Construction of subdirectly irreducible SQS-skeins of cardinality $n^2m$

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Abstract. We give a construction for subdirectly irreducible SQS-sk eins of cardinality $n^2m$ having a monolith with a congruence class of cardinality $2^m$ for each integer $m \geq 2$. Moreover, the homomorphic image of the constructed SQS-skein modulo its atom is isomorphic to the initial SQS-sk ein. Consequently, we will construct an $SK(n^2m)$ with a derived $SL(n^2m)$ such that $SK(n^2m)$ and $SL(n^2m)$ are subdirectly irreducible and have the same congruence lattice. Also, we may construct an $SK(n^2m)$ with a derived $SL(n^2m)$ in which the congruence lattice of $SK(n^2m)$ is a proper sublattice of the congruence lattice of $SK(n^2m)$.

1. Introduction

A Steiner quadruple (triple) system is a pair $(S; B)$ where $S$ is a finite set and $B$ is a collection of 4-subsets (3-subsets) called blocks of $S$ such that every 3-subset (2-subset) of $S$ is contained in exactly one block of $B$ (see [8] and [11]). Let $SQS(m)$ denote a Steiner quadruple system (briefly quadruple system) of cardinality $m$ and $STS(n)$ denote Steiner triple system (briefly triple system) of cardinality $n$. It is well-known that $SQS(m)$ exists if $m \equiv 2$ or $4 \pmod{6}$ and $STS(n)$ exists if and only if $n \equiv 1$ or $3 \pmod{6}$ [8] and [11]. Let $(S; B)$ be an $SQS$. If one considers $S_a = S - \{a\}$ for any point $a \in S$ and deletes that point from all blocks which contain it then the resulting system $(S_a; B(a))$ is a triple system, where $B(a) = \{b - \{a\} \mid b \in B, a \in b\}$. Now, $(S_a; B(a))$ is called a derived triple system (or briefly DTS) of $(S; B)$ (cf. [8] and [11]).

A sloop (briefly SL) $L = (L; \cdot, 1)$ is a groupoid with a neutral element $1$ satisfying the identities:

$$x \cdot y = y \cdot x, \quad 1 \cdot x = x, \quad x \cdot (x \cdot y) = y.$$  

A sloop $L$ is called Boolean if it satisfies the associative law. The cardinality of the Boolean sloop is equal $2^m$.

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There is one to one correspondence between STSs and Steiner loops (sloops) [8].

An SQS-skein (briefly an SK) \((Q; q)\) is an algebra with a unique ternary operation \(q\) satisfying:

\[
q(x, y, z) = q(x, z, y) = q(z, x, y), \quad q(x, x, y) = y, \quad q(x, y, q(x, y, z)) = z.
\]

An SQS-skein \((Q; q)\) is called Boolean if it satisfies in addition the identity:

\[
q(a, x, q(a, y, z)) = q(x, y, z).
\]

There is one to one correspondence between SQSs and SQS-skeins (cf. [8] and [11]).

The slope associated with a derived triple system is also called derived. A derived slope of an SQS-skein \((Q; q)\) with respect to \(a \in Q\) is the slope \((Q_a; \cdot, a)\) with the binary operation \(\cdot\) defined by \(x \cdot y = q(a, x, y)\).

A subloop \(N\) of \(L\) (sub-SQS-skein of \(Q\)) is called normal if and only if \(N = [1]_{\theta}(N = [a]_{\theta})\) for a congruence \(\theta\) on \(L\) (cf. [8] and [12]). The associated congruence \(\theta\) with the normal subloop (sub-SQS-skein) \(N\) is given by:

\[
\theta = \{(x, y) : x \cdot y \ (\text{or} \ q(a, y, z)) \in N\}.
\]

Quackenbush in [12] and similarly Armanious in [1] have proved that the congruences of slopes (SQS-skeins) are permutable, regular and uniform. Also, we may say that the congruence lattice of each of slopes and SQS-skeins is modular. Moreover, they proved that a maximal subloop (sub-SQS-skein) has the same property as in groups.

