CHARACTERIZATION FROM EXPONENTIATED GAMMA DISTRIBUTION BASED ON RECORD VALUES

A. I. Shawky

R. A. Bakoban

1Girls College of Education, P.O. Box 55002, Jeddah 21534, Saudi Arabia.
2Girls College of Education, Department of mathematics, P.O. Box 4269, Jeddah 21491, Saudi Arabia.

ABSTRACT

In this paper, we study the lower record values from an exponentiated gamma distribution and derive explicit expressions for the single, product, triple and quadruple moments. We also, establish recurrence relations for the single, product, triple and quadruple moments and moment generating function.

Key words
Lower record values; Exponentiated gamma distribution; Recurrence relations; Single moments; Product moments; Triple moments; Quadruple moments; Moment generating function.

1. Introduction

Record values are important in many real-life situations involving data relating to weather, sports, economics, and life-tests. The statistical study of record values have been pursued in different directions by several authors; for example, see, Arnold et al. (1992, 1998) and Ahsanullah (1995). For the Rayleigh and Weibull distributions by Balakrishnan and Chan (1993). Also, Sultan et al. (2002) derived moments from generalized power function based on record values. Balakrishnan and Ahsanullah (1994) and Al-Zaid and Ahsanullah (2003) have established some recurrence relations for single and product moments of record values from Lomax and Gumbel distributions respectively. Pawlas and Szynal (1999) dealt with Pareto, generalized Pareto and Burr distributions. Also, general recurrence relations based on upper record values was established by Mohie El-Din et al. (2000).

Now, let \( \{X_n, n \geq 1\} \) be an infinite sequence of i.i.d. random variables from an absolutely continuous distribution function \( F \), and probability density function (p.d.f.) \( f \). Let \( X_{i,j} \) denote the \( i \)th order statistic of the random sample \( X_1, X_2, \ldots, X_j \), and \( F_{i,j} \) be its cumulative distribution function (c.d.f.). Let \( T_k = \min\{X_1, X_2, \ldots, X_k\}, k \geq 1 \). We say that \( X_j \) is a lower record value of this sequence if \( T_j < T_{j-1}, j \geq 2 \). By definition, \( X_1 \) is a record value. Let \( L(n) = \min\{j : j > L(n-1), X_j < X_{L(n-1)}\}, n \geq 2 \) with \( L(1) = 1 \). Then \( X_{L(n)}, n \geq 1 \), denotes the sequence of lower record values. From the above definition, the sequence of record statistics can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of the observations.

Consider the exponentiated gamma (EG) distribution with p.d.f. and c.d.f., respectively,

* Permanent address: Fac. of Eng. at Shoubra, P.O. Box 1206, El Maadi 11728, Cairo, Egypt.
\[ f(x) = \theta x e^{-\theta [1 - e^{-\theta (x + 1)}]}^{\theta - 1}, \quad x > 0, \theta > 0, \]  

(1.1)

and

\[ F(x) = [1 - e^{-\theta (x + 1)}]^\theta, \quad x > 0, \theta > 0, \]  

(1.2)

for details about this distribution, see Shawky and Bakoban (2006).

In this paper, we consider the lower record values from an exponentiated gamma distribution. In section 2, we derive explicit expressions for the single, product, triple and quadruple moments. We also, establish recurrence relations for the single, product, triple and quadruple moments in section 3. Finally we derive the single, product, triple and quadruple moment generating function (MGF) and recurrence relations for the single one in section 4.

2. Moments of Lower Record Values

Let \( X_{L(1)}, X_{L(2)}, \ldots, X_{L(n)} \) be the first \( n \) lower record values from the \( EG \) distribution given in (1.1). Then the single, double, triple and quadruple moments of lower record values are given as follows.

2.1 Single moments

The p.d.f. of the \( n^{th} \) lower record value \( X_{L(n)} \) is given by (Ahsanullah (1995))

\[ f_n(x) = \frac{1}{\Gamma(n)} \left[-\log F(x)\right]^{n-1} f(x), \quad x > 0, \ n = 1, 2, \ldots, \]  

(2.1)

where \( f(.) \) and \( F(.) \) are given, respectively, by (1.1) and (1.2).

The single moments of the \( n^{th} \) lower record value, \( E(X_{L(n)}^a) \), denoted by \( \mu_n(a), n = 1, 2, \ldots \) and \( a = 0, 1, 2, \ldots \), is given by

\[ \mu_n(a) = \frac{1}{\Gamma(n)} \int_0^x x^a \left[-\log F(x)\right]^{n-1} f(x)dx. \]  

(2.2)

The exact explicit expression for the single moments of the \( n^{th} \) lower record value \( X_{L(n)} \) from \( EG \) distribution is given by the following theorem.

**Theorem 1**

For \( n = 1, 2, \ldots, a \geq 0 \) and \( \theta \) is a real value, then

\[ \mu_n(a) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k a_i (n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}}, \]  

(2.3)

where \( a_i (n-1) \) is the coefficient of \( e^{-(n-1+i)x} (x+1)^{n-1+i} \) in the expansion of \( \sum_{i=1}^{\infty} \frac{e^{-ix} (x+1)^i}{i} \) (see, the Appendix).

**Proof**

From (2.2) and (1.2) we get

\[ \mu_n(a) = \frac{\theta^{n-1}}{\Gamma(n)} \int_0^x x^a \left[-\log[1 - e^{-\theta (x + 1)}]\right]^{n-1} f(x)dx. \]  

(2.4)

Using the logarithmic expansion we get

\[ \mu_n(a) = \frac{\theta^{n-1}}{\Gamma(n)} \int_0^x x^a \left[\sum_{i=1}^{\infty} \frac{e^{-xi} (x+1)^i}{i}\right]^{n-1} f(x)dx \]
\[ \mu_n^{(a)} = \frac{\theta^{n-1}}{\Gamma(n)} \sum_{i=0}^{\infty} a_i (n-1) \int_0^\infty x^{a} e^{-(a-1+i)x} (x+1)^{n-1+i} f(x) dx, \]

where \( a_i (n-1) \) is defined in the Appendix, from (1.1) we have

\[ \mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} a_i (n-1) \int_0^\infty x^{a+i} e^{-(a+i)x} (x+1)^{n-i-1}[1-e^{-x}(x+1)]^{\theta-1} dx, \]

From the Binomial theorem we find

\[ \mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k a_i (n-1) \binom{n-1+i+k}{k} \int_0^\infty x^{a+i+k} e^{-(a+i+k)x} (x+1)^{n-i+k} dx, \]

Again, from the Binomial theorem we get

\[ \mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k a_i (n-1) \binom{n-1+i+k}{k} \int_0^\infty x^{a+j+k} e^{-(a+j+k)x} dx, \]

Since the integration is a complete gamma function, then, the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (2.3) takes the form

\[ \mu_n^{(a)} = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k a_i (n-1) \binom{n-1+i+k}{k} \Gamma(a+j+2) \frac{1}{(n+i+k)^{a+j+2}}. \] (2.5)

The single moments of record values from gamma distribution \( G(2,1) \) can be obtained from (2.5) by setting \( \theta = 1 \).

2.2 Double moments

The joint p.d.f. of \( X_{L(m)} \) and \( X_{L(n)} \), \( 1 \leq m < n \) is given by (Ahsanullah (1995))

\[ f_{m,n}(x, y) = \frac{1}{\Gamma(m)\Gamma(n-m)}[-\log F(x)]^{m-1} f(x) \frac{[-\log F(y) + \log F(x)]^{n-m-1} f(y),}{[-\log F(y) + \log F(x)]^{n-m-1} f(y), \quad x > y > 0, \] (2.6)

where \( f(\cdot) \) and \( F(\cdot) \) are given, respectively, by (1.1) and (1.2).

The double moments of the lower record values, \( E(X_{L(m)}^a X_{L(n)}^b) \), denoted by

\[ \mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_y^\infty x^a y^b [-\log F(x)]^{m-1} [-\log F(y) + \log F(x)]^{n-m-1} f(x) f(y) dx dy. \] (2.7)

The exact explicit expression for the double moments of lower record values from an EG distribution is given by the following theorem.

