ON THE MIXTURE OF EXPONENTIATED PARETO DISTRIBUTIONS

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ABSTRACT

Recently Gupta et al. (1998) introduced a new distribution, called exponentiated Pareto distribution. In this paper, we consider the exponentiated Pareto mixture distribution (EPMD) as a possible model for a lifetime distribution. The moments about zero, median, mode and measures of variation, skewness and kurtosis of random variable have EPMD are derived. We establish recurrence relation between moments about mean of random variable having EPMD. Based on Type II censored samples from a heterogeneous population that can be represented by a finite mixture of exponentiated Pareto lifetime model, the maximum likelihood and Bayes estimates of the parameters are obtained. An approximation form due to Lindley (1980) is used to obtain the corresponding Bayes estimates.

Key words: Mixture of exponentiated Pareto distribution; Moments; Median; Mode; Heterogeneous population; Type II censored samples; Maximum likelihood estimators; Bayes estimators; Lindley's approximation.

1. INTRODUCTION

Finite mixture distributions have recently received a remarkable attention in the statistical literature partly because of the interest of their mathematical properties, and mainly because they arise in an extremely wide variety of disciplines ranging from atomic physics to life testing, reliability and microbiology.


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The exponentiated Pareto model, with cumulative distribution function (cdf),

\[ F(x) = [1 - (1 + x)^{-\lambda}]^\theta, \quad x > 0, \theta, \lambda > 0, \]  

was introduced by Gupta et al. in (1998) as a lifetime model. We shall write \( X \sim \text{EP}(\theta, \lambda) \) to denote that the random variable \( X \) follows an exponentiated Pareto distribution (EPD) with two shape parameters \( \theta \) and \( \lambda \). The probability density function (pdf) is then given by

\[ f(x) = \theta \lambda [1 - (1 + x)^{-\lambda}]^{\theta-1} (1+x)^{-(\lambda+1)}, \quad x > 0, \lambda > 0, \theta > 0. \]  

When \( \theta = 1 \), the above distribution corresponds to the standard Pareto distribution of the second kind [see Johnson et al. (1994)]. For \( \theta > 1 \), the distribution has a unique mode which is \( \left(\frac{\lambda \theta + 1}{\lambda + 1}\right)^{1/\lambda} - 1 \). The median of the distribution is \( \left[1 - (0.5)^{1/\theta}\right]^{-1/\lambda} - 1 \) and its r-th moment is given by

\[ \alpha_r = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} A_r(\theta), \quad \lambda > r, \quad r = 0, 1, 2, 3, \ldots, \]  

where

\[ A_j(\theta) = \theta \, B(\theta, 1 - j / \lambda), \quad \lambda > j, \]  

\[ B(m, l) = \int_0^1 x^{m-1} (1-x)^{l-1} \, dx \quad \text{is beta function.} \]

The reliability (survival) function (RF) and hazard rate function (HRF) of \( X \sim \text{EP}(\theta, \lambda) \) are given, respectively, by

\[ R(x) = 1 - [1 - (1 + x)^{-\lambda}]^\theta, \quad x > 0, \theta, \lambda > 0, \]  

and

\[ h(x) = \frac{\theta \lambda [1 - (1 + x)^{-\lambda}]^{\theta-1} (1+x)^{-(\lambda+1)}}{1 - [1 - (1 + x)^{-\lambda}]^\theta}, \quad x > 0, \lambda > 0, \theta > 0. \]  

For more properties, see, Shawky and Abu-Zinadah (2006).

A heterogeneous population may be described by a finite mixture model with pdf

\[ g(x) = \sum_{i=1}^{k} p_i f_i(x), \]  

where, for \( i = 1, \ldots, k \), \( f_i(x) \) is the i-th pdf component and the mixing proportions, \( p_i \), satisfy the conditions \( 0 < p_i < 1 \) and \( \sum_{i=1}^{k} p_i = 1 \). The corresponding cdf is given by
\[ G(x) = \sum_{i=1}^{k} p_i F_i(x), \quad (1.7) \]

where \( F_i(x) \) is the i-th cdf component. The RF of the mixture is given by

\[ S(x) = \sum_{i=1}^{k} p_i R_i(x), \quad (1.8) \]

where \( R_i(x) \) is the i-th reliability component. The HRF of the mixture is given by

\[ H(x) = \sum_{i=1}^{k} \frac{p_i R_i(x)}{S(x)} h_i(x), \quad (1.9) \]

where \( h_i(x) \) is the i-th hazard component.

Lack of identifiability is not uncommon even for finite mixtures. We show (APPENDIX) that a finite mixture of \( k \) EP(\( \theta, \lambda \)) components is identifiable. For a comprehensive study of finite mixtures and the concept of identifiability see Titterington et al. (1985), McLachlan and Basford (1988), and Maritz and Lwin (1989).

The pdf, cdf, RF and HRF of a finite mixture of \( k \) exponentiated Pareto components are given, respectively, by (1.6) - (1.9) where, for \( i = 1, \ldots, k \), \( f_i(x), F_i(x), R_i(x) \) and \( h_i(x) \) are given by (1.2), (1.1), (1.4) and (1.5) after indexing \( \theta \) and \( \lambda \) by \( i \).

We note that

\[
g(0) = \begin{cases} 
0 & \text{if all } \theta_i > 1; \ i = 1, 2, \ldots, k \\
\infty & \text{if any } \theta_i < 1; \ i = 1, 2, \ldots, k \\
\sum_{i=1}^{k} p_i \lambda_i & \text{if all } \theta_i = 1; \ i = 1, 2, \ldots, k \\
\sum_{i} p_i \lambda_i & \text{if } \theta_i = 1, \theta_j > 1, i \neq j; i \in \{t \mid \theta_t = 1, t = 1, 2, \ldots, k\}, j \in \{t \mid \theta_t > 1, t = 1, 2, \ldots, k\} 
\end{cases}
\]

and \( g(\infty) = 0 \) for all \( \theta_i, \lambda_i > 0 ; i = 1, 2, \ldots, k \).