**Theorem 1.** (cf. [1] and [8]) Every subloop (sub-SQS-skein) of a finite slope \((L; \cdot, 1)\) (SQS-skein \((Q; q)\)) with cardinality \(\frac{1}{2} |L| \left(\frac{1}{2} |Q|\right)\) is normal. \(\square\)

A Boolean slope is a Boolean group. If \((G; +)\) is a Boolean group, then \((G; q(x, y, z) = x + y + z)\) is a Boolean SQS-skein [1].

Guelzo [10] and Armanious [2], [3] gave generalized doubling constructions for nilpotent subdirectly irreducible SQS-skeins and slopes of cardinality 2\(n\). In [6] the authors gave recursive construction theorems as \(n \rightarrow 2n\) for subdirectly irreducible slopes and SQS-skeins. All these constructions supplies us with subdirectly irreducible SQS-skeins having a monolith \(\theta\) satisfying \(|[x]_{\theta}| = 2\) (the minimal possible order of a proper normal SQS-skeins). Also in these constructions, the authors begin with a subdirectly irreducible \(\text{SK}(n)\) to construct a subdirectly irreducible \(\text{SK}(2n)\) satisfying the property that the cardinality of the congruence class of its monolith is equal 2. Armanious [5] has given another construction of a subdirectly irreducible \(\text{SK}(2n)\). He begins with a finite simple \(\text{SK}(n)\) to construct a subdirectly irreducible \(\text{SK}(2n)\) having a monolith \(\theta\) with \(|[x]_{\theta}| = n\) (the maximal possible order of a proper normal sub-SQS-skein).

In [7] the authors begin with an arbitrary \(\text{SL}(n)\) to construct subdirectly irreducible \(\text{SL}(n2^m)\) for each possible integers \(n \geq 4\) and \(m \geq 2\).
In this article, we begin with an arbitrary \( SK(n) \) for each possible value \( n \geq 4 \) to construct subdirectly irreducible \( SK(n2^m) \) for each integer \( m \geq 2 \). This construction enables us to construct subdirectly irreducible \( SQS \)-sk eins having a monolith \( \theta \) satisfying that its congruence class is \( SK(2^m) \). Moreover, its homomorphic image modulo \( \theta \) is isomorphic to \( Q \).

We will show that our construction supplies us with construction of an \( SK(n2^m) \) with a derived \( SL(n2^m) \) such that the congruence lattices of \( SK(n2^m) \) and \( SL(n2^m) \) are the same for each possible case. Moreover, we may construct an \( SK(n2^m) \) with a derived \( SL(n2^m) \) such that the congruence lattices of \( SK(n2^m) \) is a proper sublattice of the congruence lattice of \( SL(n2^m) \).

2. Subdirectly irreducible \( SQS \)-sk eins \( Q \times_{\alpha} B \)

Let \( Q := (Q; q) \) be an \( SK(n) \) and \( B := (B; \bullet, 1) \) be a Boolean \( SL(2^m) \), where \( Q = \{x_0, x_1, x_2, \ldots, x_{n-1}\} \) and \( B = \{1, a_1, a_2, \ldots, a_{2m-1}\} \). In this section we extend the \( SQS \)-skein \( Q \) to a subdirectly irreducible \( SQS \)-skein \( Q \times_{\alpha} B \) of cardinality \( n2^m \) having \( Q \) as a homomorphic image.