Theorem 2

For \( m, n = 1, 2, \ldots, m < n, a, b \geq 0 \) and \( \theta \) is a real value, then

\[ \mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{s=0}^{\infty} \sum_{l=0}^{\infty} \sum_{h=0}^{\infty} (-1)^{n+m+s+k+1} a_i (n-s-2) \]

\[ \times a_h (s) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \binom{\theta-1}{k} \binom{i_1 + s + k}{j_1} \frac{\Gamma(a+j+2)}{(n-s+i+t-1)^{a+j+2-p}} \]

\[ \times \frac{\Gamma(b + p + j_1 + 2)}{p!(n+t+i+l+k)^{b+p+j_1+2}}, \] (2.8)

where \( a_i (n-s-2) \) and \( a_h (s) \) are defined as the Appendix.
Proof

Relation (2.7) can be written as

\[
\mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty y^b I(y) f(y) dy,
\]

(2.9)

where

\[
I(y) = \int_y^\infty x^s [-\log F(x)]^{n-m-1}[\log x^{\delta} - \log F(x)]^{n-s-2} \frac{f(x)}{F(x)} dx
\]

(2.10)

\[
= \sum_{s=0}^{n-m-1} (-1)^{n-m-1} \binom{n-m-1}{s} \left[ \log F(y) \right]^s \int_y^\infty x^s [-\log F(x)]^{n-s-2} \frac{f(x)}{F(x)} dx
\]

\[
= \sum_{s=0}^{n-m-1} (-1)^{n-m-1} \theta^{n-s-2} \binom{n-m-1}{s} \left[ \log F(y) \right]^s \int_y^\infty x^s \frac{f(x)}{F(x)} \left\{ -\log(1 - e^{-x} (x+1)) \right\}^{n-s-2} dx
\]

\[
= \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{n-s+i+2} (-1)^{n-m-1} \theta^{n-s-1} a_i (n-s-2) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \times \left[ \log F(y) \right]^s \int_y^\infty x^{s+j+1} e^{-(n-s+i+t-1)x} dx.
\]

The last integration is an incomplete gamma function which can be defined as

\[
\frac{\sum_{j=0}^{r-1} \frac{(-1)^j (\lambda y)^j}{j!}}{\Gamma(r)} = \frac{\sum_{j=0}^{r-1} e^{-\lambda y} (\lambda y)^j}{\Gamma(r)}.
\]

Substituting \( r \) is an integer number, then

\[
I(y) = \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{n-s+i+2} (-1)^{n-m-1} \theta^{n-s-1} a_i (n-s-2) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \times \left[ \log F(y) \right]^s \frac{\Gamma(a+j+2)}{p! (n-s+i+t-1)^{a+j+2+p}} y^p e^{-(n-s+i+t-1)y}. \]

(2.11)

Substituting \( I(y) \) in (2.9), we get

\[
\mu_{m,n}^{(a,b)} = \frac{1}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{n-s+i+2} (-1)^{n-m-1} \theta^{n-s-1} a_i (n-s-2) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \times \left[ \log F(y) \right]^s \frac{\Gamma(a+j+2)}{p! (n-s+i+t-1)^{a+j+2+p}} y^p e^{-(n-s+i+t-1)y} f(y) dy
\]

\[
= \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{n-s+i+2} (-1)^{n-m-1} a_i (n-s-2) \binom{n-m-1}{s} \binom{n-s+i+t-2}{j} \times \left[ \log F(y) \right]^s \frac{\Gamma(a+j+2)}{p! (n-s+i+t-1)^{a+j+2+p}} \times \int_0^\infty y^{b+p} e^{-(n-s+i+t-1)y} f(y) dy
\]
\[ \mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m) \Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{i+s} a_i (n-s-2) a_j (s) \times \left( \begin{array}{c} n-m-1 \\ s \end{array} \right) \left( \begin{array}{c} n-s+i+t-2 \\ j \end{array} \right) \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+t-2-p}} \times \int_0^\infty y^{b+p+1} (y+1)^i e^{-(n+i+t)y} [1-e^{-y} (y+1)]^{\theta-1} dy, \]

where \( a_i (s) \) is defined as in the Appendix.

\[ \mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m) \Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{i+s+k} a_i (n-s-2) \times a_j (s) \left( \begin{array}{c} n-m-1 \\ s \end{array} \right) \left( \begin{array}{c} n-s+i+t-2 \\ j \end{array} \right) \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+t-2-p}} \times \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+t-2-p}} \int_0^\infty y^{b+p+j+1} e^{-(n+i+t+k)y} dy. \]

Hence the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (2.8) becomes

\[ \mu_{m,n}^{(a,b)} = \frac{\theta^n}{\Gamma(m) \Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^{i+s+k} a_i (n-s-2) a_j (s) \left( \begin{array}{c} n-m-1 \\ s \end{array} \right) \left( \begin{array}{c} n-s+i+t-2 \\ j \end{array} \right) \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+t-2-p}} \times \frac{\Gamma(a+j+2)}{p!(n-s+i+t-1)^{a+j+t-2-p}} \int_0^\infty y^{b+p+j+1} e^{-(n+i+t+k)y} dy. \]

The double moments of record values from gamma distribution \( G(2,1) \) can be obtained from (2.12) by setting \( \theta = 1 \).

2.3 Triple moments

The joint p.d.f. of \( X_{m,l} \), \( X_{n,l} \) and \( X_{l,l} \), \( 1 \leq m < n < l \) is given by (Ahsanullah (1995))

\[ f_{m,n,l}(x, y, z) = \frac{1}{\Gamma(m) \Gamma(n-m) \Gamma(l-n)} \left[ -\log F(x) \right]^{m-1} \left[ -\log F(y) + \log F(x) \right]^{n-m-1} \times \left[ -\log F(z) + \log F(y) \right]^{l-1} \frac{f(x) f(y)}{F(x) F(y)} f(z), \quad x > y > z > 0. \]

where \( f(.) \) and \( F(.) \) are given, respectively, by (1.1) and (1.2).

The triple moments of the lower record values, \( E(X_{m,l}^a X_{n,l}^b X_{l,l}^c) \), denoted by \( \mu_{m,n,l}^{(a,b,c)} \), \( m, n, l = 1, \ldots, m < n < l \) and \( a, b, c = 0, 1, 2, \ldots \), is given by
The exact explicit expression for the triple moments of lower record values from EG distribution is given by the following theorem.

**Theorem 3**

For \( m, n, l = 1, 2, \ldots, m < n < \ell, a, b, c \geq 0 \) and \( \theta \) is a real value, then

\[
\mu_{(a,b,c)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \int_0^\infty \int_0^\infty x^a y^b z^c (-\log F(x))^{m-1} (-\log F(y) + \log F(x))^{n-m-1} \times (-\log F(z) + \log F(y))^{l-n-1} \frac{f(x) f(y)}{F(x) F(y)} f(z) dx dy dz. \tag{2.14}
\]

where \( a_i (n-s_i - 2), a_j (s_i + l - n - 1 - s_2) \) and \( a_k (s_2) \) are defined in the Appendix.

**Proof**

Relation (2.14) can be written as

\[
\mu_{(a,b,c)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \int_0^\infty \int_0^\infty y^b z^c I(y) (-\log F(z) + \log F(y))^{l-n-1} \frac{f(y)}{F(y)} f(z) dy dz, \tag{2.16}
\]

where \( I(y) \) is defined in (2.10).