In this paper, we consider the exponentiated Pareto mixture distribution (EPMD) with \( k \) components and some statistical properties of this distribution are given in Section 2. Based on Type II censored samples from EPMD, the maximum likelihood estimators (MLE's) and Bayes estimators of the parameters are obtained in Sections 3 and 4 where we use Lindley's approximation to obtain the corresponding Bayes estimators.
2. STATISTICAL PROPERTIES

The statistical properties are very important to characterize the distributions. The $r$-th moment about zero, $\mu'_r = E(X^r)$, of the exponentiated Pareto mixture (EPM) random variable $X$ with pdf (1.6) in its closed form is

$$\mu'_r = E(X^r) = \sum_{i=1}^{k} \sum_{j=0}^{r} (-1)^r-j \binom{r}{j} p_i A_j(\theta_i), \quad \lambda_i > r, \quad r = 0, 1, 2, 3, \ldots, \quad (2.1)$$

where

$$A_j(\theta_i) = \theta_i B(\theta_i, 1-j) = \frac{1}{\lambda_i^j}, \quad \lambda_i > j.$$

The above closed form of $\mu'_r$ allows us to derive the following forms of statistical measures for the EPM distribution:

- Mean, denoted by $\mu$,

$$\mu = \sum_{i=1}^{k} p_i A_1(\theta_i) - 1, \quad \lambda_i > 1. \quad (2.2)$$

- Variance, denoted by $\sigma^2$,

$$\sigma^2 = \sum_{i=1}^{k} p_i A_2(\theta_i) - (\sum_{i=1}^{k} p_i A_1(\theta_i))^2, \lambda_i > 2. \quad (2.3)$$

- Coefficient of variation, denoted by $\gamma_2$,

$$\gamma_2 = \frac{\sqrt{\sum_{i=1}^{k} p_i A_2(\theta_i) - (\sum_{i=1}^{k} p_i A_1(\theta_i))^2}}{\sum_{i=1}^{k} p_i A_1(\theta_i) - 1}, \lambda_i > 2. \quad (2.4)$$

- Coefficient of skewness, denoted by $\gamma_3$,

$$\gamma_3 = \frac{\sum_{i=1}^{k} p_i A_3(\theta_i) + (\sum_{i=1}^{k} p_i A_1(\theta_i))^2 (\sum_{i=1}^{k} p_i A_2(\theta_i))^2 - 3 (\sum_{i=1}^{k} p_i A_2(\theta_i))^3}{(\sum_{i=1}^{k} p_i A_2(\theta_i) - (\sum_{i=1}^{k} p_i A_1(\theta_i))^2)^{3/2}}, \lambda_i > 3. \quad (2.5)$$
- Coefficient of kurtosis, denoted by $\gamma_4$,

$$
\gamma_4 = \frac{\sum_{i=1}^{k} p_i A_4(\theta_i) + (\sum_{i=1}^{k} p_i A_1(\theta_i))[-4 \sum_{i=1}^{k} p_i A_3(\theta_i) + 6(\sum_{i=1}^{k} p_i A_2(\theta_i))(\sum_{i=1}^{k} p_i A_1(\theta_i)) - 3(\sum_{i=1}^{k} p_i A_1(\theta_i))^3]}{\left(\sum_{i=1}^{k} p_i A_2(\theta_i) - (\sum_{i=1}^{k} p_i A_1(\theta_i))^2 \right)^2}.
$$

Remark: If we recall that the r-th moment about mean of the EPM random variable $X$ with pdf (1.6) is $\mu_r = E((X - \mu)^r)$, then

$$
\sum_{i=1}^{k} p_i A_r(\theta_i) = \sum_{j=0}^{r-2}(r-j)\mu_{r-j}(\mu+1)^{j} + (\mu+1)^r, \lambda_i > r, r = 0, 1, 2, \ldots. \quad (2.7)
$$

The median of an EPM random variable $X$, is given by

$$
m = \left[1 - \left(\frac{0.5 - B_k}{p_1}\right)^{1/\theta_1}\right]^{-1/\lambda_1}
$$

where

$$
B_k = \sum_{i=2}^{k} p_i[1-(m+1)^{-\lambda_i}]\theta_i.
$$

The mode of an EPM random variable $X$ is given by

$$
D = \left[\frac{(\theta_i-1)\lambda_i}{(\lambda_i+1)\theta_i} - C_k(1+D)\right]^{-1/\lambda_1}
$$

where

$$
C_k = \sum_{i=2}^{k} p_i E_i(D) f_i(D),
$$

$$
E_i(x) = (\theta_i-1)\lambda_i[1-(1+x)^{\lambda_i}]^{-1}(1+x)^{-(\lambda_i+1)} - (\lambda_i+1)(1+x)^{-1}, i = 1, 2, \ldots, k.
$$

3. MAXIMUM LIKELIHOOD ESTIMATION

Suppose that only the r smallest observations in a random sample of n items are observed ($1 \leq r \leq n$). That is, suppose that the data consists of the r smallest lifetimes $X_{(1)} < \ldots < X_{(r)}$ out of a random sample of n items $X_1, \ldots, X_n$ (Type II censored sample)
from a life distribution which is a mixture of \( k \) EP\((\theta_i, \lambda_i)\), \( i = 1, 2, \ldots, k \), components and assuming that the mixing proportion \( p_i, i = 1, 2, \ldots, k \), is known.

The likelihood function based on a Type II censored sample [see, Titterington et al. (1985)] is given by

\[
L(\psi|x) = \frac{n!}{(n-r)!} \left[ \prod_{j=1}^{r} g(x_{(j)}) \right] \left[ S(x_{(r)}) \right]^{n-r},
\]

where \( g(x) \) and \( S(x) \) are given, respectively, by (1.6) and (1.8), the components \( f_i(x) \) and \( R_i(x) \) are given, respectively, by (1.2) and (1.4) after indexing the parameters \( \theta \) and \( \lambda \) by \( i = 1, 2, \ldots, k \). \( \psi = (\theta_1, \lambda_1, \ldots, \theta_k, \lambda_k) \) and \( x = (X_{(1)}, \ldots, X_{(r)}) \). The \( x_{(j)} \)'s are the ordered times for \( j = 1, \ldots, r \).