We divide the set of elements of the direct product \( Q \times B \) into two subsets \( \{x_0, x_1\} \times B \) and \( \{x_2, \ldots, x_{n-1}\} \times B \). Consider the cyclic permutation \( \alpha = (a_1a_2 \ldots a_{2m-1}) \) on the set \( \{1, a_1, a_2, \ldots, a_{2m-1}\} \) and the characteristic function \( \chi \) from the direct product \( Q \times B \) to \( B \) defined as follows

\[
\chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) = \begin{cases} 
\iota_m \bullet \iota_n \bullet \alpha^{-1}(\iota_m \bullet \iota_n) & \text{for } y_m = y_n = x_0, y_k = x_1 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\
i_m \bullet i_n \bullet \alpha(i_m \bullet i_n) & \text{for } y_m = y_n = x_1, y_k = x_0 \text{ and } \{m, n, k\} = \{1, 2, 3\} \\
1 & \text{otherwise.}
\end{cases}
\]

It is clear that \( \chi((y_1, i_1), (y_2, i_2), (y_3, i_3)) = 1 \) in two cases:

(i) \( y_1 = y_2 = y_3 = x_0 \) or \( y_1 = y_2 = y_3 = x_1 \).

(ii) \( y_1, y_2 \) or \( y_3 \in Q - \{x_0, x_1\} \).

For this characteristic function we obtain the following result:

**Lemma 2.** The characteristic function \( \chi \) satisfies the properties:

(i) \( \chi((x, a), (y, b), (z, c)) = \chi((x, a), (z, c), (y, b)) = \chi((z, c), (x, a), (y, b)) \)

(ii) \( \chi((x, a), (x, a), (y, b)) = 1; \)

(iii) \( \chi((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c)))) = \chi((x, a), (y, b), (z, c)) \).

**Proof.** According to the definition of \( \chi \), we may deduce that (i) is valid.

In (ii), if \( x = x_0 \) and \( y = x_1 \) then \( \chi((x_0, a), (x_0, a), (x_1, b)) = a \bullet a \bullet a^{-1}(a \bullet a) = 1. \) If \( x = x_1 \) and \( y = x_0 \), then \( \chi((x_1, a), (x_1, a), (x_0, b)) = a \bullet a \bullet a(a \bullet a) = 1. \) Otherwise if \( x \) or \( y \) \( \neq x_0 \) or \( x_1 \), then \( \chi((x, a), (x, a), (y, b)) = 1. \)
To prove the third property, we have only three essential cases:

1. If \( x = y = x_0 \) and \( z = x_1 \) then
   \[
   \chi((x_0, a), (x_0, b), (q(x_0, x_0, x_1), a \bullet b \bullet c \bullet \chi((x_0, a), (x_0, b), (x_1, c))))
   = \chi((x_0, a), (x_0, b), (x_1, c \bullet \alpha_1^{-1}(a \bullet b)))
   = a \bullet b \bullet \alpha_1^{-1}(a \bullet b)
   = \chi((x_0, a), (x_0, b), (x_1, c)).
   \]

2. If \( x = y = x_1 \) and \( z = x_0 \) then
   \[
   \chi((x_1, a), (x_1, b), (q(x_1, x_1, x_0), a \bullet b \bullet c \bullet \chi((x_1, a), (x_1, b), (x_0, c))))
   = \chi((x_1, a), (x_1, b), (x_0, c \bullet \alpha(a \bullet b)))
   = a \bullet b \bullet \alpha(a \bullet b)
   = \chi((x_1, a), (x_1, b), (x_0, c)).
   \]

Note that
\[
\chi((x_0, a), (x_0, b), (x_1, c)) = \chi((x_0, a), (x_1, c), (x_0, b)) = \chi((x_1, c), (x_0, a), (x_0, b))
\]
and
\[
\chi((x_1, a), (x_1, b), (x_0, c)) = \chi((x_1, a), (x_0, c), (x_1, b)) = \chi((x_0, c), (x_1, a), (x_1, b)).
\]

3. Otherwise, i.e., when \( i \) \( x = y = z = x_0 \) or \( x = y = z = x_1 \)
   \( ii \) \( x, y \) or \( z \in \{x_0, x_1\} \),
we have
   \[
   \chi((x, a), (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c)))) = \chi((x, a), (y, b), (z, c) = 1.
   \]

This completes the proof of the lemma. \( \square \)

**Lemma 3.** Let \((Q; q)\) be an arbitrary \( SK(n) \), and \((B; \bullet, 1)\) be a Boolean \( S L(2^m) \) for \( m \geq 2 \). Also let \( q' \) be a ternary operation on the set \( Q \times B \) defined by :
   \[
   q'(\langle x, a, \rangle, (y, b), (z, c)) := (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))).
   \]
Then \( Q \times_B B = (Q \times B; q') \) is an \( SK(n2^m) \) for each possible \( n \geq 4 \).

