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# ON LEFT DERIVATIONS OF $BCI$-ALGEBRAS

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**Abstract**. In the present paper, we introduce the notion of left derivation of a $BCI$-algebra and investigate some related properties. A condition for left derivation to be regular is given. Finally, we give a characterization of a $p$-semisimple $BCI$-algebra which admits left derivation.

## 1. Introduction

In [3], Y. B. Jun and X. L. Xin applied the notion of derivation in ring and near-ring theory to $BCI$-algebras, and they also introduced a new concept called a regular derivation in $BCI$-algebras. They investigated some of its properties, defined a $d$-invariant ideal and gave conditions for an ideal to be $d$-invariant. In non-commutative rings, the notion of derivations is extended to $\alpha$-derivations, left derivations and central derivations. The properties of $\alpha$-derivations and central derivations were discussed in several papers with respect to the ring structures. For left derivations, M. Brešar and J. Vukman [2] used them to give some results in prime and semi-prime rings. For skew polynomial rings, all left derivations are obtained in a similar way to a polynomial rings (see A. Nakajima and M. Sapanci [8]). In [10], J. Zhan and Y. L. Liu introduced the notion of $f$-derivations of $BCI$-algebras. The objective of this paper is to define left derivation on $BCI$-algebras and then investigate a regular left derivations. Finally, we study left derivations on $p$-semisimple $BCI$-algebras.

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## 2. Preliminaries

Let $X$ be a non-empty set with a binary operation $*$ and a constant 0. The system $(X, *, 0)$ is called a $BCI$-algebra, if it satisfies the following axioms for all $x, y, z \in X$:

BCI-1 $((x * y) * (x * z)) * (z * y) = 0$,

BCI-2 $(x * (x * y)) * y = 0$,

BCI-3 $x * x = 0$,

BCI-4 $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leq$ on $X$ by putting $x \leq y$ if and only if $x*y = 0$. Then the system $(X, *, 0)$ is a partially ordered set. A $BCI$-algebra $X$ satisfying $0 \leq x$ for all $x \in X$, is called $BCK$-algebra. A non-empty subset $I$ of a $BCI$-algebra $X$ is said to be an ideal of $X$ if it satisfies for all $x, y \in X$:

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Any ideal $I$ has the property $y \in I$ and $x \leq y$ imply $x \in I$.

In any $BCI$-algebra $X$, the following properties hold for all $x, y, z \in X$:

(1) $x * 0 = x$.

(2) $(x * y) * z = (x * z) * y$.

(3) $0 * (x * y) = (0 * x) * (0 * y)$.

(4) $x * (x * (x * y)) = x * y$.

(5) $((x * z) * (y * z)) * (x * y) = 0$.

(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

(7) $x * 0 = 0$ implies $x = 0$.

For a $BCI$-algebra $X$, we denote by $X_+ = \{x \in X \mid 0 \leq x\}$, the $BCK$-part of $X$ and by $G(X) = \{x \in X \mid 0 * x = x\}$, the $BCI$-$G$-part of $X$. If $X_+ = \{0\}$, then $X$ is called a $p$-semisimple $BCI$-algebra. In a $p$-semisimple $BCI$-algebra $X$, the following hold for all $x, y, z \in X$ :

(8) $(x * z) * (y * z) = x * y$.

(9) $0 * (0 * x) = x$.

(10) $x * (0 * y) = y * (0 * x)$.

(11) $x * y = 0$ implies $x = y$.

(12) $x * a = x * b$ implies $a = b$.

(13) $a * x = b * x$ implies $a = b$.

(14) $a * (a * x) = x$.

Let $X$ be a $p$-semisimple $BCI$-algebra. We define addition $+$ as $x + y = x * (0 * y)$, for all $x, y \in X$. Then $(X, +)$ be an abelian group with identity 0 and $x - y = x * y$. Conversely, let $(X, +)$ be an abelian group with identity 0 and let $x - y = x * y$. Then $X$ is a $p$-semisimple $BCI$-algebra and $x + y = x * (0 * y)$, for all $x, y \in X$ (see [5]). We denote $x \wedge y = y * (y * x)$, $0 * (0 * x) = a_x$, and

$$L_P(X) = \{a \in X \mid x * a = 0 \text{ implies } x = a, \forall\ x \in X\}.$$