Let \( \ell = \ln L(\psi|x) \), where \( L(\psi|x) \) is given by (3.1). Setting the derivative of \( \ell \) with respect to \( \theta_i, \lambda_i, i = 1, 2, \ldots, k \), to zero, we find

\[
\frac{\partial \ell}{\partial \theta_i} = \sum_{j=1}^{r} \omega_i(x_{(j)}) \xi_i(x_{(j)}) - (n-r) \xi_i^*(x_{(r)}) [\omega_i(x_{(r)}) - 1] = 0
\]

and

\[
\frac{\partial \ell}{\partial \lambda_i} = \sum_{j=1}^{r} \xi_i(x_{(j)}) [1 - \lambda_i \ln(1 + x_{(j)}) + \left(1 - \frac{1}{\theta_i}\right) \omega_i(x_{(j)}) - (n-r) \xi_i^*(x_{(r)}) \omega_i(x_{(r)}) = 0,
\]

where, for \( j = 1, 2, \ldots, r \) and \( i = 1, 2, \ldots, k \),

\[
\xi_i(x_{(j)}) = \frac{f_i(x_{(j)})}{g(x_{(j)})}, \quad \xi_i^*(x_{(r)}) = \frac{F_i(x_{(r)})}{S(x_{(r)})}, \quad \omega_i(x_{(j)}) = 1 + \ln F_i(x_{(j)})
\]

and

\[
\omega_i'(x_{(j)}) = \frac{f_i(x_{(j)})}{F_i(x_{(j)})} (1 + x_{(j)}) \ln(1 + x_{(j)}).
\]

The \( 2k \) nonlinear likelihood equations (3.2), (3.3) can be solved by using Newton-Raphson iteration scheme to yield the MLE \( \hat{\psi} = (\hat{\theta}_1, \hat{\lambda}_1, \ldots, \hat{\theta}_k, \hat{\lambda}_k) \) of \( \psi = (\theta_1, \lambda_1, \ldots, \theta_k, \lambda_k) \). The corresponding maximum likelihood estimates of \( S(x) \) and \( H(x) \) are given, respectively, by (1.8) and (1.9) after replacing \( \theta_i \) and \( \lambda_i \), \( i = 1, 2, \ldots, k \), by their corresponding maximum likelihood estimates \( \hat{\theta}_i \) and \( \hat{\lambda}_i \), \( i = 1, 2, \ldots, k \).
4. BAYES ESTIMATION

Let \( \alpha_i = (\theta_i, \lambda_i) \) and \( \alpha_j = (\theta_j, \lambda_j) \), \( i \neq j, i, j = 1, 2, \ldots, k \), be independent random variables and assuming that the mixing proportion \( p_i, i = 1, 2, \ldots, k \), is known. The joint prior density of the random vector \( \psi = (\theta_1, \lambda_1, \ldots, \theta_k, \lambda_k) = (\alpha_1, \ldots, \alpha_k) \) is thus given by

\[
p(\psi) = p(\alpha_1, \ldots, \alpha_k) = p_1(\alpha_1) \cdots p_k(\alpha_k),
\]

where, for \( i = 1, 2, \ldots, k \), \( p_i(\alpha_i) \) is a prior density function of \( \alpha_i = (\theta_i, \lambda_i) \) and \( \theta_i, \lambda_i > 0 \).

We suggest a conditional prior distribution of \( \theta_i \) given \( \lambda_i \), for \( i = 1, 2, \ldots, k \), which may appropriately be the conjugate gamma with density function which possesses a hyper-shape-parameter \( \nu_i > 0 \) and is given by

\[
\kappa_1(\theta_i | \lambda_i) = \frac{\lambda_i^{-\nu_i}}{\Gamma(\nu_i)} \theta_i^{\nu_i - 1} e^{-\theta_i / \lambda_i}, \theta_i > 0.
\]

The scale parameter of this density is \( \lambda_i \) which is assumed to become known previously with knowledge which may be translated into an exponential distribution with density function

\[
\kappa_2(\lambda_i) = \frac{1}{d_i} e^{-\lambda_i / d_i}, \lambda_i > 0, d_i > 0,
\]

where \( d_i \) is a positive hyper-scale parameter and a mean value of \( \lambda_i \).

Multiplying \( \kappa_1(\theta_i | \lambda_i) \) by \( \kappa_2(\lambda_i) \), we obtain the prior density of \( \alpha_i = (\theta_i, \lambda_i) \) by

\[
p_i(\alpha_i) = p_i(\theta_i, \lambda_i) = \frac{\lambda_i^{-\nu_i}}{d_i \Gamma(\nu_i)} \theta_i^{\nu_i - 1} e^{-\lambda_i / d_i \theta_i}, \lambda_i > 0, \theta_i > 0.
\]

Based on the above considerations, the prior density function of \( \psi \) is given by

\[
p(\psi) = \prod_{i=1}^{k} \left( \frac{\lambda_i^{-\nu_i}}{d_i \Gamma(\nu_i)} \theta_i^{\nu_i - 1} \right) \exp\left[ - \sum_{i=1}^{k} \frac{\lambda_i^2}{d_i \theta_i} \right], \theta_i, \lambda_i > 0, \nu_i, d_i > 0. \tag{4.1}
\]

It is well known that the posterior density function of \( \psi \) given the observations, denoted by \( q(\psi | x) \), is given by

\[
q(\psi | x) = L(\psi | x) p(\psi) / \int_{\Omega} L(\psi | x) p(\psi) d\psi. \tag{4.2}
\]

It then follows that, under the squared error loss function, the Bayes estimator \( \tilde{\phi} \) of a function of the parameters \( \phi(\psi) \) is given by the ratio
where \( L(\psi|x) \) is given by (3.1), \( p(\psi) \) by (4.1) and \( \Omega \) is the specified parameter space of \( \psi \) on which the posterior density \( q(\psi|x) \) is positive.