**Proof.** Let \( Q = \{x_0, x_1, x_2, \ldots, x_{2^{n-1}}\} \) and \( B = \{1, a_1, a_2, \ldots, a_{2^m-1}\} \). For all \( (x, a), (y, b), (z, c) \in Q \times B \), according to Lemma 2 \( i \) and the properties of the operations "\( q' \)" and "\( \bullet \)" we find that:
   \[
   q'(\langle x, a, \rangle, (y, b), (z, c)) = q'(\langle x, a, \rangle, (z, c), (y, b)) = q'(\langle z, c, \rangle, (x, a), (y, b)).
   \]
By using Lemma 2 \( ii \)
   \[
   q'(\langle x, a, \rangle, (x, a), (y, b) = (q(x, x, y), a \bullet a \bullet b \bullet \chi((x, a), (x, a), (y, b))) = (y, b).
   \]
Also, Lemma 2 \( iii \) gives us that
   \[
   q'(\langle x, a, \rangle, (y, b), (q'(\langle x, a, \rangle, (y, b), (z, c)))
   = q'(\langle x, a, \rangle, (y, b), (q(x, y, z), a \bullet b \bullet c \bullet \chi((x, a), (y, b), (z, c))) = (z, c).
   \]
Hence \( Q \times_B B = (Q \times B; q') \) is an \( SQS \)-skew. \( \square \)
In the next theorem we prove that the constructed $Q \times \alpha B$ is a subdirectly irreducible SQS-sklein having a monolith $\theta_1$ satisfying that the cardinality of its congruence class equal $2^m$.

**Theorem 4.** The constructed sloop $Q \times \alpha B = (Q \times B; q')$ is a subdirectly irreducible SQS-sklein.

**Proof.** The projection $\Pi : (x, a) \rightarrow x$ from $Q \times B$ into $Q$ is an onto homomorphism and the congruence Ker $\Pi := \theta_1$ on $Q \times B$ is given by:

$$\theta_1 = \bigcup_{i=0}^{n-1} \{(x_i, 1), (x_i, a_1), \ldots, (x_i, a_{m-1})\}^2,$$

so one can directly see that $[(x_0, 1)]_{\theta_1} = \{(x_0, 1), (x_0, a_1), \ldots, (x_0, a_{m-1})\}$.

Now $C(Q) \cong C((Q \times \alpha B)/\theta_1) \cong [\theta_1 : 1]$. Our proof will now be complete if we show that $\theta_1$ is the unique atom of $C(Q \times \alpha B)$.

First, assume that $\theta_1$ is not an atom of $C(Q \times \alpha B)$, then we can find an atom $\gamma$ satisfying that: $\gamma < \theta_1$ and $|[(x_i, a_i)]_{\gamma}| = r < |[(x_i, a_i)]_{\theta_1}| = 2^m$. In the following we get a contradiction by proving $[(x_1, 1)]_{\gamma}$ is not a normal sub-SQS-sklein of $Q \times \alpha B$.

