Length of curves

Definition. If $y = f(x)$ is a smooth curve on the interval $[a,b]$, then the arc length L of this curve over $[a,b]$ is defined as

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx .$$

Moreover, for a curve expressed in form $x \in g(y)$ where g' is continuous on $[a,b]$ the arc length L from $y = c$ to $y = d$ can be defined as

$$L = \int_c^d \sqrt{1+(g'(y))^2}\, dy .$$

Example. Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2,\sqrt{2})$?

Solution. $y' = \frac{3}{2}x^{1/2}$, then

$$L = \int_1^2 \sqrt{1+(\tfrac{3}{2}x^{1/2})^2}\, dx = \int_1^2 \sqrt{1+\tfrac{9}{4}x}\, dx$$

$$= \frac{1}{2}\int_1^2 \sqrt{4+9x}\, dx = \frac{1}{27}(4+9x)^{3/2}\Big|_1^2$$

$$= \frac{1}{27}\left((22)^{3/2} - (13)^{3/2}\right) = \frac{1}{27}\left(22\sqrt{22} - 13\sqrt{13}\right).$$

Example. Find the arc length of the curve $y = \ln|\sec x|$ from $x = 0$ to $x = p/4$?

Solution. $y' = \tan x$, then

$$L = \int_0^{p/4} \sqrt{1+\tan x^2}\, dx = \int_0^{p/4} \sec x\, dx$$

$$= \ln\left|\tan x + \sec x\right|_0^{p/4} = \ln|1+\sqrt{2}|.$$

Example. Find the arc length of the curve $y = 3x^{3/2} - 1$, from $x = 1$ to $x = 4$?

Solution. Let $x = (y+1)^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3}(y+1)^{-1/3}$, then

$$L = \int_0^7 \sqrt{1+\frac{4}{9(y+1)^{2/3}}}\, dy$$

Let $u = (y+1)^{2/3}$, then

$$L = \int_0^7 u\sqrt{4+9u^2}\, du = \frac{1}{27}\left|(4+9u^2)^{3/2}\right|_1^2 = 7.63.$$

Problems. Find the arc length of the following curves:

1)	$y = 3x^{3/2} - 1$	from $x = 0$ to $x = 1$
2)	$y = x^{2/3}$	from $x = 1$ to $x = 8$
3)	$y = \ln|\sin x|$	from $x = p/4$ to $x = p/2$
4)	$24xy = y^4 + 48$	from $y = 1$ to $y = 4$
5)	$y = \cosh x$	$-1 \le x \le 1$
6)	$y = e^{-x}$	$-8 \le x \le -5$
7)	$y = x^2 + x$	$1 \le x \le 2$