For any $x \in X$, $V(a) = \{a \in X \mid a * x = 0\}$ is called the branch of $X$ with respect to $a$. We have $x * y \in V(a * b)$, whenever $x \in V(a)$ and $y \in V(b)$, for all $x, y \in X$ and all $a, b \in L_P(X)$. Note that $L_P(X) = \{x \in X \mid a_x = x\}$ which is the $p$-semisimple part of $X$, and $X$ is a $p$-semisimple $BCI$-algebra if and only if $L_P(X) = X$. We note that $a_x \in L_P(X)$, for $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_P(X)$, for all $y \in X$. It is clear that $G(X) \subset L_p(X)$ and $x * (x * a) = a$ and $a * x \in L_P(X)$, for all $a \in L_P(X)$ and all $x \in X$. For more details, we refer to [1, 4, 7, 9, 11].

**Definition 2.1.**([6]) A $BCI$-algebra $X$ is said to be commutative if $x \wedge y = y \wedge x$, for all $x, y \in X$.

**Definition 2.2.**([3]) Let $X$ be a $BCI$-algebra. By a $(l, r)$-derivation of $X$, we mean a self map $d$ of $X$ satisfying the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)), \text{ for all } x, y \in X.$$

If $X$ satisfies the identity

$$d(x * y) = (x * d(y)) \wedge (d(x) * y), \text{ for all } x, y \in X,$$

then we say that $d$ is a $(r, l)$-derivation of $X$.

Moreover, if $d$ is both a $(l, r)$-derivation and $(r, l)$-derivation of $X$, we say that $d$ is a derivation of $X$.

**Definition 2.3.**([3]) A self-map $d$ of a $BCI$-algebra $X$ is said to be regular if $d(0) = 0$.

**Definition 2.4.**([3]) Let $d$ be a self-map of a $BCI$-algebra $X$. An ideal $A$ of $X$ is said to be $d$-invariant, if $d(A) = A$.

## 3. Left Derivations

In this section, we define the left derivations.

**Definition 3.1.** Let $X$ be a $BCI$-algebra. By a left derivation of $X$, we mean a self-map $D$ of $X$ satisfying

$$D(x * y) = (x * D(y)) \wedge (y * D(x)), \text{ for all } x, y \in X.$$

**Example 3.2.** Let $X = \{0, 1, 2\}$ be a $BCI$-algebra with Cayley table defined by

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 1 | 0 | 2 |
| 2 | 2 | 2 | 0 |

.

Define a map $D : X \to X$ by

$$D(X) = \begin{cases} 2, & \text{if } x = 0, 1 \\ 0, & \text{if } x = 2. \end{cases}$$

Then it is easily checked that $D$ is a left derivation of $X$.

**Proposition 3.3.** *Let $D$ be a left derivation of a $BCI$-algebra $X$. Then for all $x, y \in X$, we have*

(1) $x * D(x) = y * D(y)$.

(2) $D(x) = a_{D(x) \wedge x}$.

(3) $D(x) = D(x) \wedge x$.

(4) $D(x) \in L_P(X)$.

**Proof.** (1) Let $x, y \in X$. Then

$$D(0) = D(x * x) = (x * D(x)) \wedge (x * D(x)) = x * D(x).$$

Similarly, $D(0) = y * D(y)$. So, $x * D(x) = y * D(y)$.
(2) Let $x \in X$. Then

$$\begin{aligned} D(x) &= D(x * 0) \\ &= (x * D(0)) \wedge (0 * D(x)) \\ &= (0 * D(x)) * ((0 * D(x)) * (x * D(0))) \\ &\leq 0 * (0 * (x * D(0))) \\ &= 0 * (0 * (x * (x * D(x)))) \\ &= 0 * (0 * (D(x) \wedge x)) \\ &= a_{D(x) \wedge x}. \end{aligned}$$

Thus $D(x) \leq a_{D(x)\wedge x}$. But

$$a_{D(x)\wedge x} = 0 * (0 * (D(x) \wedge x)) \leq D(x) \wedge x \leq D(x).$$

Therefore, $D(x) = a_{D(x)\wedge x}$.
(3) Let $x \in X$. Then using (2), we have

$$D(x) = a_{D(x)\wedge x} \leq D(x) \wedge x,$$

but we know that $D(x) \wedge x \leq D(x)$, and hence (3) holds.
(4) Since $a_x \in L_P(X)$, for all $x \in X$, we get $D(x) \in L_P(X)$ by (2).