Suppose $[(x_1, 1)]_{\gamma} = \{(x_1, 1), (x_1, a_1), \ldots, (x_1, a_{r-1})\}$. If $\{a_1, a_2, \ldots, a_{r-1}\}$ is an increasing successive subsequence of $\{a_1, a_2, \ldots, a_{m-1}\}$ and if $\alpha(a_i) = a_i$, for all $i = 1, 2, \ldots, r - 1$, then $\alpha(a_{r-1}) = a_r \notin \{a_1, a_2, \ldots, a_{r-1}\}$. If $\{a_1, a_2, \ldots, a_{r-1}\}$ is an increasing and not successive subsequence selected from $\{a_1, a_2, \ldots, a_{m-1}\}$ then there exists an element $a_j \in \{a_1, a_2, \ldots, a_{r-1}\}$ such that $\alpha(a_j) = a_{j+1} \notin \{a_1, a_2, \ldots, a_{r-1}\}$. For both cases, we can always find an element $(x_1, a_k) \in [(x_1, 1)]_{\gamma}$ such that $(x_1, a_k) \notin [(x_1, 1)]_{\gamma}$ ($a_k = a_{r-1}$ for the first case, and $a_k = a_j$ for the second case).

We can determine the class containing $(x_0, 1)$ when we use the fact that $[(x_0, 1)]_{\gamma} = q'([(x_1, 1)]_{\gamma}, (x_1, 1), (x_0, 1))$, hence we will find that

$$[(x_0, 1)]_{\gamma} = \{(x_0, 1), (x_0, \alpha(a_1)), (x_0, \alpha(a_2)), \ldots, (x_0, \alpha(a_{r-1}))\}.$$

By the same way $[(x_2, 1)]_{\gamma} = q'([(x_1, 1)]_{\gamma}, (x_1, 1), (x_2, 1))$, and this leads to

$$[(x_2, 1)]_{\gamma} = \{(x_2, 1), (x_2, a_1), (x_2, a_2), \ldots, (x_2, a_{r-1})\}.$$

From the other side $[(x_2, 1)]_{\gamma} = q'([(x_0, 1)]_{\gamma}, (x_0, 1), (x_2, 1))$, here we will find that

$$[(x_2, 1)]_{\gamma} = \{(x_2, 1), (x_2, \alpha(a_1)), (x_2, \alpha(a_2)), \ldots, (x_2, \alpha(a_{r-1}))\}.$$

This means that for each $a_k \in \{a_1, a_2, \ldots, a_{r-1}\}$ $\alpha(a_k) \in \{a_1, a_2, \ldots, a_{r-1}\}$. This contradicts the assumption that $(x_1, \alpha(a_k)) \notin [(x_1, 1)]_{\gamma}$. Hence, we may say that there is no atom $\gamma$ of $C(Q \times \alpha B)$ satisfying $\gamma < \theta_1$. Therefore, $\theta_1$ is an atom of the lattice $C(Q \times \alpha B)$. 

Secondly, to prove that $\theta_1$ is the unique atom of $C(Q \times_\alpha B)$. Assume that $\delta$ is another atom of $C(Q \times_\alpha B)$, then $\theta_1 \cap \delta = 0$. Hence, one can easily see that there is only one element $(x, a_i) \in [(x, a_i)]\delta$ with the first component $x$ (note that $[(x, a_i)]\theta_1 = \{(x, 1), (x, a_1), \ldots, (x, a_i), \ldots, (x, a_{m-1})\}$). For this reason we may say that the class $[(x, 0)]\delta$ has at most one pair $(x, a_i)$ with first component $x_1$. So we have two possibilities; either

(i) $[(x, 0)]\delta$ contains only one pair $(x, a_i)$ with first component $x_1$, or

(ii) $[(x, 0)]\delta$ has not any pairs with first component $x_1$.

For the first case, let $((x, a), (x_1, a_i)) \in \delta$ such that $x_0 \neq x \neq x_1$, and $a_s \neq a_i$. Then

$$q'(\{(x, 0), (x, a), (x_1, a_s)\}) \in \{(x, 0), (x_1, a_s)\}.$$

In this case $(x_1, a_i) \in [(x, 0)]\delta$. Thus

$$q'(\{(x, 0), (x, a), (x_1, a_s)\}) \in [(x, 0)]\delta.$$

Hence, $(x, a \cdot a \cdot a_i) \in [(x, 0)]\delta$.

