Numerical solutions for a generalized Ito system by using Adomian decomposition method

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ABSTRACT

In this paper, we used the Adomian decomposition method for solving a generalized Ito system to construct the numerical solutions for this system. The solutions in the form of a rapidly convergent power series with easily computable components, the accuracy of the proposed numerical scheme is examined by comparison with other analytical and numerical results.

Keywords: Adomian Decomposition Method, Generalized Ito System, Numerical Solutions, Analytical Solutions.

2000 Mathematics Subject Classification: 49K05, 49K15, 49S05.

1 Introduction

In this paper, we implemented the Adomian decomposition method [1], [2] for finding the exact and approximate solutions of a generalized Ito system [3] and [4]

\[
\begin{align*}
  u_t &= v_x, \\
  v_t &= -2v_{xxx} - 6(uv)_x - 6(wp)_x, \\
  w_t &= w_{xxx} + 3uw_x, \\
  p_t &= p_{xxx} + 3up_x.
\end{align*}
\]

which is generalization of the well-known integrable Ito system [5]

\[
\begin{align*}
  u_t &= v_x, \\
  v_t &= -2v_{xxx} - 6(uv)_x.
\end{align*}
\]
Hirota bilinear representation of the system (1.1)-(1.4) 3-soliton solutions were found in ref [3] and [4] passes the Painlevé test for integrability in five distinct cases. In recent years, large amounts of effort have been directed towards finding exact solutions. Various methods for seeking explicit traveling solutions to nonlinear partial differential equations have been proposed, such as Darboux transformation [6], [7], Hirota bilinear method [8], [9], Lie group method [10], [11], the homogeneous balance method [12], the tanh method [13], the tanh-coth method [14]. In the beginning of the 1980, a so-called Adomian decomposition method (ADM) [15], [16], [1] Adomian has been used to solve effectively, easily, and accurately a large class of linear and nonlinear equations, solutions partial, deterministic or stochastic differential equations with approximates which converge rapidly.

Body Math Unlike classical techniques, the nonlinear equations are solved easily and elegantly without transforming the equation by using the ADM. The technique has many advantages over the classical techniques, mainly, it avoids linearization and perturbation in order to find explicit solutions of a given nonlinear equations. Many authors have developed this method such as [17], [18], [19], [20]Wazwaz11 and applied this method in numerous of nonlinear partial differential equations, for example [21] used the ADM to solve a generalized fifth order KdV equations, [22] applied it to find the solution the Kaup-Kupershmidt equation, [23] for approximating the solution of the Korteweg-de Vries equation, [24] to construct the solution for the shallow water equations, [25] solved the generalized Burgers-Huxley equation, [26] solved multi-term linear and non-linear diffusion-wave equations of fractional order by using Adomian decomposition method, [27] used the ADM to solve coupled system of nonlinear physical equations, coupled system of diffusion-reaction equation and integro-differential diffusion-reaction equation and [28] used the Adomian decomposition method for solving the Laplace equation with Newmann and Dirichlet boundary conditions in annular domains.

Body Math In this paper, we would like to implement the Adomian decomposition method to a generalized Ito system (1.1)-(1.4).

2 Analysis of the method (ADM)

Body Math In this section, we outline the main steps for applying our method in a generalized Ito system (1.1)-(1.4). For this purposes, using Adomian decomposition method in order to obtain a solution for this system. Let us consider the standard form of Eqs.(1.1)-(1.4) in an operator form:

Body Math

\[
\begin{align*}
L_t u &= L_x v, \quad (2.1) \\
L_t v &= -2L_{xxx}v - 6L_x(uv) - 6L_x(wp), \quad (2.2) \\
L_t w &= L_{xxx}w + 3uL_xw, \quad (2.3) \\
L_t p &= L_{xxx}p + 3uL_xp. \quad (2.4)
\end{align*}
\]

where the notation \( L_t = \frac{\partial}{\partial t}, L_x = \frac{\partial}{\partial x}, L_{xx} = \frac{\partial^2}{\partial x^2} \) and \( L_{xxx} = \frac{\partial^3}{\partial x^3} \), symbolizes the linear differential operators. Assuming the inverse of the operator \( L_t^{-1} \) exists and it can conveniently be taken as the definite integral with respect to \( t \) from 0 to \( t \), i.e.,
Body Math

