Darcy’s Law

In 1856, Darcy investigated the flow of water through sand filters for water purification purposes. His experimental apparatus is shown below:

Where \( q \) is the volume flow rate of water downward through the cylindrical sand pack. The sand pack has a length \( L \) and a cross-sectional area \( A \). \( h_1 \) is the height above a datum of water in a manometer located at the input face. \( h_2 \) is the height above a datum of water in a manometer located at the output face. The following assumptions are implicit in Darcy’s experiment:

1. Single-phase flow (only water)
2. Homogeneous porous medium (sand)
3. Vertical flow.
4. Non-reactive fluid (water)
5. Single geometry.

From this experiment, Darcy concluded the following points:

The volume flow rate is directly proportional to the difference of water level in the two manometers; i.e.:
\[ q \propto h_1 - h_2 \]

The volume flow rate is directly proportional to the cross-sectional area of the sand pack; i.e.:

\[ q \propto A \]

The volume flow rate is inversely proportional to the length of the sand pack; i.e.:

\[ q \propto \frac{1}{L} \]

Thus we write:

\[ q = C \frac{A}{L} (h_1 - h_2) \]

Where:
A is the cross-sectional area of the sand pack
L is the length of the sand pack
h₁ is the height above a datum of water in a manometer located at the input face
h₂ is the height above a datum of water in a manometer located at the output face
C is the proportionality constant which depends on the rock and fluid properties. For the fluid effect, C is directly proportional to the fluid specific weight; i.e.

\[ C \propto \gamma \]

and inversely proportional to the fluid viscosity; i.e.:

\[ C \propto \frac{1}{\mu} \]

Thus:

\[ C \propto \frac{\gamma}{\mu} \]

For the rock effect, C is directly proportional to the square of grain size; i.e.:

\[ C \propto (\text{grain size})^2 = d^2 \]

It is inversely proportional to tortuosity; i.e.:

\[ C \propto \frac{1}{\text{tortuosity}} = \frac{1}{\tau} \]
and inversely proportional to the specific surface; i.e.:

\[ C \propto \frac{1}{\text{specific surface}} = \frac{1}{S_s} \]

where \( S_s \) is given by:

\[ S_s = \frac{\text{Interstitial surface area}}{\text{bulk volume}} \]

Combine the above un-measurable rock properties into one property, call it permeability, and denote it by \( K \), we get:

\[ q = K \frac{\gamma}{\mu} \frac{A}{L} (h_1 - h_2) \]

Since:

\[ q = \nu A \]

Thus we can write:

\[ \nu = \frac{q}{A} = \frac{K \gamma}{\mu} \frac{(h_1 - h_2)}{L} \]

Now, let us consider the more realistic flow; i.e. the tilted flow for the same sand pack:
Note that the fluid flows from point 1 to point 2, which means that the pressure at point 1 is higher than the pressure at point 2. Since:

$$h_1 = D_1 - \frac{p_1}{\gamma} \quad \& \quad h_2 = D_2 - \frac{p_2}{\gamma}$$

Thus:

$$h_1 - h_2 = \left( D_1 - \frac{p_1}{\gamma} \right) - \left( D_2 - \frac{p_2}{\gamma} \right) = \left( D_1 - D_2 \right) - \frac{1}{\gamma} (p_1 - p_2)$$

which can be written in a difference form as:

$$h_1 - h_2 = \Delta D - \frac{1}{\gamma} \Delta p$$

Substituting (2) into (1) and rearranging yields:

$$v = K \frac{\gamma}{\mu} \left[ \frac{\Delta D}{L} - \frac{1}{\gamma} \frac{\Delta p}{L} \right] = - \frac{K}{\mu} \left[ \frac{\Delta p}{L} - \gamma \frac{\Delta D}{L} \right]$$

The differential form of Darcy’s equation for single-phase flow is written as follows:

$$v = -\frac{K}{\mu} \left[ \frac{\partial p}{\partial L} - \gamma \frac{\partial D}{\partial L} \right]$$

For multi-phase flow, we write Darcy’s equation as follows:

$$v = -K \frac{k_r}{\mu} \left[ \frac{\partial p}{\partial L} - \gamma \frac{\partial D}{\partial L} \right]$$

More conveniently, the compact differential form of Darcy’s equation is written as follows:

$$\tilde{v} = -K \frac{k_r}{\mu} (\nabla p - \gamma \nabla D)$$

Where K is the absolute permeability tensor which must be determined experimentally. It is written as follows:
\[ K = \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \]

Substitute (5) into (4), we obtain:

\[
\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = -\left( \frac{k_r}{\mu} \right) \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial p}{\partial x} - \gamma \frac{\partial D}{\partial x} \\ \frac{\partial p}{\partial y} - \gamma \frac{\partial D}{\partial y} \\ \frac{\partial p}{\partial z} - \gamma \frac{\partial D}{\partial z} \end{bmatrix}
\]