By using the properties of congruences, $((x_0, 1), (x_1, a_i)), ((x, a_s), (x, a))$ and $((x_1, a_s), (x, a)) \in \delta$, we shall find that $q'(\{(x_0, 1), (x_1, a_s), (x, a)\}) \in \delta$. This means that

$$q'(\{(x_0, 1), (x, a), (x_1, a_i), (x, a_s)\}) \in [(x_0, 1)]\delta.$$

So,

$$(x, a \cdot a_i \cdot a_s) \in [(x, 0)]\delta.$$

Since the class $[(x_0, 1)]\delta$ contains at most one element with a first component $x$, it follows that $\alpha(a_0 \cdot a_s) = a_i \cdot a_s$ hence $a_i \cdot a_s = 1$, which contradicts the choice that $a_s \neq a_i$. This implies that $[(x_0, 1)]\delta$ is not a normal sub-$\text{S\text{-}S\text{-}S\text{-}k\text{e\text{-}i\text{n}}}$ of $Q \times_\alpha B$.

For the second case (ii) when $[(x_0, 1)]\delta$ has not any pair with first component $x_1$. Let $(x, a) \in [(x_0, 1)]\delta$ such that $x_0 \neq x \neq x_1$, and let $(x, b)$ and $(x, c)$ are two elements in $Q \times B$ such that $a \neq b$. Then

$$q'(\{(x_0, 1), (x, a), (x_0, 1), (x, c), (x, b)\}) \in [q'(\{(x_0, 1), (x, c), (x, b)\})].$$

This means that $\{(x_1, c \cdot a \cdot b) \in [q'(\{(x_0, 1), (x_1, c), (x, b)\})]\delta$. Also,

$$q'(\{(x_0, 1), (x, c), q'(\{(x_0, 1), (x, a), (x, b)\}) \in \alpha^{-1}(a \cdot b) \cdot \delta.$$

Therefore $\{(x_1, c \cdot a^{-1}(a \cdot b)) \in [q'(\{(x_0, 1), (x_1, c), (x, b)\})]\delta$.

By using the fact that the class $[q'(\{(x_0, 1), (x_1, c), (x, b)\})\delta$ contains only one element with the first component $x_1$, we may say that $\alpha^{-1}(a \cdot b) = a \cdot b$, hence $a \cdot b = 1$, which contradicts that $a \neq b$. Thus $[(x_0, 1)]\delta$ is not a normal sub-$\text{S\text{-}S\text{-}S\text{-}k\text{e\text{-}i\text{n}}}$ of $Q \times_\alpha B$. This means that there is no another atom $\delta$, and $\theta_1$ is the unique atom of $C(Q \times_\alpha B)$. Therefore, $Q \times_\alpha B$ is a subdirectly irreducible $\text{S\text{-}S\text{-}S\text{-}k\text{e\text{-}i\text{n}}}$.

\[\square\]
Corollary 5. Let $B$ be a Boolean $SL(2^m)$ for an integer $m \geq 2$. Then the congruence class $[(x_0, 1)]/\theta_1$ of the monolith $\theta_1$ of the constructed subdirectly irreducible $SQS$-skein $Q \times_\alpha B$ is a Boolean $SK(2^m)$.

Also, Theorem 3 enable us to construct a subdirectly irreducible $SQS$-skein $Q \times_\alpha B$ having a monolith $\theta_1$ satisfying that $(Q \times_\alpha B)/\theta_1 \cong Q$.

Corollary 6. Every $SQS$-skein $Q$ is isomorphic to the homomorphic image of the subdirectly irreducible $SQS$-skein $Q \times_\alpha B$ over its monolith, for each Boolean sloop $B$.

Remark: The $SQS$-skein $Q \times_\alpha B$ having $L \times_\alpha B$ as a derived sloop.