The ratio of the integrals (4.3) may thus be approximated by using a form due to Lindley (1980) which reduces, in the case of 2k parameters, to the form

\[
\tilde{\phi} = E(\phi(\psi)|x) = \phi + \frac{1}{2} \sum_{i=1}^{2k} \sum_{j=1}^{2k} (\phi_{ij} + 2\phi_i \rho_j) \sigma_{ij} + \frac{1}{2} \sum_{i=1}^{2k} \sum_{j=1}^{2k} \sum_{s=1}^{2k} \sum_{e=1}^{2k} \ell_{ijs} \phi \sigma_{ijs} \phi \sigma_{se},
\]

(4.4)

where, for \( i, j, s = 1, 2, \ldots, 2k, \)

\[
\rho_i = \frac{\partial \rho}{\partial \tau_i}, \quad \rho = \ln[p(\psi)], \quad \psi = (\tau_1, \tau_2, \ldots, \tau_{2k-1}, \tau_{2k}) = (\theta_1, \ldots, \theta_k, \lambda_k),
\]

\[
\phi_i = \frac{\partial \phi}{\partial \tau_i}, \quad \phi = \phi(\psi), \quad \phi_{ij} = \frac{\partial^2 \phi}{\partial \tau_i \partial \tau_j}, \quad \ell_{ijs} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s}, \quad \ell = \ln[L(\psi|x)], \quad \text{and}
\]

\( \sigma_{ij} = (i,j)-\text{th element in the matrix } \Sigma, \) where \( \Sigma \) is the inverse of the matrix \( \{-\ell_{ij}\}, \)

such that \( \ell_{ij} = \frac{\partial^2 \ell}{\partial \tau_i \partial \tau_j}. \)

Now, we obtain \( \rho_i, \ell_{ij} \text{ and } \ell_{ijs}, \) for \( i, j, s = 1, 2, \ldots, 2k, \) as follows

\[
\rho_i = \frac{\partial \rho}{\partial \tau_i} = \frac{\partial \rho}{\partial \theta_m} = \frac{\theta_m - 1}{\theta_m} - \frac{1}{\theta_m}, \quad m = 1, 2, \ldots, k, \quad i = 1, 3, \ldots, 2k - 1,
\]

\[
\rho_i = \frac{\partial \rho}{\partial \lambda_m} = \frac{\theta_m - v_m}{\lambda_m} - \frac{1}{\lambda_m}, \quad m = 1, 2, \ldots, k, \quad i = 2, 4, \ldots, 2k,
\]

\[
\ell_{ii} = \frac{\partial^2 \ell}{\partial \tau_i^2} = \frac{\partial^2 \ell}{\partial \theta_i^2}, \quad m = 1, 2, \ldots, k, \quad i = 1, 3, \ldots, 2k - 1,
\]
$$\ell_{ii} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^2} \xi^m(x(j)) \{ -1 + \omega^2_m(x(j)) [1 - p_m \xi^m(x(j))] \}$$

$$-(n-r) \frac{p_m}{\theta_m^2} \xi^*_m(x(r)) [1 + p_m \xi^*_m(x(r))] [\omega_m(x(r)) - 1]^2,$$

$$\ell_{ij} = \frac{\partial^2 \ell}{\partial \tau_i \partial \tau_j}, \quad m \neq e, \quad m, e = 1, 2, \ldots, k, \quad i \neq j, \quad i, j = 1, 3, \ldots, 2k - 1,$$

$$\ell_{ij} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^2} \xi^m(x(j)) \{ \omega^m_m(x(j)) + [1 - p_m \xi^m(x(j))] \omega_m(x(j)) \}$$

$$\times [1 + (\frac{\theta_m - 1}{\theta_m}) \omega^m_m(x(j)) - \lambda_m \ln(1 + x(j))]$$

$$-(n-r) \frac{p_m}{\theta_m^2} \xi^*_m(x(r)) \omega^m_m(x(r)) [1 + p_m \xi^*_m(x(r))] [\omega_m(x(r)) - 1],$$

$$\ell_{ij} = \frac{\partial^2 \ell}{\partial \tau_i \partial \tau_j}, \quad m \neq e, \quad e, m = 1, 2, \ldots, k, \quad i \neq j, \quad i = 2, 4, \ldots, 2k, \quad j = 1, 3, \ldots, 2k - 1,$$

$$\ell_{ij} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^2} \xi^m(x(j)) \xi^e(x(j)) \omega^m_m(x(j)) \{ 1 + (\frac{\theta - 1}{\theta}) \omega^e_e(x(j)) - \lambda_e \ln(1 + x(j)) \}$$

$$-(n-r) \frac{p_m}{\theta_m^2} \xi^*_m(x(r)) \xi^*_e(x(r)) \omega^e_e(x(r)) [\omega^e_e(x(r)) - 1],$$

$$\ell_{ii} = \frac{\partial^2 \ell}{\partial \tau_i^2}, \quad m = 1, 2, \ldots, k, \quad i = 2, 4, \ldots, 2k,$$
\[\ell_{ii} = \sum_{j=1}^{r} \frac{p_m}{\lambda_m^2} \xi_m(x(j))[-1 + [1 - p_m \xi_m(x(j))]\{1 + (\frac{1}{\theta_m} - 1)\omega_m(x(j)) - \lambda_m \ln(1 + x(j))^2\}
- (\frac{\theta_m - 1}{\theta_m})\omega_m(x(j))\{\frac{1}{\theta_m} \omega_m(x(j)) + \lambda_m \ln(1 + x(j))\}\}
- (n-r)\frac{p_m}{\lambda_m^2} \xi^*_m(x(r))\omega_m(x(r))\{\omega_m(x(r))[1 - \frac{1}{\theta_m} + p_m \xi^*_m(x(r))] - \lambda_m \ln(1 + x(r))\},\]

\[\ell_{ij} = \frac{\partial^2 \ell}{\partial \tau_i \partial \tau_j}, m \neq e, e, m = 1, 2, \ldots, k, i \neq j, i, j = 2, 4, \ldots, 2k,\]

