DIFFERENCE EQUATIONS

Linear Constant-Coefficient Difference Equations

Difference Equations

In discrete-time systems, essential features of input and output signals appear only at specific instants of time, and they may not be defined between discrete time steps or they may be constant. These systems are also called the **sequential** systems. *tn*, *xtn*, *xn* or *x(n)* are used to represent the input. They are described by **difference equations**. A general Nth-order linear constant-coefficient differential equation can be written as

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

that can be written in compact form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Solutions of difference equations are very similar to those of the differential equations counterparts. If we substitute $y[n] = y_h[n] + y_p[n]$, $\sum_{k=0}^{N} a_k y_h[n-k] = 0$ yielding

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The complete solution requires adding the particular solution to the homogeneous one and evaluating the coefficients of the homogeneous part using initial (auxiliary) conditions. We will be mainly focusing on systems that are initially at rest yielding x[n] = 0 and y[n] = 0 for $n<n_0$. These systems are also called as **causal** systems.

The equation can be rearranged to express the current output as the sum of current and previous inputs and past values of the output.

$$y[n] = \frac{1}{a_0}\left(\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right)$$

The equation clearly indicates the necessity for initial conditions y[-N], y[-N+1], … , y[-1] to evaluate y[n].

The "z" Operator

We can express the unit delay (shift) by using the z operator as y[n+1] = zy[n] and y[n-1] = z^{-1}y[n]. Then, the difference equation can be expressed as

$$\left(a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1} + a_N z^{-N}\right) y[n] = \left(b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{-M+1} + b_M z^{-M}\right) x[n]$$

Calling $\left(a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-N+1} + a_N z^{-N}\right) = D(z)$ and
$\left(b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{-M+1} + b_M z^{-M}\right) = N(z)$,
the output can be expressed as

$$y[n] = \frac{N(z)}{D(z)} x[n]$$

D(z) is the characteristic polynomial of the system. Its roots are used in obtaining the homogeneous solution (hence the impulse response) of the system. The roots can be real or complex. Roots might be distinct or repeated roots. A non-repeating root (z_1) will contribute to the solution by a $(z_1)^n$ function. Repeated roots also contribute by similar functions, but having "n" as multiplier for repetitions.

For a system that was initially at rest, H(z) = N(z)/D(z) is defined as the transfer function of the system (z transform of the impulse response h[n])

Example – 1

Find the complete solution of the system represented by

4y[n] – 4y[n-1] + y[n-2] = 2x[n] – x[n-1]

for x[n] = u[n] assuming that the system is at initial rest (i.e. y[-1] = y[-2] = 0)

$D(z) = 4 - 4z^{-1} + z^{-2} = 0$, multiplying both sides by z^2: $4z^2 - 4z + 1 = 0$ yields $z_{1,2}$ = 1/2. Hence the homogeneous solution is

$$y_h[n] = \left\{ c_1 \left(\frac{1}{2}\right)^n + c_2 n \left(\frac{1}{2}\right)^n \right\} u[n]$$

The input is $x[n] = u[n] = (1)^n u[n]$. The particular solution is

$$y_p[n] = \left. \frac{2 - z^{-1}}{4 - 4z^{-1} + z^{-2}} \right|_{z=1} u[n] = u[n]$$

Complete solution $y[n] = \left\{ c_1 \left(\frac{1}{2}\right)^n + c_2 n \left(\frac{1}{2}\right)^n + 1 \right\} u[n]$

Coefficients c_1 and c_2 must be evaluated using initial conditions y[-1] = y[-2] = 0

The equation can be rewritten as 4y[n] = 2x[n] – x[n-1] + 4y[n-1] - y[n-2]

4y[0] = 2x[0] – x[-1] + 4y[-1] - y[-2] = 2 $\rightarrow$ y[0] = 1/2

4y[1] = 2x[1] – x[0] + 4y[0] - y[-1] = 2 – 1 + 2 = 3 $\rightarrow$ y[1] = 3/4

y[0] = c_1 + 1 = 1/2 $\rightarrow$ c_1 = -1/2

y[1] = c_1(1/2) + c_2(1/2) + 1 = -1/4 + 1 +c_2(1/2) = 3/4 $\rightarrow$ c_2 = 0

Therefore $y[n] = \{-(1/2)(1/2)^n + 1\}u[n]$
Complex roots are expressed in polar form as $\alpha \pm j\beta = re^{\pm j\Omega}$ and the corresponding solution is defined as $r^n\cos(n\Omega)$. Repeated conjugate roots will produce same functions with a multiplier "n" added for every repetition.

The Impulse Response

The impulse response of the system can easily found using the homogenous solution and taking the input as $x[n] = \delta[n]$

Example – 2

Find the impulse response of the system in example -1

h[n] = {c1(1/2)n + c2n(1/2)n}u[n]

$4h[n] = 2\delta[n] - \delta[n-1] + 4h[n-1] - h[n-2]$

The first impulse is applied at n = 0. Therefore all values of x[n] and h[n] prior to n = 0 are considered as zero.