Let $(Q; q)$ be an $SK(n)$ and $(L; \ast, x_0)$ be a derived $SL(n)$ of $Q$ with respect to the element $x_0$ with the same congruence lattice. This means that for $L = Q = \{x_0, x_1, \ldots, x_{n-1}\}$, the binary operation "\ast" is defined by $x \ast y = q(x_0, x, y)$.

By using the construction in [7], we construct subdirectly irreducible $SL(n2^m)$. This means that if we begin with our derived sloop $L := (L; \ast, x_0)$ of cardinality $n$ and the Boolean sloop $B := (B; \bullet, 1)$ of cardinality $2^m$, we get subdirectly irreducible sloop $L \times_\alpha B = (L \times B; \circ, (x_0, 1))$, where

$$(x, a) \circ (y, b) := (x \ast y, a \ast b \ast \chi((x, a), (y, b)))$$

and

$\chi((x, a), (y, b)) = \begin{cases} a \ast a^{-1}(a) & \text{for } x = x_0, y = x_1, \\ b \ast a^{-1}(b) & \text{for } x = x_1, y = 1, \\ c \ast a(c) & \text{for } x = x_1 = y \text{ and } a \ast b = c, \\ 1 & \text{otherwise.} \end{cases}$

It is easy to see that $\chi((x, a), (y, b))L = \chi((x_0, 1), (x, a), (y, b))$ (the characteristic function of our construction) for all $x, y \in L = Q$. Hence $(x, a) \circ (y, b) = q'(\chi((x_0, 1), (x, a), (y, b)))$ for all $(x, a), (y, b) \in L \times B = Q \times B$, this means directly that the constructed sloop $L \times_\alpha B$ is a derived sloop of the constructed $SQS$-skein $Q \times_\alpha B$. Therefore, we have the following result:

Corollary 7. Let $L$ be a derived sloop of the $SQS$-skein $Q$ with respect to the element $x_0$, then the sloop $L \times_\alpha B$ is a derived sloop of the $SQS$-skein $Q \times_\alpha B$ with respect to $(x_0, 1)$.

Note that $Q$ is isomorphic to the homomorphic image of $Q \times_\alpha B$ over its monolith Corollary 5) and also $L$ is isomorphic to the homomorphic image of $L \times_\alpha B$ over its monolith [7]. Hence according to [7], Theorem 4 and Corollary 6, we may say that:

There is always an $SQS$-skein $Q \times_\alpha B$ with a derived sloop $L \times_\alpha B$, in which both $Q \times_\alpha B$ and $L \times_\alpha B$ are subdirectly irreducible of cardinality $n2^m$ having the same congruence lattice for each possible integers $n \geq 4$ and $m \geq 2$. 

Construction of subdirectly irreducible $SQS$-skeins
The construction of a semi-Boolean SQS-skein (each derived sloop $L$ of $Q$ is Boolean) given in [9] satisfies that $C(Q)$ is a proper sublattice of the congruence lattice of its derived sloop $C(L)$. This means that we may begin with SQS-skein $Q$ with a derived sloop $L$ in which the congruence lattice of $Q$ is a proper sublattice of the congruence lattice of $L$, this leads to $C(L \times \alpha B)$ is a proper sublattice of $C(Q \times \alpha B)$.

Consequently, we may construct SQS-skein $Q \times \alpha B$ with a derived sloop $L \times \alpha B$ such that $Q \times \alpha B$ and $L \times \alpha B$ are subdirectly irreducible of cardinality $n2^m$ and have the same congruence lattice, if we begin with $L$ derived sloop of $Q$ with the same congruence lattice. Also, we may construct SQS-skein $Q \times \alpha B$ with a derived sloop $L \times \alpha B$ in which the congruence lattice of $Q \times \alpha B$ is a proper sublattice of the congruence lattice of $L \times \alpha B$, if we begin with $L$ derived sloop of $Q$ such that the congruence lattice of $Q$ is a proper sublattice of the congruence lattice of $L$.

References


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