Solve for velocity components yields:

\[
v_x = -\left( \frac{k_r}{\mu} \right) k_{xx} \left( \frac{\partial p}{\partial x} - \gamma \frac{\partial D}{\partial x} \right) + k_{xy} \left( \frac{\partial p}{\partial y} - \gamma \frac{\partial D}{\partial y} \right) + k_{xz} \left( \frac{\partial p}{\partial z} - \gamma \frac{\partial D}{\partial z} \right)
\]

\[
v_y = -\left( \frac{k_r}{\mu} \right) k_{yx} \left( \frac{\partial p}{\partial x} - \gamma \frac{\partial D}{\partial x} \right) + k_{yy} \left( \frac{\partial p}{\partial y} - \gamma \frac{\partial D}{\partial y} \right) + k_{yz} \left( \frac{\partial p}{\partial z} - \gamma \frac{\partial D}{\partial z} \right)
\]

\[
v_z = -\left( \frac{k_r}{\mu} \right) k_{zx} \left( \frac{\partial p}{\partial x} - \gamma \frac{\partial D}{\partial x} \right) + k_{zy} \left( \frac{\partial p}{\partial y} - \gamma \frac{\partial D}{\partial y} \right) + k_{zz} \left( \frac{\partial p}{\partial z} - \gamma \frac{\partial D}{\partial z} \right)
\]

In most practical problems, it is necessary to assume that \( K \) is a diagonal tensor; i.e.:

\[ K = \begin{bmatrix} k_{xx} & 0 & 0 \\ 0 & k_{yy} & 0 \\ 0 & 0 & k_{zz} \end{bmatrix} \]

and thus velocity components of equation (6) become:

\[
v_x = -k_{xx} \left( \frac{k_r}{\mu} \right) \left( \frac{\partial p}{\partial x} - \gamma \frac{\partial D}{\partial x} \right)
\]

\[
v_y = -k_{yy} \left( \frac{k_r}{\mu} \right) \left( \frac{\partial p}{\partial y} - \gamma \frac{\partial D}{\partial y} \right)
\]

\[
v_z = -k_{zz} \left( \frac{k_r}{\mu} \right) \left( \frac{\partial p}{\partial z} - \gamma \frac{\partial D}{\partial z} \right)
\]
Unit Analysis

Since:

\[ q = \frac{KA \Delta p}{\mu \Delta L} \quad \Rightarrow \quad K = \frac{q \mu \Delta L}{A \Delta p} \]

When \( q \) in cc/sec, \( \mu \) in cp, \( \Delta L \) in cm, \( A \) in cm\(^2\), and \( \Delta p \) in atm, then \( K \) is in the units of Darcy, where:

\[ 1 \text{ Darcy} = \frac{(1 \text{ cm}^3)(1 \text{ cp})(1 \text{ cm})}{(1 \text{ atm})(1 \text{ s})} = \frac{(1 \text{ cm}^2)(1 \text{ cp})}{(1 \text{ atm})(1 \text{ s})} \]

Since

\[ 1 \text{ cp} = \frac{1}{100} \text{ poise} = \frac{1}{100} \frac{\text{dyne} \cdot \text{s}}{\text{cm}^2} \]

and

\[ 1 \text{ atm} = 1,013,250 \frac{\text{dyne}}{\text{cm}^2} \]

thus we have:

\[ 1 \text{ Darcy} = 1000 \text{ md} = \frac{1 \text{ cm}^2 \cdot 1 \text{ dyne} \cdot \text{s} \cdot \text{cm}^2}{(100) \text{ cm}^2 \cdot (1,013,250) \text{ dyne} \cdot \text{s} \cdot \text{cm}^2} = \frac{1}{101,325,000} \text{ cm}^2 \]

or

\[ 1 \text{ md} = \frac{10^{-9}}{101.325} \text{ cm}^2 = 986.923 \times 10^{-14} \text{ cm}^2 \]

Since:

\[ q = \frac{KA \Delta p}{\mu \Delta L} \]

Thus:

\[ \frac{\text{bbl}}{\text{day}} = \frac{(md)(ft)^2(psi)}{(cp)(ft)} = \frac{(md)(ft)(psi)}{(cp)} \]

\[ = \frac{10^{-9}}{101.325} \left( \frac{1}{2.54 \times 12} \right)^2 \left( \frac{1}{5.6146} \right) \text{bbl} \left( \frac{144 \text{ lb}}{ft^2} \right) \]

\[ = \frac{1}{47,880} \left( \frac{1}{86,400} \right) \left( \frac{lb - \text{day}}{ft^2} \right) = 1127106.663 \times 10^{-9} \]

\[ \approx 0.001127107 \]