By integrating \( I(y) \) as it was shown in theorem 2, then by substituting (2.11) into (2.16) we get

\[
\mu_{(a,b,c)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \int_0^\infty \int_0^\infty x^a y^b z^c \frac{1}{\Gamma(a + j_1 + 2)} \times \frac{\Gamma(b + p_1 + j_2 + 2)}{p_2! (t_1 + t_2 + i_1 + i_2 + l - s_2 - 1)^{b+p_1+j_2+2}} \times \frac{\Gamma(c + p_2 + j_3 + 2)}{(t_1 + t_2 + i_1 + i_2 + i_3 + l + k)^{c+p_2+j_3+2}} \int_0^\infty I(z) f(z) dz, \tag{2.17}
\]

where

\[
I(z) = \int_0^\infty y^{b+p_1} [\log F(y)]^{n} e^{-(n-s_i + s_i - 1)y} (-\log F(z) + \log F(y))^{l-n-1} \frac{f(y)}{F(y)} dy. \tag{2.18}
\]

We can find \( J(z) \) in (2.18) in the similar way that was shown in theorem 2, then by substituting the result into (2.17) we get
In the same way of theorem 2, we can get (2.15). Hence the theorem is proved. If $q$ is a positive integer number, then the relation (2.15) becomes

$$
\mu_{m,n,l}^{(a,b,c)} = \frac{1}{\Gamma(m) \Gamma(n-m) \Gamma(l-n)} \sum_{s_1=0}^{n-m-1} \sum_{s_2=0}^{\infty} \sum_{j_1=0}^{n-s_1-s_2} \frac{n!}{s_1! j_1!} \frac{(l-n-1)!}{s_2! j_2!} \theta^{l-s_1-1} \times (-1)^{l-m-s_2} a_h(n-s_1-2) a_i(s_1+l-n-1-s_2) \left( n-s_1+i_1+t_1-2 \right)_{s_1} \left( l-n-1+s_1-s_2+i_2+t_2 \right)_{s_2} \left( \theta-1 \right)_{j_1} \left( \theta+1 \right)_{j_2} \left( \theta+1 \right)_{j_3} \left( \theta+1 \right)_{j_4} \times \prod_{p=0}^{\infty} \frac{\Gamma(a+j_1+2)}{p_1!} \left( n-s_1+i_1+t_1-2 \right)^{p_1} \times \prod_{p=0}^{\infty} \frac{\Gamma(b+p_1+j_2+2)}{p_2!} \left( l-n-1+s_1-s_2+i_2+t_2 \right)^{p_2} \times \prod_{p=0}^{\infty} \frac{\Gamma(c+p_2+j_3+2)}{p_3!} \left( l-n-1+s_1-s_2+i_2+t_2 \right)^{p_3} \times \prod_{p=0}^{\infty} \frac{\Gamma(d+p_3+j_4+2)}{p_4!} \left( l-n-1+s_1-s_2+i_2+t_2 \right)^{p_4},
$$

where $\theta = 1$.

The triple moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.19) by setting $\theta = 1$.

### 2.4 Quadruple moments

The joint p.d.f. of $X_{L(m)}, X_{L(n)}, X_{L(l)}$ and $X_{L(v)}$, $1 \leq m < n < l < v$ is given by (Ahsanullah (1995))

$$
f_{m,n,l,v}(x, y, z, w) = \frac{1}{\Gamma(m) \Gamma(n-m) \Gamma(l-n) \Gamma(v-l)} \left[ -\log F(x) \right]^{m-1} \left[ -\log F(y) + \log F(x) \right]^{n-m-1} \times \left[ -\log F(z) + \log F(y) \right]^{l-1} \left[ -\log F(w) + \log F(z) \right]^{v-l-1} \frac{f(x) f(y) f(z) f(w)}{F(x) F(y) F(z) f(w)},
$$

where $f(.)$ and $F(.)$ are given, respectively, by (1.1) and (1.2).

The quadruple moments of the lower record values, $E(X_{L(m)}^a X_{L(n)}^b X_{L(l)}^c X_{L(v)}^d)$, denoted by $\mu_{m,n,l,v}^{(a,b,c,d)}$, $m, n, l, v = 1, 2, \ldots, m < n < l < v$ and $a, b, c, d = 0, 1, 2, \ldots$, is given by
\[
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \int_0^\infty \int_y^\infty x^a y^b z^c w^d \left[ -\log F(x) \right]^{m-1} \left[ -\log F(y) + \log F(x) \right]^{n-1} \\
\times \left[ -\log F(z) + \log F(y) \right]^{l-1} \left[ -\log F(w) + \log F(z) \right]^{v-1} f(x) f(y) f(z) f(w) \, dx \, dy \, dz \, dw.
\]

(2.21)

The exact explicit expression for the quadruple moments of lower record values from EG distribution is given by the following theorem.

**Theorem 4**

For \( m, n, l, v = 1, 2, \ldots, m < n < l < v, a, b, c, d \geq 0 \) and \( \theta \) is a real value, then

\[
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{\theta^m}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{i_0=0}^{n-1} \sum_{i_1=0}^{n-i_0-1} \sum_{i_2=0}^{n-i_0-i_1-1} \sum_{i_3=0}^{n-i_0-i_1-i_2} \frac{\Gamma(a+j+1)\Gamma(b+p_j+j+2)\Gamma(c+p_j+j+2)\Gamma(d+p_j+j+2)}{p_j!(n-s_i-1)! j_i! (v-l-1)! s_i! l_i!} \times a_i (n-s_i-2) a_3 (s_i + l - n - 1 - s_2) a_{v-1} (v-l-1 + s_2 - s_3)
\]

and \( a_i (n-s_i-2), a_3 (s_i + l - n - 1 - s_2), a_{v-1} (v-l-1 + s_2 - s_3) \) are defined in the Appendix.

**Proof**

Relation (2.21) can be written as

\[
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \int_0^\infty \int_y^\infty x^a y^b z^c w^d I(y) \left[ -\log F(z) + \log F(y) \right]^{l-1} \\
\times \left[ -\log F(w) + \log F(z) \right]^{v-1} f(y) f(z) f(w) \, dy \, dz \, dw,
\]

(2.23)

where \( I(y) \) is defined in (2.10).

By integrating \( I(y) \) as it was shown in theorem 2, then by substituting (2.11) into (2.23) we get

\[
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{i_0=0}^{n-1} \sum_{i_1=0}^{n-i_0-1} \sum_{i_2=0}^{n-i_0-i_1-1} \sum_{i_3=0}^{n-i_0-i_1-i_2} \frac{\Gamma(a+j+1)\Gamma(b+p_j+j+2)\Gamma(c+p_j+j+2)\Gamma(d+p_j+j+2)}{p_j!(n-s_i-1)! j_i! (v-l-1)! s_i! l_i!} \times a_i (n-s_i-2)
\]

and \( a_i (n-s_i-2), a_3 (s_i + l - n - 1 - s_2), a_{v-1} (v-l-1 + s_2 - s_3) \) are defined in the Appendix.

(2.24)

where \( J(z) \) is defined in (2.18). By integrating \( J(z) \) as it was shown in theorem 3, then by substituting the result into (2.24) we get
\begin{equation}
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \frac{n-m-1}{s_1} \frac{n-s_1 + i_1 + t_1 - 2}{j_1} \frac{l-n-1}{s_2} \frac{l-n-1+s_1 - s_2 + i_2 + t_2}{j_2} \frac{\Gamma(a+j_1+2)}{j_1!} \frac{\Gamma(b+p_1 + j_2 + 2)}{j_2!} \frac{\Gamma(c+p_2 + j_3 + 2)}{j_3!} \times \int_{0}^{\infty} w^d K(w) f(w) dw,
\end{equation}

where

\begin{equation}
K(w) = \int_{v}^{\infty} e^{-wz} \log F(z)^{\frac{1}{2}} \left[ -\log F(w) + \log F(z) \right]^{\frac{1}{2} - 1} f(z) \frac{dz}{F(z)}.
\end{equation}

We can find \( K(w) \) in (2.26) in the similar way that was shown in theorem 2, then by substituting the result into (2.25) we get

\begin{equation}
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \frac{n-m-1}{s_1} \frac{n-s_1 + i_1 + t_1 - 2}{j_1} \frac{l-n-1}{s_2} \frac{l-n-1+s_1 - s_2 + i_2 + t_2}{j_2} \frac{\Gamma(a+j_1+2)}{j_1!} \frac{\Gamma(b+p_1 + j_2 + 2)}{j_2!} \frac{\Gamma(c+p_2 + j_3 + 2)}{j_3!} \times \int_{0}^{\infty} w^d \log F(w) \left( e^{-v} - e^{-(v-s_1 + i_1 + t_1 + i_2 + t_2 + v-s_1 + i_1 + t_1 + 1)} \right) f(w) dw.
\end{equation}

In the same way of theorem 2, we can get (2.22). Hence the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (2.22) becomes

\begin{equation}
\mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} \frac{n-m-1}{s_1} \frac{n-s_1 + i_1 + t_1 - 2}{j_1} \frac{l-n-1}{s_2} \frac{l-n-1+s_1 - s_2 + i_2 + t_2}{j_2} \frac{\Gamma(a+j_1+2)}{j_1!} \frac{\Gamma(b+p_1 + j_2 + 2)}{j_2!} \times \int_{0}^{\infty} w^d \log F(w) \left( e^{-v} - e^{-(v-s_1 + i_1 + t_1 + i_2 + t_2 + v-s_1 + i_1 + t_1 + 1)} \right) f(w) dw.
\end{equation}
The quadruple moments of record values from gamma distribution $G(2,1)$ can be obtained from (2.27) by setting $\theta = 1$.