**Remark 3.4.** Proposition 3.3(4) implies that $D(X)$ is a subset of $L_P(X)$.

**Proposition 3.5.** *Let $D$ be a left derivation of a $BCI$-algebra $X$. Then for all $x, y \in X$, we have*
(1) $y * (y * D(x)) = D(x)$.
(2) $D(x) * y \in L_P(X)$.

**Proposition 3.6.** *Let $D$ be a left derivation of a $BCI$-algerbra $X$. Then*
(1) $D(0) \in L_P(X)$.
(2) $D(x) = 0 + D(x)$, *for all* $x \in X$.

(3) $D(x+y) = x + D(y)$, *for all* $x, y \in L_P(X)$.
(4) $D(x) = x$, *for all* $x \in X$ *if and only if* $D(0) = 0$.
(5) $D(x) \in G(X)$, *for all* $x \in G(X)$.

**Proof.** (1) Follows by Proposition 3.3(4).
(2) Let $x \in X$. From Proposition 3.3(4), we get $D(x) = a_{D(x)}$, so we have

$$D(x) = a_{D(x)} = 0 * (0 * D(x)) = 0 + D(x).$$

(3) Let $x, y \in L_P(X)$. Then

$$\begin{aligned} D(x+y) &= D(x * (0 * y)) \\ &= (x * D(0 * y)) \wedge ((0 * y) * D(x)) \\ &= ((0 * y) * D(x)) * (((0 * y) * D(x)) * (x * D(0 * y))) \\ &= x * D(0 * y) \\ &= x * ((0 * D(y)) \wedge (y * D(0))) \\ &= x * D(0 * y) \\ &= x * (0 * D(y)) \\ &= x + D(y). \end{aligned}$$

(4) Let $D(0) = 0$ and $x \in X$. Then

$$D(x) = D(x) \wedge x = x * (x * D(x)) = x * D(0) = x * 0 = x.$$

Conversely, let $D(x) = x$, for all $x \in X$. So it is clear that $D(0) = 0$.
(5) Let $x \in G(X)$. Then $0 * x = x$ and so

$$\begin{aligned} D(x) &= D(0 * x) \\ &= (0 * D(x)) \wedge (x * D(0)) \\ &= (x * D(0)) * ((x * D(0)) * (0 * D(x))) \\ &= 0 * D(x). \end{aligned}$$

This gives $D(x) \in G(X)$.

**Remark 3.7.** Proposition 3.6(4) shows that a regular left derivation of a $BCI$-algebra is the identity map. So we have the following:

**Proposition 3.8.** *A regular left derivation of a BCI-algebra is trivial.*

**Remark 3.9.** Proposition 3.6(5) gives that $D(x) \in G(X) \subseteq L_P(X)$.

**Definition 3.10.** An ideal $A$ of a $BCI$-algebra $X$ is said to be $D$-invariant, if $D(A) \subset A$.

Now, Proposition 3.8 helps to prove the following theorem.

**Theorem 3.11.** *Let $D$ be a left derivation of a BCI-algebra $X$. Then $D$ is regular if and only if every ideal of $X$ is $D$-invariant.*

**Proof.** Let $D$ be a regular left derivation of a $BCI$-algebra $X$. Then Proposition 3.8 gives that $D(x) = x$, for all $x \in X$. Let $y \in D(A)$, where $A$ is an ideal of $X$. Then $y = D(x)$, for some $x \in A$. Thus

$$y * x = D(x) * x = x * x = 0 \in A.$$

Then $y \in A$ and $D(A) \subset A$. Therefore, $A$ is $D$-invariant.

Conversely, let every ideal of $X$ be $D$-invariat. Then $D(\{0\}) \subset \{0\}$, and hence $D(0) = 0$ and $D$ is regular.

Finally, we give a characterization of a left derivation of a $p$-semisimple $BCI$-algebra.

**Proposition 3.12.** *Let $D$ be a left derivation of a $p$-semisimple BCI-algebra. Then the following hold for all $x, y \in X$ :*

(1) $D(x * y) = x * D(y)$.

(2) $D(x) * x = D(y) * y$.

(3) $D(x) * x = y * D(y)$.