$4h[0] = 2 \rightarrow h[0] = 1/2 = c1$

$4h[1] = -1 + 2 = 1 \rightarrow h[1] = 1/4 = (1/2)(1/2) + c2(1/2) \rightarrow c2 = 0$

Therefore, $h[n] = (1/2)(1/2)^n u[n]$. This is an infinite impulse response (IIR) system.

Impulse and Step Responses

Impulse and step responses for linear systems can be driven easily form each other. For a continuous-time system, the unit impulse can be obtained from unit step and the unit step from unit impulse by differentiation and integration respectively. For a discrete-time system, the unit impulse (sample pulse) can be obtained from unit step and the unit step from unit impulse by differencing and summation respectively. Table-1 summarizes the operations.

Table 1 Derivation of impulse and step responses from each other

System type	Input		Output	
	Step to impulse	Impulse to step	Step to impulse	Impulse to step
Continuous-time	$\delta(t) = {du(t)}/{dt}$	$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$	$h(t) = {dg(t)}/{dt}$	$g(t) = \int_{-\infty}^{t} h(\tau)d\tau$
Discrete-time	$\delta[n] = u[n] - u[n-$	$u[n] = \sum_{k=-\infty}^{n} \delta[k]$	$h[n] = g[n] - g[n-1]$	$g[n] = \sum_{k=-\infty}^{n} h[k]$

Example – 3
Determine the step response of the system in example-2.

$$g[n] = \sum_{k=-\infty}^{n} h[k] = \sum_{k=0}^{n} \frac{1}{2}\left(\frac{1}{2}\right)^k = \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = \left(1-\frac{1}{2}\left(\frac{1}{2}\right)^n\right)u[n]$$

The result is the same as what we obtained in example-1 from direct solution to x[n] = u[n]

Stability

The bounded-input bounded-output (BIBO) stability of the system requires $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

This condition is satisfied only of the roots of the characteristic polynomial remains within a unit circle as illustrated in Fig.1. Roots on the circle itself are stable provided that the system is not disturbed at one of its natural modes (i.e. the input doesn't have one of the functions forming the impulse response).

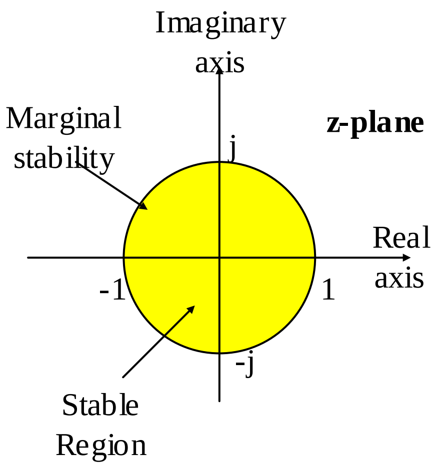

Fig. 1 Illustration of stability for discrete-time systems

The Frequency Response Function

If the input is defined as x[n] = $X_m e^{j\Omega}$, then y[n] can be found as

$$y[n] = H(z)\big|_{z=e^{j\Omega}} X_m e^{jn\Omega + j\theta x} = H(e^{j\Omega}) X_m e^{jn\Omega + j\theta x} = \left|H(e^{j\Omega})\right| X_m e^{jn\Omega t + j\theta x + j\angle H(e^{j\Omega})} = Y_m e^{jn\Omega + j\theta y}$$

$H(e^{j\Omega}) = \left|H(e^{j\Omega})\right| e^{j\angle H(e^{j\Omega})}$ is called the frequency response function of the discrete-time system.

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Questions on Discrete-Time System Descriptions

1. For a discrete-time system represented by the difference equation:

   6y[n+2] – y[n+1] – y[n] = 2x[n+1] – x[n]

a. Draw the representative block diagram using constant multiplier, unit delay and summing junction elements

b. Find the impulse response sequence of the system and express it in implicit form.

c. Is the system stable? Does it have finite impulse response (FIR)?

d. Find the output sequence y[n] using convolution operation for x[n] = u[n] – u[n-4]

e. Draw carefully the sequences for x[n] and y[n].

2. Find and sketch the output for the following systems using convolution:
   a. $x(t)=u(t)-u(t-4)$; $h(t)=e^{-t}u(t)$
   b. $x(n)=2^{-n}u(n)$; $h(n)=[3*2^{-n}+2*3^{-n}]u(n)$

3. A discrete-time is represented by the difference equation

$$y[n] - 0.2y[n-1] - 0.35y[n-2] = x[n] - 0.4x[n-1]$$

   a. Draw a representative block diagram of the system using delay elements, constant multipliers and summing junction
   b. Find out and draw the impulse response of the system (first 5 nonzero elements only)
   c. Is the system stable? Why?
   d. Find the output sequence for x=u[n] using convolution

Hint: $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$

4. A discrete-time filter is designed so that the output is obtained by adding the input and 30% of the output two steps before and subtracting 10% of the output a step before.

   a. Write down the difference equation representing the system and draw a representative block diagram of the system using delay elements, constant multipliers and summing junction
   b. Find out and draw the impulse response of the system (first 5 nonzero elements only)
   c. Is the system stable? Why?