3. Recurrence Relations Between Moments

For the exponentiated gamma distribution, it easily observed that

$$ F(x) = \frac{x^{-1}}{\theta} [e^x - (x+1)] f(x) \quad (3.1) $$

By using this relation, we establish below some recurrence relations satisfied by the single, product, triple and quadruple moments of record values.

3.1 Recurrence relations between single moments

**Theorem 5**

For $n = 1, 2, ..., a = 0, 1, 2, ...$, and $\theta$ is a real value, then

$$ \mu_n^{(a)} - (1 + \frac{a}{2\theta}) \mu_{n+1}^{(a)} = a \sum_{i=0}^{\infty} \frac{1}{i!} \mu_{n+1}^{(a+i-2)} \quad (3.2) $$

**Proof**

Let us replay equation (2.2)

$$ \mu_n^{(a)} = \frac{1}{\Gamma(n+1)} \int_{0}^{\infty} x^n [-\log F(x)]^a f(x) dx. $$

Upon integrating by parts, we obtain

$$ \mu_n^{(a)} - \mu_{n+1}^{(a)} = \frac{a}{\Gamma(n+1)} \int_{0}^{\infty} x^{a-1} [-\log F(x)]^a F(x) dx $$

$$ = \frac{a}{\theta \Gamma(n+1)} \left\{ \int_{0}^{\infty} x^{a-2} e^x - (x+1)[-\log F(x)]^a f(x) dx \right\} \quad \text{(using (3.1))} $$

$$ = \frac{a}{\theta \Gamma(n+1)} \left\{ \int_{0}^{\infty} x^{a-2} e^x [-\log F(x)]^a f(x) dx - \int_{0}^{\infty} x^{a-2} (x+1)[-\log F(x)]^a f(x) dx \right\} $$

$$ = \frac{a}{\theta \Gamma(n+1)} \left\{ \sum_{i=0}^{\infty} \frac{1}{i!} \int_{0}^{\infty} x^{a+i-2} [-\log F(x)]^a f(x) dx \right\} $$

$$ - \int_{0}^{\infty} x^{a-1} [-\log F(x)]^a f(x) dx - \int_{0}^{\infty} x^{a-2} [-\log F(x)]^a f(x) dx \right\} $$

$$ = \frac{a}{\theta} \left\{ \sum_{i=0}^{\infty} \frac{1}{i!} \mu_{n+1}^{(a+i-2)} - \mu_{n+1}^{(a-1)} - \mu_{n+1}^{(a-2)} \right\} $$

After some simplification, we get (3.2). Hence the theorem is proved.

3.2 Recurrence relations between double moments

**Theorem 6**

For $m < n, m,n = 1, 2, ...$ and $a,b = 0, 1, 2, ...$, and $\theta$ is a real value, then

$$ \mu_{m,n}^{(a,b)} - \mu_{m+1,n}^{(a,b)} = \frac{a}{\theta} \sum_{i=0}^{\infty} \frac{1}{i!} \mu_{m+1,n+1}^{(a+i-2,b)} \quad (3.3) $$
Proof
Let us consider (2.9), then by integrating $I(y)$, we get
\[
I(y) = \frac{n-m-1}{m} \int_{y}^{\infty} x^{\alpha} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-2} \frac{f(x)}{F(x)} \, dx \\
+ \frac{a}{m} \int_{y}^{\infty} x^{\alpha-1} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-1} \, dx
\]
From (3.1) we get
\[
I(y) = \frac{n-m-1}{m} \int_{y}^{\infty} x^{\alpha} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-2} \frac{f(x)}{F(x)} \, dx \\
+ \frac{a}{m} \int_{y}^{\infty} x^{\alpha-1} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-1} \frac{f(x)}{F(x)} x^{-[e^{x}-(x+1)]} \, dx \quad (3.4)
\]
By substituting $I(y)$ in (2.9), we get
\[
\mu_{m,n}^{(a,b)}(x) = \frac{1}{\Gamma(m+1)\Gamma(n-m-1)} \int_{y}^{\infty} x^{\alpha} y^{b} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-2} \frac{f(x)}{F(x)} \, dx \\
\times \frac{f(x)}{F(x)} f(y) \, dx \, dy + \frac{a}{\theta \Gamma(m+1)\Gamma(n-m)} \int_{y}^{\infty} x^{\alpha-2} y^{b} \left[ e^{x} - (x+1) \right] \, dx \, dy.
\]
\[
\mu_{m,n}^{(a,b)} - \mu_{m+1,n}^{(a,b)} = \frac{a}{\theta \Gamma(m+1)\Gamma(n-m)} \left\{ \sum_{i=0}^{\infty} \frac{1}{i!} \int_{y}^{\infty} x^{\alpha+i-2} y^{b} \left[ -\log F(x) \right]^{m} \left[ -\log F(y) + \log F(x) \right]^{n-m-1} \frac{f(x)}{F(x)} \, dx \, dy \right\} \times \frac{f(x)}{F(x)} f(y) \, dx \, dy.
\]
In the same manner of theorem 5, we can get
\[
\mu_{m,n}^{(a,b)} - \mu_{m+1,n}^{(a,b)} = \frac{a}{\theta} \left( \sum_{i=0}^{\infty} \frac{1}{i!} \mu_{m+1,n+1}^{(a+i-2,b)} - \mu_{m,n+1}^{(a-1,b)} - \mu_{m+1,n+1}^{(a-2,b)} \right).
\]
After some simplification, we get (3.3). Hence the theorem is proved.

3.3 Recurrence relations between triple moments

Theorem 7
For $m < n < l$, $m,n,l = 1,2,...$ and $a,b,c = 0,1,2,...$, and $\theta$ is a real value, then
\[
\mu_{m,n,l}^{(a,b,c)} - \mu_{m+1,n,l}^{(a,b,c)} = \frac{a}{\theta} \sum_{i=2}^{\infty} \frac{1}{i!} \mu_{m+1,n+1,i+1}^{(a+i-2,b,c)}.
\]
Proof
Let us consider (2.16), then by integrating $I(y)$ in (2.10) by parts as it was shown in theorem 6 and substituting the result (3.4) in (2.16), we get
\[ \mu_{m,n,l}^{(a,b,c)} = \frac{1}{\Gamma(m+1) \Gamma(n-m-1) \Gamma(l-n)} \int_0^\infty \int_0^\infty \int_0^\infty x^a y^b z^c \left[-\log F(x)\right]^m \left[-\log F(y) + \log F(x)\right]^{n-m-2} \times \left[-\log F(z) + \log F(y)\right]^{l-1} f(x) f(y) f(z) dx dy dz \]

\[ + \frac{a}{\theta \Gamma(m+1) \Gamma(n-m) \Gamma(l-n)} \int_0^\infty \int_0^\infty \int_0^\infty x^{a-1} y^b z^c \left[e^x - (x+1)\right][-\log F(x)]^m \times \left[-\log F(y) + \log F(x)\right]^{n-m-1} \left[-\log F(z) + \log F(y)\right]^{l-1} f(x) f(y) f(z) dx dy dz. \]

In the same manner of theorem 6, we can get
\[ \mu_{m,n,l}^{(a,b,c)} - \mu_{m+1,n,l}^{(a,b,c)} = \frac{a}{\theta} \sum_{i=0}^\infty \frac{1}{i!} \mu_{m+i,n+1,l+1}^{(a-1,b,c)} - \mu_{m+1,n+i+1,l+i+1}^{(a-2,b,c)}. \]

After some simplification, we get (3.6). Hence the theorem is proved.

