - v f(0)$$

$$(ii) \quad E[f''(t)] = \frac{T(v)}{v^2} - f(0) - v f'(0)$$

$$(iii) \quad E[f^{(n)}(t)] = \frac{T(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k} f^{(k)}(0)$$

Proof:

$$(i) \quad E[f'(t)] = v \int_0^{\infty} f'(t) e^{\frac{-t}{v}} dt$$

Integrating by parts to find that:

$$E[f'(t)] = \frac{T(v)}{v} - v f(0)$$

(ii) Let $g(t) = f'(t)$, then:

$$E[g'(t)] = \frac{1}{v} E[g(t)] - v g(0)$$

we find that by using (i),

$$E[f''(t)] = \frac{T(v)}{v^2} - f(0) - v f'(0)$$

(iii) Can be proof by mathematical induction

Application of ELzaki Transform of Ordinary Differential Equations.

As stated in the introduction of this paper, the ELzaki transform can be used as an effective tool. For analyzing the basic characteristics of a linear system governed by the differential equation in response to initial data. The following examples illustrate the use of the ELzaki transform in solving certain initial value problems described by ordinary differential equations.

Consider the first-order ordinary differential equation.

$$\frac{dx}{dt} + px = f(t) \quad , \quad t > 0 \tag{3}$$

With the initial Condition

$$x(0) = a \tag{4}$$

Where p and a are constants and $f(t)$ is an external input function so that its ELzaki transform exists.

Applying ELzaki transform of the equation (3) we have:

$$\frac{1}{v}\bar{x}(v) - vx(0) + p\bar{x}(v) = \bar{f}(v)$$

$$\bar{x}(v) = \frac{v\bar{f}(v)}{1+pv} + \frac{av^2}{1+pv}$$

The inverse ELzaki transform leads to the solution.

The second order linear ordinary differential equation has the general form.

$$\frac{d^2y}{dx^2} + 2p\frac{dy}{dx} + qy = f(x) \quad , x > 0 \quad (5)$$

The initial conditions are

$$y(0) = a \quad , \quad \frac{dy}{dx}(0) = b \quad (6)$$

Are constants. Application of the ELzaki transforms b and p, q, a where to this general initial value problem gives

$$\frac{1}{v^2}\bar{y}(v) - y(0) - vy'(0) + 2p\left[\frac{1}{v}\bar{y}(v) - vy(0)\right] + g\bar{y}(v) = \bar{f}(v)$$

The use of (6) leads to the Solution for $\bar{y}(v)$ as

$$\bar{y}(v) = \frac{v^2\bar{f}(v)}{qv^2 + 2pv + 1} + \frac{av^2}{qv^2 + 2pv + 1} + \frac{(b + 2pa)v^3}{qv^2 + 2pv + 1}$$

The inverse transform gives the solution.

Example (1):

Consider the first order differential equation

$$\frac{dy}{dx} + y = 0 \quad , \quad y(0) = 1$$

Take ELzaki transform to this equation gives:

$$\frac{1}{v}E(y) - vy(0) + E(y) = 0$$

$$E(y) = \frac{v^2}{1+v} \quad \text{and} \quad y(x) = e^{-x}$$

Where $E(y)$ is the ELzaki transform of the function $y(x)$,

Example (2):

Solve the differential equation

$$y' + 2y = x \quad , \quad y(0) = 1$$

ELzaki transform of this equation is

$$\frac{1}{v} E(y) - vy(0) + 2E(y) = v^3$$

$$E(y) = \frac{(v^3 + v)v}{1 + 2v}$$

$$E(y) = \frac{1}{2}v^3 + \frac{5}{4} \left(\frac{v^2}{1 + 2v} \right) - \frac{1}{4}v^2$$

The inverse transform of this equation gives the Solution:

$$y(x) = \frac{1}{2}x + \frac{5}{4}e^{-2x} - \frac{1}{4}$$

Example (3):