\[\ell_{ij} = - \sum_{j=1}^{r} \frac{p_m p_e}{\lambda_m^2 \lambda_e} \xi_m(x(j))\xi_e(x(j))\{1 + (\frac{1}{\theta_m} - 1)\omega_m(x(j)) - \lambda_m \ln(1 + x(j))\}
- (\frac{\theta_e - 1}{\theta_e})\omega_e(x(j)) - \lambda_e \ln(1 + x(j))\}
- (n-r)\frac{p_m p_e}{\lambda_m^2 \lambda_e} \xi_m^*(x(r))\xi_e^*(x(r))\omega_m(x(j))\omega_e(x(j)),\]

\[\ell_{iii} = \frac{\partial^3 \ell}{\partial \tau_i^3}, m = 1, 2, \ldots, k, i = 1, 3, \ldots, 2k - 1,\]

\[\ell_{iii} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^3} \xi_m(x(j))\{2 + [1 - p_m \xi_m(x(j))]\{-3 \omega_m(x(j)) + \omega_m^3(x(j))[1 - 2 p_m \xi_m(x(j))]\}\}
- (n-r)\frac{p_m}{\theta_m^3} \{\omega_m(x(r)) - 1\}^3 \xi_m^*(x(r))\{1 + p_m \xi_m^*(x(r))\}[1 + 2 p_m \xi_m^*(x(r))],\]

\[\ell_{jj} = \frac{\partial^3 \ell}{\partial \tau_j^3}, m \neq e, e, m = 1, 2, \ldots, k, i \neq j, i, j = 1, 3, \ldots, 2k - 1,\]

\[\ell_{jj} = \sum_{j=1}^{r} \frac{p_m p_e}{\theta_m^2 \theta_e} \xi_m(x(j))\xi_e(x(j))\omega_e(x(j))\{1 - \omega_m^2(x(j))[1 - 2 p_m \xi_m(x(j))]\}
- (n-r)\frac{p_m p_e}{\theta_m^2 \theta_e} \xi_m^*(x(r))\xi_e^*(x(r))\{\omega_e(x(r)) - 1\}^2 [\omega_e(x(r)) - 1][1 + 2 p_m \xi_m^*(x(r))],\]
\[
\ell_{ij} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s} = \frac{\partial^3 \ell}{\partial \theta_i \partial \theta_j \partial \theta_m}, \quad m \neq e \neq v, \quad m, e, v = 1, 2, \ldots, k, \quad i \neq j \neq s, \quad i, j, s = 1, 3, \ldots, 2k - 1,
\]

\[
\ell_{ij} = 2 \sum_{j=1}^{r} \frac{p_m p_e p_v \xi_m(x(j)) \xi_e(x(j)) \xi_v(x(j)) \omega_m(x(j)) \omega_e(x(j)) \omega_v(x(j))}{m e v},
\]

\[
-2(n-r) \frac{p_m p_e p_v \xi^*_m(x(r)) \xi^*_e(x(r)) \xi^*_v(x(r)) [\omega_m(x(r)) - 1][\omega_e(x(r)) - 1][\omega_v(x(r)) - 1],
\]

\[
\ell_{ii} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_i^2} = \frac{\partial^3 \ell}{\partial \lambda \partial \theta_i^2 m}, \quad m = 1, 2, \ldots, k, \quad i \neq j, \quad j = 2, 4, \ldots, 2k, \quad i = 1, 3, \ldots, 2k - 1,
\]

\[
\ell_{ij} = \sum_{j=1}^{r} \frac{p_m}{\theta_m \theta_i^2} \xi_m(x(j)) [1 - p_m \xi_m(x(j))] [2 \omega_m(x(j)) \omega' on m(x(j))]
\]

\[
+ \left[1 + \left(\frac{m}{\theta_m^2} - 1\right) \omega_m(x(j)) - \lambda_m \ln(1 + x(j))\right] \left[1 + \omega_m^2(x(j))\right] \left[-2 p_m \xi_m(x(j))\right]
\]

\[
- \left(1 - n - r\right) p_m \xi^*_m(x(r)) \omega' m(x(r)) [\omega_m^2(x(r)) - 1][1 + p_m \xi^*_m(x(r))]
\]

\[
\times \left[2 + [\omega_m(x(r)) - 1][1 + 2 p_m \xi^*_m(x(r))]ight],
\]

\[
\ell_{ii} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_i^2} = \frac{\partial^3 \ell}{\partial \lambda \partial \theta_i^2 m}, \quad m \neq e, \quad m, e = 1, 2, \ldots, k, \quad i \neq j, \quad j = 2, 4, \ldots, 2k, \quad i = 1, 3, \ldots, 2k - 1,
\]

\[
\ell_{ij} = \sum_{j=1}^{r} \frac{p_e p_m \xi_m(x(j)) \xi_e(x(j)) \left[1 + \left(\frac{e}{\theta_e^2} - 1\right) \omega_e(x(j)) - \lambda_e \ln(1 + x(j))\right]}{e \theta_m}
\]

\[
\times \left[1 - \omega_m^2(x(j))\right] \left[1 - 2 p_m \xi_m(x(j))\right]
\]

\[
- \left(1 - n - r\right) p_e p_m [\omega_m(x(r)) - 1]^2 \omega_e(x(r)) \xi^*_m(x(r)) \xi^*_e(x(r)) [1 + 2 p_m \xi^*_m(x(r))],
\]

\[
\ell_{ij} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s} = \frac{\partial^3 \ell}{\partial \lambda \partial \theta_i \partial \theta_m}, \quad m \neq e, \quad m, e = 1, 2, \ldots, k, \quad i \neq j \neq s, \quad i = 2, 4, \ldots, 2k,
\]

\[
j, s = 1, 3, \ldots, 2k - 1,
\]
\[
\ell_{ij} = - \sum_{j=1}^{r} \frac{P_e P_m}{\lambda m} \omega_e (x(j)) \xi_m(x(j)) \xi_e(x(j)) \omega_m(x(j)) \\
+ [1 + (\frac{\theta}{\theta m} - 1) \omega_m(x(j)) - \lambda m \ln(1 + x(j))] \omega_m(x(j)) [1 - 2 \epsilon_m \xi_m(x(j))]
\]