3.4 Recurrence relations between quadruple moments

**Theorem 8**

For \( m < n < l < v, m,n,l,v = 1,2,... \) and \( a,b,c,d = 0,1,2,... \), and \( \theta \) is a real value, then
\[ \mu_{m,n,l,v}^{(a,b,c,d)} - \mu_{m+1,n,l,v}^{(a,b,c,d)} = \frac{a}{\theta} \sum_{i=0}^\infty \frac{1}{i!} \mu_{m+i,n+1,l+1,v+1}^{(a-1,b,c,d)} - \mu_{m+1,n+i+1,l+i+1,v+i+1}. \]

**Proof**

Let us consider (2.23), then by integrating \( I(y) \) in (2.10) by parts as it was shown in theorem 6 and substituting the result (3.4) in (2.23), we get
\[ \mu_{m,n,l,v}^{(a,b,c,d)} = \frac{1}{\Gamma(m+1) \Gamma(n-m-1) \Gamma(l-n) \Gamma(v-l)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^a y^b z^c w^d \left[-\log F(x)\right]^m \times \left[-\log F(y) + \log F(x)\right]^{n-m-1} \times \left[-\log F(w) + \log F(z)\right]^{l-1} f(x) f(y) f(z) f(w) dx dy dz dw \]

\[ + \frac{a}{\theta \Gamma(m+1) \Gamma(n-m) \Gamma(l-n) \Gamma(v-l)} \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x^{a-1} y^b z^c w^d \left[e^x - (x+1)\right][-\log F(x)]^m \times \left[-\log F(y) + \log F(x)\right]^{n-m-1} \times \left[-\log F(w) + \log F(z)\right]^{l-1} \times f(x) f(y) f(z) f(w) dx dy dz dw. \]

In the same manner of theorem 6, we can get
\[ \mu_{m,n,l,v}^{(a,b,c,d)} - \mu_{m+1,n,l,v}^{(a,b,c,d)} = \frac{a}{\theta} \sum_{i=0}^\infty \frac{1}{i!} \mu_{m+i,n+1,l+1,v+1}^{(a-1,b,c,d)} - \mu_{m+1,n+i+1,l+i+1,v+i+1}^{(a-2,b,c,d)}. \]

After some simplification, we get (3.6). Hence the theorem is proved.

4. Moment Generating Function

Let \( X_{L(1)}, X_{L(2)},..., X_{L(n)} \) be the first \( n \) lower record values from the \( EG \) distribution given in (1.1). Then moment generating function (MGF) for the single, double, triple and quadruple moments of lower record values are given as follows.
4.1 Moment generating function for single moments

The MGF of the lower record value \( X_{L(n)} \) denoted by \( M_n(t) \) is given (see Mohie El-Din et al. (2000)) by

\[
M_n(t) = E(e^{tx_{L(n)}}) = \int_0^\infty e^{tx} f_n(x) \, dx, \tag{4.1}
\]

where \( f_n(x) \) is defined in (2.1).

The exact explicit expression for the MGF for single moments of lower record values from \( EG \) distribution is given by the following theorem.

**Theorem 9**

For \( n = 1, 2, \ldots, a \geq 0 \) and \( \theta \) is a real value, then

\[
M_n^{(a)}(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^\infty x^{-i} \sum_{k=0}^\infty (-1)^k a_i (n-1) \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}} \tag{4.2}
\]

where \( a_i (n-1) \) is defined as the Appendix.

**Proof**

From (2.1) and (4.1), we get

\[
M_n(t) = \frac{1}{\Gamma(n)} \int_0^\infty e^{tx} [\log F(x)]^{n-1} f(x) \, dx. \tag{4.3}
\]

By using (1.2), we get

\[
M_n(t) = \frac{\theta^{n-1}}{\Gamma(n)} \int_0^\infty e^{tx} [\log [1-e^{-x}(x+1)]]^{n-1} f(x) \, dx.
\]

From the logarithmic expansion, we get

\[
M_n(t) = \frac{\theta^{n-1}}{\Gamma(n)} \sum_{i=0}^\infty x^{-i} \sum_{k=0}^\infty \frac{e^{-ix}(x+1)^j}{i} \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}} \tag{4.4}
\]

where \( a_i (n-1) \) is defined in the Appendix, from (1.1), we get

\[
M_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^\infty x^{-i} \sum_{k=0}^\infty (-1)^k a_i (n-1) \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n+i+k)^{a+j+2}} \tag{4.4}
\]

By differentiating both sides of (4.4) w.r.t. \( t \), \( a \) times, we can easily obtain (4.2). Note that by putting \( t=0 \) in (4.2), we get (2.3).
If \( \theta \) is a positive integer number, then the relation (4.2) becomes
\[
M^{(a)}_n(t) = \frac{\theta^n}{\Gamma(n)} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^k a_i(n-1) \binom{\theta-1}{k} \binom{n-1+i+k}{j} \frac{\Gamma(a+j+2)}{(n-1+i+k)^{a+j+2}}.
\]

The MGF for single moments of lower record value from gamma distribution \( G(2,1) \) can be obtained from (4.5) by setting \( \theta = 1 \).

4.2 Recurrence relations for single MGF

**Theorem 10**

For \( n = 1, 2, \ldots, a \geq 0 \) and \( \theta \) is a real value, then
\[
M^{(a)}_n(t) + \frac{t-\theta}{\theta} M^{(a)}_{n+1}(t) + \frac{a}{\theta} M^{(a-1)}_{n+1}(t) = \frac{t}{\theta} E_{n+1}(X^{a-1}(e^{(t+1)X} - e^{\theta X})) + \frac{a}{\theta} E_{n+1}(X^{a-2}(e^{(t+1)X} - e^{\theta X})). \tag{4.6}
\]

**Proof**

Let us consider equation (4.3), upon integrating by parts, we obtain
\[
M_n(t) = \frac{1}{\Gamma(n+1)} \int_0^{\infty} \frac{[\log F(x)]^n}{x^n} \{ e^{\theta x} F(x) + e^{\theta x} f(x) \} dx
\]
\[
M_n(t) - M_{n+1}(t) = \frac{t}{\Gamma(n+1)} \int_0^{\infty} e^{\theta x} \{ -\log F(x) \}^n F(x) dx
\]
\[
= \frac{t}{\theta \Gamma(n+1)} \int_0^{\infty} e^{\theta x} x^{-1} \{ -\log F(x) \}^n f(x) dx
\]
\[
= \frac{t}{\theta \Gamma(n+1)} \{ \int_0^{\infty} x^{-1} e^{(t+1)x} \{ -\log F(x) \}^n f(x) dx - \int_0^{\infty} x^{-1} e^{\theta x} (x+1) \{ -\log F(x) \}^n f(x) dx \}
\]
\[
= \frac{t}{\theta} E_{n+1}(e^{(t+1)X}) - \frac{t}{\theta} [M_{n+1}(t) + E_{n+1}(e^{\theta X})].
\]

By rearranging the last equation, we obtain
\[
M_n(t) + \frac{t-\theta}{\theta} M_{n+1}(t) = \frac{t}{\theta} E_{n+1}(\frac{e^{(t+1)X}}{X} - \frac{e^{\theta X}}{X}). \tag{4.7}
\]

By differentiating both sides of (4.7) w.r.t. \( t \), \( a \) times, we can easily obtain (4.6).

Note that by putting \( t = 0 \) in (4.6), we get (3.2).

4.3 Moment generating function for double moments

The joint MGF of \( X_{L(m)} \) and \( X_{L(n)} \), \( E(e^{t_{1}X_{L(m)}+t_{2}X_{L(n)}}) \), denoted by \( M_{m,n}(t_{1},t_{2}) \) is given by
\[
M_{m,n}(t_{1},t_{2}) = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^{\infty} \int_0^{\infty} e^{t_{1}X_{L(m)}+t_{2}X_{L(n)}} f_{m,n}(x,y) dy dx, \tag{4.8}
\]
where \( f_{m,n}(x,y) \) is defined in (2.6).

The exact explicit expression for the MGF for double moments of lower record values from \( EG \) distribution is given by the following theorem.
Theorem 11

For \( m, n = 1, 2, \ldots, m < n \), and \( \theta \) is a real value, then

\[
M_{m,n}(t_1, t_2) = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_0=0}^{\infty} \sum_{j_0=0}^{\infty} \sum_{p=j_0+2}^{\infty} \sum_{q=j_0+2}^{\infty} \sum_{t_1=0}^{n-s+i_0+v-2} \sum_{j_2=0}^{\infty} (-1)^{n-m-s+k-1} \\
\times \alpha_i(s) \alpha_j(n-s-2) \left( \frac{n-m-1}{s} \right) \left( \frac{\theta-1}{k} \right) \left( \frac{i_1+s+k}{j_1} \right) \left( \frac{n-s+i_2+v-2}{j_2} \right) \\
\times \frac{\Gamma(j_1+2)}{\Gamma(p+j_2+2)} \\
\times \frac{1}{p!(1+k+i_1-t_2+s)^{h+2-p}} \frac{(n+v+i_2-t_1-t_2+k)^{p+j_2+2}}{F(x)}.
\] (4.9)

where \( \alpha_i(s) \) and \( \alpha_j(n-s-2) \) is defined as the Appendix.