**Proof.** (1) Let $x, y \in X$. Then

$$D(x * y) = (x * D(y)) \wedge (y * D(x)) = x * D(y).$$

(2) We know that

$$(x * y) * (x * D(y)) \leq D(y) * y$$

and

$$(y * x) * (y * D(x)) \leq D(x) * x.$$

This means that

$$((x * y) * (x * D(y))) * (D(y) * y) = 0,$$

and

$$((y * x) * (y * D(x))) * (D(x) * x) = 0.$$

So

$$((x * y) * (x * D(y))) * (D(y) * y) = ((y * x) * (y * D(x))) * (D(x) * x). \quad \text{(I)}$$

Using Proposition 3.3(1), we get

$$(x * y) * D(x * y) = (y * x) * D(y * x). \quad \text{(II)}$$

By (1), (II) yields

$$(x * y) * (x * D(y)) = (y * x) * (y * D(x)).$$

Since $X$ is a $p$-semisimple $BCI$-algebra. (I) implies that

$$D(x) * x = D(y) * y.$$

(3) We have, $D(0) = x * D(x)$. From (2), we get $D(0) * 0 = D(y) * y$ or $D(0) = D(y) * y$. So $D(x) * x = y * D(y)$.

**Theorem 3.13.** *In a $p$-semisimple $BCI$-algebra $X$, a self-map $D$ of $X$ is left derivation if and only if it is derivation.*

**Proof.** Assume that $D$ is a left derivation of a $BCI$-algebra $X$. First, we show that $D$ is a $(r, l)$-derivation of $X$. Then

$$\begin{aligned} D(x * y) &= x * D(y) \\ &= (D(x) * y) * ((D(x) * y) * (x * D(y))) \\ &= (x * D(y)) \wedge (D(x) * y). \end{aligned}$$

Now, we show that $D$ is a $(r,l)$-derivation of $X$. Then

$$\begin{aligned}
D(x*y) &= x*D(y)\\
&= (x*0)*D(y)\\
&= (x*(D(0)*D(0))*D(y)\\
&= (x*((x*D(x))*(D(y)*y)))*D(y)\\
&= (x*((x*D(y))*(D(x)*y)))*D(y)\\
&= (x*D(y))*((x*D(y))*(D(x)*y))\\
&= (D(x)*y)\wedge(x*D(y)).
\end{aligned}$$

Therefore, $D$ is a derivation of $X$.
Conversely, let $D$ be a derivation of $X$. So it is a $(r,l)$-derivatin of $X$. Then

$$\begin{aligned}
D(x*y) &= (x*D(y))\wedge(D(x)*y)\\
&= (D(x)*y)*((D(x)*y)*(x*D(y)))\\
&= x*D(y) = (y*D(x))*((y*D(x))*(x*D(y)))\\
&= (x*D(y))\wedge(y*D(x)).
\end{aligned}$$

Hence, $D$ is a left derivation of $X$.

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## References

[1] M. Aslam and A. B. Thaheem, *A note on p-semisimple $BCI$-algebras*, Math. Japon., 36:1(1991), 39-45.
[2] M. Brešar and J. Vukman, *On left derivations and related mappings,* Proc. Amer. Math. Soc., 100(1990), 7-16.
[3] Y. B. Jun and X. L. Xin , *On derivations of $BCI$-algebras*, Inform. Sci., 159(2004), 167-176.
[4] T. D. Lei and C. C. Xi, *p-Radical in $BCI$-algebras*, Math. Japon., 30:4(1985), 511-517.
[5] D. J. Meng, *$BCI$-algebras and abelian groups*, Math. Japon., 32:5(1987), 693-696.
[6] J. Meng and X. L. Xin, *Commutative $BCI$-algebras*, Math. Japon., 37:3(1992), 569-572.
[7] J. Meng, Y. B. Jun and E. H. Roh, *$BCI$-algebras of order* 6, Math. Japon., 47:1(1998), 33-43.

[8] A. Nakajima and M. Sapanci, *Left derivations on skew polynomial rings*, Bull. Greek Math. Soc., 36(1991), 113-119.
[9] X. L. Xin, E. H. Roh and J. C. Li, *Some results on the $BCI$-$G$-part of $BCI$-algebras*, Far East J. Math. Sci., Special Volume (Part III), (1997), 363-370.
[10] J. Zhan and Y. L. Liu, *On $f$-derivations of $BCI$-algebras*, Internat. J. of Math. & Math. Sci., 11(2005), 1675- 1684.
[11] Q. Zhang, *Some other characterizations of $p$-semisimple $BCI$-algebras*, Math. Japon., 36:5(1991), 815-817.

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