\[-(n-r) \frac{P_e P_m}{\lambda m} \omega_e(x(r)) \xi_m(x(r)) \xi_e(x(r)) \omega_m(x(r)) [1 + [1 + 2 \epsilon_m \xi_m(x(r))][\omega_m(x(r)) - 1]],\]

\[
\ell_{ij} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s}, \quad m \neq e \neq v, \quad m, e, v = 1, 2, ..., k, \quad i \neq j \neq s, \quad i, j = 2, 4, ..., 2k, \\
\quad j, s = 1, 3, ..., 2k - 1,
\]

\[
\ell_{ij} = 2 \sum_{j=1}^{r} \frac{P_e P_m}{\lambda v} \omega_e(x(j)) \xi_m(x(j)) \xi_e(x(j)) \omega_e(x(j)) \omega_m(x(j)) \\
\times [1 + (\frac{\theta}{\theta v} - 1) \omega_v(x(j)) - \lambda v \ln(1 + x(j))] \\
- 2(n-r) \frac{P_e P_m}{\lambda v} \omega_v(x(r)) \xi_m(x(r)) \xi_e(x(r)) \xi_v(x(r)) \omega_v(x(r)) [\omega_m(x(r)) - 1][\omega_e(x(r)) - 1],
\]

\[
\ell_{ii} = \frac{\partial^3 \ell}{\partial \tau_i^3}, \quad m = 1, 2, ..., k, \quad i = 2, 4, ..., 2k,
\]
\[
\ell_{iii} = \sum_{j=1}^{r} \frac{p}{\lambda^3_m} \xi_{m}^{m} (x_{(j)}) \{2 + \left[1 - \frac{1}{2} \sum_{i \neq e} \frac{1}{\lambda^2_{e}} \right] \left[1 + \frac{\theta_m - 1}{\theta_m} \omega_{m}^{m} (x_{(j)}) - \lambda_m \ln(1 + x_{(j)}) \right] \\
\times \left[-3 + \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) \left[1 + \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) + \lambda_m \ln(1 + x_{(j)}) \right] \right] \\
+ \left[1 - \frac{1}{2} \sum_{i \neq e} \frac{1}{\lambda^2_{e}} \right] \left[1 + \frac{\theta_m - 1}{\theta_m} \omega_{m}^{m} (x_{(j)}) - \lambda_m \ln(1 + x_{(j)}) \right]^2 \\
+ \left(- \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) \right) \left[\frac{2}{\theta_m} + \frac{1}{\theta_m} \frac{3}{\lambda_m} \ln(1 + x_{(j)}) \right] \\
- \lambda_m \ln(1 + x_{(j)}) \left[1 + \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) - \lambda_m \ln(1 + x_{(j)}) \right] \left[1 - \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) \right] \right),
\]

\[
\ell_{ijj} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j}, m \neq e, m, e = 1, 2, \ldots, k, i \neq j, i, j = 2, 4, \ldots, 2k,
\]

\[
\ell_{ijj} = \sum_{j=1}^{r} \frac{p}{\lambda^3_m} \xi_{m}^{m} (x_{(j)}) \xi_{e}^{e} (x_{(j)}) \left[1 + \frac{\theta_e - 1}{\theta_e} \omega_{e}^{e} (x_{(j)}) - \lambda_e \ln(1 + x_{(j)}) \right] \\
\times \left[1 - \frac{1}{2} \sum_{i \neq e} \frac{1}{\lambda^2_{e}} \right] \left[1 + \frac{\theta_m - 1}{\theta_m} \omega_{m}^{m} (x_{(j)}) - \lambda_m \ln(1 + x_{(j)}) \right] \\
+ \left(- \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) \right) \left[\frac{2}{\theta_m} + \frac{1}{\theta_m} \frac{3}{\lambda_m} \ln(1 + x_{(j)}) \right] \\
- \lambda_m \ln(1 + x_{(j)}) \left[1 + \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) - \lambda_m \ln(1 + x_{(j)}) \right] \left[1 - \frac{1}{\theta_m} \omega_{m}^{m} (x_{(j)}) \right] \right),
\]

\[
\ell_{ijj} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s}, m \neq e \neq v, m, e, v = 1, 2, \ldots, k, i \neq j \neq s, i, j, s = 2, 4, \ldots, 2k,
\]
\[ \ell_{ijs} = 2 \sum_{j=1}^{r} \frac{p_m p_e p_v}{\lambda^* e^* m} \xi_m(x(j)) \xi_e(x(j)) \xi_v(x(j)) \left[ 1 + \left( \frac{\theta - 1}{\theta} \right) \omega_e(x(j)) - \lambda_e \ln(1 + x(j)) \right] \\
\times \left[ 1 + \left( \frac{\theta}{\lambda^*} \right) \omega^* m(x(j)) - \lambda_m \ln(1 + x(j)) \right] \left[ 1 + \left( \frac{\theta}{\lambda^*} \right) \omega^* v(x(j)) - \lambda_v \ln(1 + x(j)) \right] \\
- 2(n-r) \frac{p_m p_e p_v}{\lambda^* e^* m} \xi^*_m(x(r)) \xi^*_e(x(r)) \xi^*_v(x(r)) \omega^* m(x(r)) \omega^* e(x(r)) \omega^* v(x(r)), \]

\[ \ell_{ijs} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s}, \quad m \neq e, \quad m,e = 1,2,\ldots,k, \quad i \neq j \neq s, \quad i = 1,3,\ldots,2k-1, \quad j,s = 2,4,\ldots,2k, \]

\[ \ell_{ijs} = - \sum_{j=1}^{r} \frac{p_m p_e p_v}{\lambda e^* m} \xi_m(x(j)) \xi_e(x(j)) \left[ 1 + \left( \frac{\theta - 1}{\theta} \right) \omega_e(x(j)) - \lambda_e \ln(1 + x(j)) \right] \\
\times \left[ \omega^* m(x(j)) + \omega^* m(x(j)) \right] \left[ 1 + \left( \frac{\theta}{\lambda^*} \right) \omega^* m(x(j)) - \lambda_m \ln(1 + x(j)) \right] \left[ 1 - 2p_m \xi^*_m(x(j)) \right] \\
- (n-r) \frac{p_m p_e p_v}{\lambda e^* m} \xi^*_m(x(r)) \xi^*_e(x(r)) \omega^* m(x(r)) \omega^* e(x(r)) \\
\times \left[ \omega^* m(x(r)) + 2p_m \xi^*_m(x(r)) \right] \left[ \omega^* m(x(r)) - 1 \right] \],