Proof

From (2.6) and (4.8), we get

\[
M_{m,n}(t_1, t_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \int_0^\infty \int_0^\infty e^{t_1} e^{t_2} [-\log F(x)]^{n-m-1} [-\log F(y) + \log F(x)]^{n-m-1} \frac{f(x)}{F(x)} f(y) dy dx.
\]

where

\[
Q_i(x) = \int_0^x e^{t} [-\log F(y) + \log F(x)]^{n-m-1} f(y) dy
\]

\[
= \sum_{s=0}^{n-m-1} \left( \frac{n-m-1}{s} \right) \left( \frac{\theta-1}{k} \right) \left( \frac{i_1+s+k}{j_1} \right) \left( \frac{n-s+i_2+v-2}{j_2} \right) \left( \frac{\theta-1}{k} \right) \left( \frac{i_1+s+k}{j_1} \right) \left( \frac{n-s+i_2+v-2}{j_2} \right) \frac{\Gamma(j_1+2)}{\Gamma(p+j_2+2)} \\
\times \frac{1}{p!(1+k+i_1-t_2+s)^{h+2-p}} \frac{(n+v+i_2-t_1-t_2+k)^{p+j_2+2}}{F(x)}.
\] (4.11)

Substituting (4.11) into (4.10), we have

\[
M_{m,n}(t_1, t_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \sum_{s=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_0=0}^{\infty} \sum_{j_0=0}^{\infty} \sum_{p=j_0+2}^{\infty} \sum_{q=j_0+2}^{\infty} \sum_{t_1=0}^{n-s+i_0+v-2} \sum_{j_2=0}^{\infty} (-1)^{n-m-s+k-1} \theta^{s+i} \\
\times \alpha_i(s) \alpha_j(n-s-2) \left( \frac{n-m-1}{s} \right) \left( \frac{\theta-1}{k} \right) \left( \frac{i_1+s+k}{j_1} \right) \left( \frac{n-s+i_2+v-2}{j_2} \right) \\
\times \frac{\Gamma(j_1+2)}{\Gamma(p+j_2+2)} \\
\times \frac{1}{p!(1+k+i_1-t_2+s)^{h+2-p}} \frac{(n+v+i_2-t_1-t_2+k)^{p+j_2+2}}{F(x)}.
\]
\[
M_{m,n}(t_1, t_2) = \frac{1}{\Gamma(m)\Gamma(n-m)} \sum_{i_0=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{p+j_1+2}^{\infty} (-1)^{n-m-1+k-1} \theta^{p+2} a_i(s) \left( n-m-1 \right) \frac{(\theta-1)^{k+i_1+s}}{k!} \frac{\Gamma(j_1 + 2)}{\Gamma(j_1 + i_1 + s + t_1 - t_2 + s)^{h+2-p}} \\
	imes \int_0^\infty x^{p+1} e^{-\left(2k+i_1-i_2+s+s\right)x} \left[ -\log[1-e^{-s}(x+1)] \right]^{n-s-2} \ dx
\]

\[
= \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{i_0=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{p+j_1+2}^{\infty} (-1)^{n-m-1+k-1} \ a_i(s) \ a_{i_1}(n-s-2) \\
\times \left( \frac{(n-m-1)}{s} \right)^{\theta-1} \frac{(k+i_1+s)}{j_1} \frac{\Gamma(j_1 + 2)}{\Gamma(j_1 + i_1 + s + t_1 - t_2 + s)^{h+2-p}} \\
\times \int_0^\infty x^{p+1} e^{-\left(k+i_1+i_2-t_1-t_2+s+s\right)x} \left( x+1 \right)^{n-i_1+i_2+s-2} \ dx
\]

\[
= \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{i_0=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{p+j_1+2}^{\infty} (-1)^{n-m-1+k-1} \\
\times a_i(s) a_{i_1}(n-s-2) \left( \frac{n-m-1}{s} \right)^{\theta-1} \frac{(i_1+s+k)}{j_1} \frac{\Gamma(j_1 + 2)}{\Gamma(j_1 + i_1 + s + t_1 - t_2 + s)^{h+2-p}} \\
\times \int_0^\infty x^{p+j_1+s} e^{-\left(k+i_1+i_2-t_1-t_2+s+s\right)x} \ dx
\]

Since the integration is a complete gamma function, we get (4.9). Hence the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (4.9) becomes

\[
M_{m,n}(t_1, t_2) = \frac{\theta^n}{\Gamma(m)\Gamma(n-m)} \sum_{i_0=0}^{n-m-1} \sum_{k=0}^{\infty} \sum_{i_1=0}^{\infty} \sum_{j_1=0}^{\infty} \sum_{p+j_1+2}^{\infty} (-1)^{n-m-1+k-1} \\
\times a_i(s) a_{i_1}(n-s-2) \left( \frac{n-m-1}{s} \right)^{\theta-1} \frac{(i_1+s+k)}{j_1} \frac{\Gamma(j_1 + 2)}{\Gamma(j_1 + i_1 + s + t_1 - t_2 + s)^{h+2-p}} \\
\times \int_0^\infty x^{p+j_1+s} e^{-\left(k+i_1+i_2-t_1-t_2+s+s\right)x} \ dx
\]

(4.12)

The MGF for double moments of lower record values from gamma distribution \( G(2,1) \) can be obtained from (4.12) by setting \( \theta = 1 \).
4.4 Moment generating function for triple moments

The MGF for triple moments of lower record values, \( E(e^{tX_1(t_1)+tX_2(t_2)+tX_3(t_3)}) \), denoted by \( M_{m,n,l}(t_1,t_2,t_3) \) is given by

\[
M_{m,n,l}(t_1,t_2,t_3) = \int_0^\infty \int_0^y \int_0^x e^{t(x+y+z)} f_{m,n,l}(x,y,z) \, dz \, dy \, dx, \tag{4.13}
\]

where \( f_{m,n,l}(x,y,z) \) is defined in (2.13).

The exact explicit expression for the MGF for triple moments of lower record values from EG distribution is given by the following theorem:

**Theorem 12**

For \( m, n, l = 1, 2, \ldots, m < n < l \), and \( \theta \) is a real value, then

\[
M_{m,n,l}(t_1,t_2,t_3) = \frac{\theta^l}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \sum_{i_1=0}^m \sum_{j_1=0}^m \sum_{k=0}^{m+n-1} \sum_{i_2=0}^{i_1+k} \sum_{j_2=0}^{n-m-1} \sum_{i_3=0}^{i_2+j_2} \sum_{j_3=0}^{l-n-1} \frac{1}{\Gamma(j_1+2)} \frac{\Gamma(j_2+2)}{p_1!(s_1+i_1+k+1-t_1)^{s_1+i_2+i_3+s_2+j_2+2}} \times \frac{\Gamma(p_1+j_2+2)}{(v_1+i_1+i_2+i_3+l+k-t_1-t_2-t_3)^{p_1+j_2+2}} \times \frac{1}{(v_1+v_2+i_1+i_2+i_3+l+k-t_1-t_2-t_3)^{p_1+j_2+2}}, \tag{4.14}
\]

where \( a_i(s_i), a_i(l-n-1-s_i+s_2) \) and \( a_i(n-s_2-2) \) are defined as in appendix.

