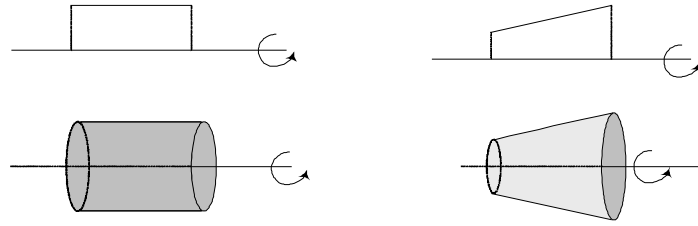


## Surface Areas of Revolution

A surface of revolution is a surface that is generated by revolving a plane curve about an axis that lies in the same plane as the curve.



### Some surface of revolution

**Definition.** If  $f$  is a smooth non-negative function on the interval  $[a, b]$ , then the Surface area  $S$  of revolution that is generated by revolving the portion of the curve  $y = f(x)$  between  $x = a$  to  $x = b$  about  $x$  - axis is defined as

$$S_{x\text{-axis}} = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Moreover, if  $g$  is non-negative and  $x = g(y)$  is a smooth curves on the interval  $[a, b]$ , then the Surface area  $S$  of revolution that is generated by revolving the portion of the curve  $x = g(y)$  between from  $y = c$  to  $y = d$  about  $y$  - axis is defined as

$$S_{y\text{-axis}} = 2\pi \int_c^d x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy.$$

**Example.** Find the area of the surface that is generated by the portion of the curve  $y = x^2$  between  $x = 1$  to  $x = 2$  about  $y$  - axis.

**Solution.** If  $x = \sqrt{y}$ , then  $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$

$$S_{y\text{-axis}} = 2\pi \int_1^4 x \sqrt{1 + \left(\frac{1}{2\sqrt{y}}\right)^2} dy = \pi \int_1^4 \sqrt{4y + 1} dy = \frac{\pi}{4} \left[ \frac{2}{3} (4y + 1)^{3/2} \right]_1^4 = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}).$$

**Example.** Find the area of the surface that is generated by the portion of the curve  $y = \sqrt{1-x^2}$  between  $x = -1$  to  $x = 1$  about  $x$  - axis.

**Solution.**  $\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}}$

$$S_{x\text{-axis}} = 2\pi \int_a^b f(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = 2\pi \int_{-1}^1 \sqrt{1-x^2} \sqrt{1 + \frac{x^2}{1-x^2}} dx = 2\pi \int_{-1}^1 dx = 4\pi.$$

**Problems.** Find the area of the surface that is generated by the portion of the following curves:

- |   |                     |                  |
|---|---------------------|------------------|
| 1) $y = 9x + 1$                         | $0 \leq x \leq 2$   | about $x$ - axis |
| 2) $x = y^3$                            | $0 \leq y \leq 1$   | about $y$ - axis |
| 3) $y =  x - 1 $                        | $0 \leq x \leq 1$   | about $x$ - axis |
| 4) $y = \frac{x^4}{16} + \frac{1}{x^2}$ | $-3 \leq x \leq -1$ | about $x$ - axis |
| 5) $y = \cosh x$                        | $0 \leq y \leq 1$   | about $y$ - axis |
| 6) $y = e^{-x}$                         | $0 \leq x \leq 1$   | about $x$ - axis |