\[ \ell_{ijs} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau_s}, \quad m \neq e \neq v, \quad m,e,v = 1,2,\ldots,k, \quad i \neq j \neq s, \quad i = 1,3,\ldots,2k-1, \quad j,s = 2,4,\ldots,2k, \]

\[ \ell_{ijs} = 2 \sum_{j=1}^{r} \frac{p_m p_e p_v}{\lambda^* e^* v} \xi_m(x(j)) \xi_e(x(j)) \xi_v(x(j)) \left[ 1 + \left( \frac{\theta - 1}{\theta} \right) \omega_e(x(j)) - \lambda_e \ln(1 + x(j)) \right] \\
\times \left[ 1 + \left( \frac{\theta}{\lambda^*} \right) \omega^* m(x(j)) - \lambda_m \ln(1 + x(j)) \right] \omega^* v(x(j)) \\
- 2(n-r) \frac{p_m p_e p_v}{\lambda^* e^* v} \xi^*_m(x(r)) \xi^*_e(x(r)) \xi^*_v(x(r)) \omega^* m(x(r)) \omega^* e(x(r)) \omega^* v(x(r)) \left[ \omega^* v(x(r)) - 1 \right], \]

\[ \ell_{ijs} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j \partial \tau^2_s}, \quad m = 1,2,\ldots,k, \quad i \neq j, \quad i = 1,3,\ldots,2k-1, \quad j = 2,4,\ldots,2k, \]
\[ \ell_{ij} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^2} \xi_m(x(j)) \left[ (1 - p_m \xi_m(x(j))) \left( \frac{1}{\theta_m} \frac{\omega_m(x(j))}{\theta_m^2} \right) \omega_m(x(j)) - \frac{1}{\theta_m} \omega_m(x(j)) \right] \]

\[ + \left[ 1 - 2 p_m \xi_m(x(j)) \right] \left[ 1 + \left( \frac{\theta_m - 1}{\theta_m} \right) \omega_m(x(j)) - \frac{1}{\theta_m} \omega_m(x(j)) \right] \lambda_m \ln(1 + x(j)) \]

\[ - \left( \frac{\theta_m - 1}{\theta_m} \right) \omega_m(x(j)) \left[ \frac{1}{\theta_m} \omega_m(x(j)) + \frac{1}{\theta_m} \omega_m(x(j)) \lambda_m \ln(1 + x(j)) \right] \]

\[ + 2 \omega_m(x(j)) \left[ 1 + \left( \frac{\theta_m - 1}{\theta_m} \right) \omega_m(x(j)) - \frac{1}{\theta_m} \omega_m(x(j)) \right] \lambda_m \ln(1 + x(j)) \]

\[ - \frac{1}{\theta_m} \omega_m(x(j)) \left[ \frac{1}{\theta_m} \omega_m(x(j)) + \frac{1}{\theta_m} \omega_m(x(j)) \lambda_m \ln(1 + x(j)) \right] \]

\[ -(n-r) \frac{p_m}{\theta_m^2 \lambda_m} \xi_m^*(x(r)) \omega_m(x(r)) \left[ \frac{1}{\theta_m} \omega_m(x(r)) + p_m \xi_m^*(x(r)) \left[ \omega_m(x(r)) - 1 \right] \right] \]

\[ \times \left[ \omega_m(x(r)) \left[ 1 - \frac{\theta_m - 1}{\theta_m} + 2 p_m \xi_m^*(x(r)) \lambda_m \ln(1 + x(r)) \right] \right], \]

\[ \ell_{ij} = \frac{\partial^3 \ell}{\partial \tau_i \partial \tau_j} = \frac{\partial^3 \ell}{\partial \theta_m \partial \lambda_m^2}, \quad m \neq e, m, e = 1,2,\ldots,k, \quad i \neq j, \quad i = 1,3,\ldots,2k-1, \quad j = 2,4,\ldots,2k, \]

\[ \ell_{ij} = \sum_{j=1}^{r} \frac{p_m}{\theta_m^2} \xi_m(x(j)) \xi_e(x(j)) \omega_e(x(j)) \]

\[ \times \left[ 1 - \left[ 1 - 2 p_m \xi_m(x(j)) \right] \left[ 1 + \left( \frac{\theta_m - 1}{\theta_m} \right) \omega_m(x(j)) - \frac{1}{\theta_m} \omega_m(x(j)) \right] \lambda_m \ln(1 + x(j)) \right] \]

\[ + \left( \frac{\theta_m - 1}{\theta_m} \right) \omega_m(x(j)) \left[ \frac{1}{\theta_m} \omega_m(x(j)) + \frac{1}{\theta_m} \omega_m(x(j)) \lambda_m \ln(1 + x(j)) \right] \]

\[ - \frac{1}{\theta_m} \omega_m(x(j)) \left[ \frac{1}{\theta_m} \omega_m(x(j)) + \frac{1}{\theta_m} \omega_m(x(j)) \lambda_m \ln(1 + x(j)) \right] \]

\[ -(n-r) \frac{p_m}{\theta_m^2 \lambda_m} \omega_m(x(r)) \xi_m^*(x(r)) \xi_e(x(r)) \left[ \omega_e(x(r)) - 1 \right] \]

\[ \times \left[ \omega_m(x(r)) \left[ 1 - \frac{\theta_m - 1}{\theta_m} + 2 p_m \xi_m^*(x(r)) \lambda_m \ln(1 + x(r)) \right] \right], \]

Put \( \phi = \phi(\psi) = \tau_i = \theta_m, \quad i = 1,3,\ldots,2k-1, \quad m = 1,2,\ldots,k, \), we get

\[ \phi_i = \frac{\partial \phi}{\partial \tau_i} = \frac{\partial \phi}{\partial \theta_m} = 1, \quad \phi_j = \frac{\partial \phi}{\partial \tau_j} = 0, \quad i \neq j, \quad j = 1,2,\ldots,2k \]

and
\[ \phi_{sj} = \frac{\partial^2 \phi}{\partial \tau^s \partial \tau^j} = 0, \ s, j = 1, 2, \ldots, 2k. \]

The Bayes estimator \[ \tilde{\phi} = \tilde{\phi}(\psi) = \tilde{\tau}_i = \tilde{\theta}_m, \ i = 1, 3, \ldots, 2k-1, \ m = 1, 2, \ldots, k, \] can be obtained by computing all terms in (4.4) according to their definitions above.