**Proof**

From (2.13) and (4.13), we get

\[
M_{m,n,l}(t_1,t_2,t_3) = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \int_0^x \int_0^y \int_0^z e^{t(x+y+z)} f(x) f(y) f(z) \, dz \, dy \, dx \times [-\log F(z) + \log F(x)]^{n-m-1}
\]

\[
\times [-\log F(z) + \log F(y)]^{l-n-1} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} f(z) \, dz \, dy \, dx.
\]

\[
= \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \int_0^x \int_0^y e^{t(x+y)} Q_2(y) [-\log F(y)]^{n-1} \times [-\log F(y) + \log F(x)]^{l-n-1} \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} dy \, dx, \tag{4.15}
\]

where

\[
Q_2(y) = \int_0^y e^{t(y-z)} [-\log F(z) + \log F(y)]^{l-n-1} f(z) \, dz \tag{4.16}
\]
We can find $Q_2(y)$ in (4.16) in the similar way that was shown in theorem 11 to find $Q_1(x)$, then we get

$$Q_2(y) = \sum_{s_i=0}^{l_n+1} \sum_{s_1=0}^{l_n+1} \sum_{k=0}^{l_n+1} \sum_{j=0}^{l_n+1} \sum_{p_1=0}^{l_n+1} (\theta s_1 l_n + 1) a_i (s_1) \left( l_n + 1 \right) \left( \theta - 1 \right) \left( i_1 + s_1 + k \right) \left( j_1 \right) \right)$$

$$\times \frac{\Gamma(j_1 + 2)}{p_1! (s_1 + i_1 + k + 1 - t_3)^{j_1+2-p_1}} \ [\log F(y)]^{y_p e^{-(\xi_1 + i_1 + k + l - t_3) y}}$$

(4.17)

Substituting (4.17) into (4.15), we get

$$M_{m,n,l}(t_1, t_2, t_3) = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \sum_{s_i=0}^{l_n+1} \sum_{s_1=0}^{l_n+1} \sum_{k=0}^{l_n+1} \sum_{j=0}^{l_n+1} \sum_{p_1=0}^{l_n+1} (\theta s_1 l_n + 1) a_i (s_1) \left( l_n + 1 \right) \left( \theta - 1 \right) \left( i_1 + s_1 + k \right) \left( j_1 \right)$$

$$\times \frac{\Gamma(j_1 + 2)}{p_1! (s_1 + i_1 + k + 1 - t_3)^{j_1+2-p_1}} \int_0^\infty \int_0^\infty e^{t_1 x} e^{t_2 y} \ [\log F(x)]^{m-1} [\log F(y) + \log F(x)]^{n-m-1}$$

$$\times \ [\log F(y)]^{y_p e^{-(\xi_1 + i_1 + k + l - t_3) y}} \frac{f(x) f(y)}{F(x) F(y)} dy dx$$

(4.18)

$$= \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \sum_{s_i=0}^{l_n+1} \sum_{s_1=0}^{l_n+1} \sum_{k=0}^{l_n+1} \sum_{j=0}^{l_n+1} \sum_{p_1=0}^{l_n+1} (\theta s_1 l_n + 1) a_i (s_1) \left( l_n + 1 \right) \left( \theta - 1 \right) \left( i_1 + s_1 + k \right) \left( j_1 \right)$$

$$\times \frac{\Gamma(j_1 + 2)}{p_1! (s_1 + i_1 + k + 1 - t_3)^{j_1+2-p_1}} \int_0^\infty e^{t_1 x} Q_3(x) [\log F(x)]^{m-1} \frac{f(x)}{F(x)} dx$$

(4.19)

where

$$Q_3(x) = \int_0^x \ [\log F(y) + \log F(x)]^{m-1} [\log F(y)]^{y_p e^{-(\xi_1 + i_1 + k + l - t_3) y}} \frac{f(y)}{F(y)} dy.$$
In similar way that was shown in theorem 2, we can get (4.14). Hence the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (4.14) becomes

\[
M_{m,n,l}(t_1,t_2,t_3) = \frac{\theta^\ell}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)} \sum_{i_1=0}^{\ell-m-1} \sum_{i_2=0}^{\ell-1} \sum_{j_1=0}^{i_1} \sum_{j_2=0}^{i_2} \sum_{p_1=0}^{j_1+2} \sum_{p_2=0}^{j_2+2} \times (-1)^{l-m-s_1-s_2+k-2} a_{i_1}(i_1,s_1)a_{i_2}(i_2,l-n-s_1+s_2) \\
\times a_{s_1}(i_1,n-s_2-2) \binom{l-n-1}{s_1} \binom{i_1+s_1+k}{j_1} \binom{n-m-1}{s_2} \times \binom{i_2+s_2+i_1+v_1}{j_2} \binom{n-s_2+i_1+v_2-2}{j_3} \frac{\Gamma(j_1+2)}{p_1!(s_1+i_1+k+1-t_3)^{l+j_3+2+p_1}} \\
\times \frac{\Gamma(p_2+j_2+2)}{p_2!(v_1+k+i_1+i_2+l-n+s_2+1-t_2-t_3)^{l+j_2+2+p_1}} \\
\times \frac{\Gamma(p_3+j_3+2)}{(v_1+v_2+i_1+i_2+i_3+l+k-t_1-t_2-t_3)^{l+j_3+2+p_1}},
\] (4.22)

The MGF for triple moments of lower record values from gamma distribution \( G(2,1) \) can be obtained from (4.22) by setting \( \theta = 1 \).

### 4.5 Moment generating function for quadruple moments

The MGF for quadruple moments of lower record values, \( E(e^{X(t_1,t_2,t_3,t_4)}) \), denoted by \( M_{m,n,l,v}(t_1,t_2,t_3,t_4) \) is given by

\[
M_{m,n,l,v}(t_1,t_2,t_3,t_4) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{t_1+x+y+z+w} f_{m,n,l,v}(x,y,z,w) dw dz dy dx.
\] (4.23)

where \( f_{m,n,l,v}(x,y,z,w) \) is defined in (2.20).

The exact explicit expression for the MGF for quadruple moments of lower record values from \( EG \) distribution is given by the following theorem.

**Theorem 13**

For \( m, n, l, v = 1, 2, \ldots, m < n < l < v \), and \( \theta \) is a real value, then
\[ M_{m,n,i,(t_1,t_2,t_3,t_4)} = \frac{\theta^r}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{s_i=0}^{v-l-1} \sum_{i_0=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_i=0}^{\infty} \frac{(-1)^{k} a_i(s_i)}{p_i!} \frac{\theta^{s_i+1}e^{l-n-1}s_i}{k!} \frac{\theta^{-l-1}s_i}{j_i!} \]
\[ \times \frac{\theta^r}{\Gamma(j_1+2)} \frac{\Gamma(p_1+j_2+2)}{p_2!} \frac{\Gamma(p_2+j_3+2)}{p_3!} \frac{\Gamma(p_3+j_4+2)}{p_4!} (k+v_{i_1}+v_{i_2}+v_{i_3}+v_{i_4}) ]^{r_{i_1}+r_{i_2}+r_{i_3}+r_{i_4}}. \]

where \( a_i(s_i), a_{i_2}(s_2+v-l-1-s_i), a_{i_3}(s_3+l-n-1-s_2) \) and \( a_{i_4}(n-s_3-2) \) are defined as in appendix.

**Proof**

From (2.20) and (4.23), we get

\[ M_{m,n,i,(t_1,t_2,t_3,t_4)} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{l-n-1s_i} [-\log F(x)]^{m-1} \times [-\log F(y)+\log F(x)]^{n-1} \times [-\log F(w)+\log F(z)+\log F(y)]^{l-1} \times f(x)f(y)f(z) F(x) F(y) f(w)dw dz dy dx. \]

\[ = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} e^{l-n-1s_i} [-\log F(x)]^{m-1} \times [-\log F(y)+\log F(x)]^{n-1} \times [-\log F(z)+\log F(y)]^{l-1} \times f(x)f(y)f(z) dz dy dx. \]

where

\[ Q_4(z) = \int_{0}^{\infty} e^{u} [-\log F(w)+\log F(z)]^{l-1} f(w) dw. \]