Let us consider the second-order differential equation

$$y'' + y = 0 \quad , \quad y(0) = y'(0) = 1$$

we take ELzaki transform to this equation gives

$$\frac{1}{v^2} E(y) - 1 + E(y) - v = 0$$

We solve this equation for $E(y)$ to get

$$E(y) = \frac{v^2}{v^2 + 1} + \frac{v^3}{v^2 + 1}$$

The inverse ELzaki transform of this equation is simply obtained as

$$y(x) = \sin x + \cos x$$

Example (4):

Consider the following equation

$$y'' - 3y' + 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 4$$

Take ELzaki transform of this equation we find that:

$$E(y) = \frac{v^2(v+1)}{(2v-1)(v-1)} = v^2 \left[\frac{2}{v-1} - \frac{3}{2v-1} \right]$$

$$E(y) = \frac{-2v^2}{1-v} + \frac{3v^2}{1-2v}$$

Then the solution is $y(x) = -2e^{-x} + 3e^{2x}$

Example (5):

Let the second order differential equation:

$$y'' + 9y = \cos 2t \quad \text{if } y(0) = 1, \quad y\left(\frac{\pi}{2}\right) = -1$$

Since $y'(0)$ is not known, let $y'(0) = c$.

Take Elzaki transform of this equation and using the conditions, we have

$$\frac{T(v)}{v^2} - 1 - cv + 9T(v) = \frac{v^2}{4v^2 + 1} \quad \text{or } T(v) + 9v^2 T(v) = v^2 \left[\frac{v^2}{4v^2 + 1} \right] + cv^3 + v^2$$

We can write this equation in the form,

$$T(v) = v^2 \left[\frac{v^2}{(1+4v^2)(1+9v^2)} \right] + \frac{cv^3}{1+9v^2} + \frac{v^2}{1+9v^2} = v^2 \left[\frac{4}{5(1+9v^2)} + \frac{cv}{3(1+9v^2)} + \frac{1}{5(1+4v^2)} \right]$$

And invert to find the solution.

$$y = \frac{4}{5} \cos 3t + \frac{c}{3} \sin 3t + \frac{1}{5} \cos 2t$$

To determine c note that $y\left(\frac{\pi}{2}\right) = -1$ then we find $c = \frac{12}{5}$ then,

$$y = \frac{4}{5} \cos 3t + \frac{4}{5} \sin 3t + \frac{1}{5} \cos 2t$$

Example (6):

Solve the differential equation:

$$y'' - 3y' + 2y = 4e^{3t}, \quad y(0) = -3, \quad y'(0) = 5$$

Taking the Elzaki transforms both side of the differential equation and using the given conditions we have,

$$\frac{T(v)}{v^2} + 3 - 5v - 3 \left[\frac{T(v)}{v} + 3v \right] + 2T(v) = \frac{4v^2}{1-3v}$$

$$\left[\frac{1}{v^2} - \frac{3}{v} + 2 \right] T(v) = \frac{4v^2}{1-3v} + 14v - 3$$

Or

$$T(v) = v^2 \left[\frac{4}{1-2v} + \frac{2}{1-3v} - \frac{9}{1-v} \right]$$

Inverting to find the solution in the form.

$$y(t) = 4e^{2t} + 2e^{3t} - 9e^t$$

Example (7):

Find the solution of the following initial value problem:

$$y'' + 4y = 9t, \quad y(0) = 0, \quad y'(0) = 7$$

Applying Elzaki transform of this problem and using the given constants we get,

$$\frac{T(v)}{v^2} - 7v + 4T(v) = 9v^3, \quad \text{or} \quad T(v) = \frac{9v^5}{1+4v^2} - \frac{7v^3}{1+4v^2} = 3v^3 + \frac{2v^3}{1+4v^2}$$

Inverting to find the solution in the form:

$$y(t) = 3t + 2 \sin 2t$$

Conclusion

The definition and application of the new transform " Elzaki transform" to the solution of ordinary differential equations has been demonstrated;

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