Also, setting \[ \phi = \phi(\psi) = \tau_i = \lambda_m, \ i = 2, 4, \ldots, 2k, \ m = 1, 2, \ldots, k, \] we find

\[ \phi_i = \frac{\partial \phi}{\partial \tau_i} = \frac{\partial \phi}{\partial \lambda_m} = 1, \ \phi_j = \frac{\partial \phi}{\partial \tau_j} = 0, \ i \neq j, \ j = 1, 2, \ldots, 2k \]

and

\[ \phi_{sj} = \frac{\partial^2 \phi}{\partial \tau^s \partial \tau^j} = 0, \ s, j = 1, 2, \ldots, 2k. \]

The Bayes estimator \[ \tilde{\phi} = \tilde{\phi}(\psi) = \tilde{\tau}_i = \tilde{\lambda}_m, \ i = 2, 4, \ldots, 2k, \ m = 1, 2, \ldots, k, \] can be obtained by computing all terms in (4.4) according to their definitions above.

**APPENDIX**

We show that a finite mixture of k EP(\( \theta, \lambda \)) components is identifiable.

**Proof:** Teicher (1961) showed that a finite mixture of k exponential components is identifiable. If \( Y \sim \text{Exp}(\theta) \) and \( Z = \text{1-exp(-Y)} \), then \( Z \sim \text{B}(\theta,1) \) where the pdf of \( Y \) and the pdf of \( Z \) are given, respectively, by

\[ f(y;\theta) = \theta e^{-\theta y}, \ y > 0, \ \theta > 0 \] and \( g(z;\theta) = \theta (1-z)^{\theta-1}, \ 0 < z < 1, \ \theta > 0. \)

The transformation is one to one and onto, so a finite mixture of \( \text{B}(\theta_i,1), \ i = 1, 2, \ldots, k, \) components is identifiable. It follows that

\[ \sum_{i=1}^{k} \ p_i \theta_i (1-z) \theta_i \theta_i^{-1} = \sum_{j=1}^{k'} \ p' j \theta' \theta' j \theta' j^{-1}, \] \hspace{1cm} (A.1)

implies that \( k = k' \) and for all \( i \), there exists some \( j \) such that \( p_i = p' j \) and \( \theta_i = \theta' j \).

Now, suppose that
The transformation \( Z = (1 + X)^{-\lambda_i} \) transforms the component densities of \( \text{EP}(\theta_i, \lambda_i) \),
\[
\theta_i \lambda_i [1 - (1 + x)]^{-\lambda_i} \theta_i^{-1} (1 + x)^{-1} - (\lambda_i + 1)
\]
to the component densities of \( \text{B}(\theta_i, 1), \theta_i (1 - z)^{-1} \), and similarly for the component densities on the right hand side of (A.2). So that (A.2) reduces to (A.1), implying that \( k = k' \) and for all \( i \), there exists some \( j \) such that \( p_i = p'_j \) and \( \theta_i = \theta'_j \). By putting \( k = k' \) and for all \( i \), there exists some \( j \) such that \( p_i = p'_j \) and \( \theta_i = \theta'_j \) in (A.2), we find
\[
\sum_{i=1}^{k} p_i \theta_i \lambda_i [1 - (1 + x)]^{-\lambda_i} \theta_i^{-1} (1 + x)^{-1} - (\lambda_i + 1)
\]
\[
= \sum_{i=1}^{k} p_i \theta_i \lambda_i [1 - (1 + x)]^{-\lambda_i} \theta_i^{-1} (1 + x)^{-1} - (\lambda_i + 1)
\]
Comparing coefficients of \( p_i \theta_i \), \( i = 1, 2, ..., k \) of both sides, yields
\[
\lambda_i [1 - (1 + x)]^{-\lambda_i} \theta_i^{-1} (1 + x)^{-1} - (\lambda_i + 1) = \lambda'_i [1 - (1 + x)]^{-\lambda'_i} \theta_i^{-1} (1 + x)^{-1} - (\lambda'_i + 1), \quad i = 1, 2, ..., k.
\]
Expanding as power series, (A.3) becomes
\[
\lambda_i (1 + x)^{-\lambda_i + 1} \left[ 1 + \sum_{j=1}^{\infty} \frac{(-\lambda_i - 1)(\theta_i - 2) ... (\theta_i - j)}{j!} (-1)^j (1 + x)^{-j} \lambda_i \right]
\]
\[
= \lambda_i' (1 + x)^{-\lambda_i' + 1} \left[ 1 + \sum_{j=1}^{\infty} \frac{(-\lambda_i' - 1)(\theta_i' - 2) ... (\theta_i' - j)}{j!} (-1)^j (1 + x)^{-j} \lambda_i' \right], \quad i = 1, 2, ..., k.
\]
Comparing coefficients of \( (\theta_i - 1)^0 \) and \( (\theta_i' - 1) \), \( i = 1, 2, ..., k \) of both sides, yields \( \lambda_i = \lambda'_i \). In other words, if (A.2) holds, then \( k = k' \) and for all \( i \), there exists some \( j \) such that \( p_i = p'_j \) and \( \theta_i = \theta'_j \), \( \lambda_i = \lambda'_i \). Therefore, a finite mixture of \( k \) \( \text{EP}(\theta_i, \lambda_i), \ i = 1, 2, ..., k \), components is identifiable.
REFERENCES