We can find \( Q_4(z) \) in (4.26) in the similar way that was shown in theorem 11 to find \( Q_4(x) \), then we get

\[ Q_4(z) = \sum_{s=0}^{v-l-1} \sum_{i_0=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_i=0}^{\infty} (-1)^{k} a_i(s_i) \theta^{s_i+1} \frac{\theta^{-l-1}s_i}{k!} \frac{\theta^{-l-1}s_i}{j_i!} \]
\[ \times \frac{\theta^{r}}{\Gamma(j_1+2)} \frac{\Gamma(p_1+j_2+2)}{p_2!} \frac{\Gamma(p_2+j_3+2)}{p_3!} \frac{\Gamma(p_3+j_4+2)}{p_4!} (k+v_{i_1}+v_{i_2}+v_{i_3}+v_{i_4}) ]^{r_{i_1}+r_{i_2}+r_{i_3}+r_{i_4}}. \]
Substituting (4.27) into (4.25), we get
\[
M_{n,m,l,v}(t_1,t_2,t_3,t_4) = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{s_l=0}^{v-l-1} \sum_{s_v=0}^{v-1} \frac{(-1)^{l-1-s_v} a_l(s_v)}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \times \left[ \sum_{j_1=0}^{j_2+2} \sum_{j_2=0}^{j_2+2} \frac{\Gamma(j_1+2)}{j_1!} e^{x t s v y} \times Q_s(y) \left[ -\log F(x) \right]^{j_1+2} \left[ -\log F(x) \right]^{j_2+2} \right] dy dx \tag{4.28}
\]
where
\[
Q_s(y) = \int_0^y \left[ -\log F(z) \right]^{j_1+2} \left[ -\log F(z) \right]^{j_2+2} d\zeta \left[ -\log F(z) \right]^{j_1+2} \left[ -\log F(z) \right]^{j_2+2} \right] dy \tag{4.29}
\]
In similar way that was shown in theorem 2 to find \( I(y) \), we can find \( Q_s(y) \), then we get
\[
Q_s(y) = \sum_{s_l=0}^{v-l-1} \sum_{s_v=0}^{v-1} \frac{(-1)^{l-1-s_v} a_l(s_v)}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \times \left[ \sum_{j_1=0}^{j_2+2} \sum_{j_2=0}^{j_2+2} \frac{\Gamma(j_1+2)}{j_1!} e^{x t s v y} \times \left[ \log F(x) \right]^{j_1+2} \left[ \log F(x) \right]^{j_2+2} \right] dy \tag{4.30}
\]
Substituting (4.30) into (4.28), we get
\[
M_{n,m,l,v}(t_1,t_2,t_3,t_4) = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \sum_{s_l=0}^{v-l-1} \sum_{s_v=0}^{v-1} \frac{(-1)^{l-1-s_v} a_l(s_v)}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \times \left[ \sum_{j_1=0}^{j_2+2} \sum_{j_2=0}^{j_2+2} \frac{\Gamma(j_1+2)}{j_1!} e^{x t s v y} \times \left[ \log F(x) \right]^{j_1+2} \left[ \log F(x) \right]^{j_2+2} \right] dy \tag{4.31}
\]
where
\[
Q_s(x) = \int_0^x \left[ -\log F(y) \right]^{j_1+2} \left[ -\log F(y) \right]^{j_2+2} d\zeta \left[ -\log F(y) \right]^{j_1+2} \left[ -\log F(y) \right]^{j_2+2} \right] dy \tag{4.32}
\]
In similar way that was shown in theorem 2 to find \( I(y) \), we can find \( Q_s(x) \), then we get
\[
Q_s(x) = \sum_{s_l=0}^{v-l-1} \sum_{s_v=0}^{v-1} \frac{(-1)^{l-1-s_v} a_l(s_v)}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \frac{\theta v l - 1 - s_v}{s_v} \times \left[ \sum_{j_1=0}^{j_2+2} \sum_{j_2=0}^{j_2+2} \frac{\Gamma(j_1+2)}{j_1!} e^{x t s v y} \times \left[ \log F(x) \right]^{j_1+2} \left[ \log F(x) \right]^{j_2+2} \right] dy \tag{4.33}
\]
Substituting (4.33) into (4.31), we get
\[ M_{m,n,t_1,t_2,t_3} = \frac{1}{\Gamma(m)\Gamma(n-m)\Gamma(l-n)\Gamma(v-l)} \times \sum_{i=0}^{\nu-1} \sum_{i_1=0}^{\nu_{-1}} \sum_{j=0}^{\nu_{-1}} \sum_{j_1=0}^{\nu_{-1}} \sum_{j_2=0}^{\nu_{-1}} \sum_{j_3=0}^{\nu_{-1}} \sum_{j_4=0}^{\nu_{-1}} a_i(s_i) a_i(s_i + v - l - 1 - s_i) \times \left( \frac{\theta!}{(\nu_1 - 1)!} \right) \frac{\Gamma(j_1 + 2)}{p_1!(s_i + i + k - t_4 + 1)^{\nu_1 + 2 - \nu_1}} \times \frac{\Gamma(j_2 + 2)}{p_2!(v_i + k + i + v - l + s_i - t_3 - t_4 + 1)^{\nu_2 + 2 - \nu_2}} \times \frac{\Gamma(j_3 + 2)}{p_3!(k + s_i - n + v + v_i + i + i_4 - t_i - t_4 - 1)^{\nu_3 + 2 - \nu_3}} \times \int_0^\infty x^n e^{-(k+s_i-n+v+v_i+i+i_4-v_i-i_i-t_i-t_4-1)x} [-\log F(x)]^n x^{n-2} \frac{f(x)}{F(x)} dx \] (4.34)

In similar way that was shown in theorem 2, we can get (4.24). Hence the theorem is proved.

If \( \theta \) is a positive integer number, then the relation (4.34) becomes

\[ M_{m,n,t_1,t_2,t_3} = \frac{\theta!}{(\nu - 1)!} \frac{\Gamma(j_1 + 2)}{p_1!(s_i + i + k - t_4 + 1)^{\nu_1 + 2 - \nu_1}} \times \frac{\Gamma(j_2 + 2)}{p_2!(v_i + k + i + v - l + s_i - t_3 - t_4 + 1)^{\nu_2 + 2 - \nu_2}} \times \frac{\Gamma(j_3 + 2)}{p_3!(k + s_i - n + v + v_i + i + i_4 - t_i - t_4 - 1)^{\nu_3 + 2 - \nu_3}} \times \int_0^\infty x^n e^{-(k+s_i-n+v+v_i+i+i_4-v_i-i_i-t_i-t_4-1)x} [-\log F(x)]^n x^{n-2} \frac{f(x)}{F(x)} dx \] (4.35)

The MGF for quadruple moments of lower record values from gamma distribution \( G(2,1) \) can be obtained from (4.35) by setting \( \theta = 1 \).

**Appendix**

Let us consider

\[ \log(1 - x) = -\sum_{i=1}^\infty \frac{x^i}{i}, |x| < 1, \text{ then} \]

\[ [-\log(1 - x)]^m = \left[ \sum_{i=1}^\infty \frac{x^i}{i} \right]^m \]

\[ = x + \frac{x^2}{2} + \frac{x^3}{3} + ... + \frac{x^m}{m} \]

\[ = \sum_{s=0}^m a_s(m)x^{m+s}, \]
where \( a_s(m) \) is the coefficient of \( x^{m+s} \) in the expansion of \( \left[ \sum_{i=1}^{\infty} \frac{x^i}{i} \right]^m \). (see, Balakrishnan and Cohen (1991)).

The coefficient \( a_s(m) \) can be generated very easily as follows. First of all, we see that

\[
a_0(1) = 1, a_1(1) = \frac{1}{2}, a_2(1) = \frac{1}{3}, \ldots, a_s(1) = \frac{1}{s+1}, \ldots \tag{A.1}
\]

Next, let us consider \( a_s(m) \) for \( m \geq 2 \), given by

\[
a_s(m) = \text{coefficient of } x^s \text{ in } \left[ \sum_{i=1}^{\infty} \frac{x^i}{i} \right]^m
\]

\[
= \sum_{i=1}^{\infty} \{ \text{coefficient of } x^i \text{ in } \left[ \sum_{i=1}^{\infty} \frac{x^i}{i} \right] \} \times \{ \text{coefficient of } x^{s-i} \text{ in } \left[ \sum_{i=1}^{\infty} \frac{x^i}{i} \right]^{m-1} \}
\]

\[
= \sum_{i=1}^{\infty} \frac{1}{i} a_{s-i}(m-1). \tag{A.2}
\]

Thus, by starting with the value of \( a_s(1) \) given in (A.1), we can compute the coefficients \( a_s(m) \) for any value of \( m \) by repeated application of the recurrence relation in (A.2). After computing the coefficients \( a_s(m) \) by this way, one may either directly compute the required moments or MGF's.

Note that, if \( m = 0 \), then the summation \( \left[ \sum_{i=1}^{\infty} (.) \right]^m \) is equal the unity, so \( a_s(0) = 1, s = 0 \) and \( a_s(0) = 0, s > 0 \).

**References**