\[ L^{-1}_t = \int_0^t (\, \cdot \, ) \, dt, \]

to the system in Eqs.(2.1)-(2.4) yields:

\[ u(x, t) = f_1(x) + L^{-1}_t [L_x v], \] (2.5)
\[ v(x, t) = f_2(x) + L^{-1}_t [-2L_{xxx} v - 6(\phi_1 + \phi_2) - 6(\phi_3 + \phi_4)], \] (2.6)
\[ w(x, t) = f_3(x) + L^{-1}_t [L_{xxx} w + 3\phi_5], \] (2.7)
\[ p(x, t) = f_4(x) + L^{-1}_t [L_{xxx} p + 3\phi_6]. \] (2.8)

Where

\[ \phi_1(E_1, E_2) = E_1^2 E_2, \quad \phi_2(E_2) = E_2^3, \]
\[ \phi_3(E_2, n) = E_2 n, \quad \phi_4(E_1) = E_1^3, \]
\[ \phi_5(E_1, E_2) = E_1 E_2^2, \quad \phi_6(E_1, n) = E_1 n, \]
\[ \phi_7(E_1) = E_1^2, \quad \phi_8(E_2) = E_2^2. \]

are related to the nonlinear terms. The ADM [1] assume an infinite series of the unknown functions \( u(x, t), v(x, t), w(x, t) \) and \( p(x, t) \) are given by:

\[ u(x, t) = \sum_{k=0}^{\infty} u_k(x, t), \quad v(x, t) = \sum_{k=0}^{\infty} v_k(x, t), \]
\[ w(x, t) = \sum_{k=0}^{\infty} w_k(x, t), \quad p(x, t) = \sum_{k=0}^{\infty} p_k(x, t). \] (2.9)

We can express \( \phi_1, \phi_2, \phi_3, \phi_4, \phi_5 \) and \( \phi_6 \) in terms of the infinite series of Adomian polynomials as follows:

\[ \phi_1(u, v) = \sum_{k=0}^{\infty} A_k, \quad \phi_2(u, v) = \sum_{k=0}^{\infty} B_k, \]
\[ \phi_3(w, p) = \sum_{k=0}^{\infty} C_k, \quad \phi_4(w, p) = \sum_{k=0}^{\infty} D_k, \]
\[ \phi_5(u, w) = \sum_{k=0}^{\infty} E_k, \quad \phi_6(u, p) = \sum_{k=0}^{\infty} F_k. \] (2.10)

where \( A_n, B_n, C_n, D_n, E_n \) and \( F_n \) are called the appropriate Adomian’s polynomials. These polynomials can be calculated for all forms of nonlinearity according to specific algorithms constructed and given in [1]. For problem (2.5)-(2.8), we use the general form of formula for Adomian polynomials \( A_n, B_n, C_n, D_n, E_n \) and \( F_n \) as:
Construct the solution simple calculations gives us constructed in a form of the recursive relations given by:

\[
A_k(u_0, ..., u_k, v_0, ..., v_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_1 \left( \sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k v_k \right) \right]_{\lambda = 0} \\
B_k(u_0, ..., u_k, v_0, ..., v_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_2 \left( \sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k v_k \right) \right]_{\lambda = 0} \\
C_k(w_0, ..., w_k, p_0, ..., p_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_3 \left( \sum_{k=0}^{\infty} \lambda^k w_k, \sum_{k=0}^{\infty} \lambda^k p_k \right) \right]_{\lambda = 0} \\
D_k(w_0, ..., w_k, p_0, ..., p_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_4 \left( \sum_{k=0}^{\infty} \lambda^k w_k, \sum_{k=0}^{\infty} \lambda^k p_k \right) \right]_{\lambda = 0} \\
E_k(u_0, ..., u_k, w_0, ..., w_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_5 \left( \sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k w_k \right) \right]_{\lambda = 0} \\
F_k(u_0, ..., u_k, p_0, ..., p_k) = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[ \phi_6 \left( \sum_{k=0}^{\infty} \lambda^k u_k, \sum_{k=0}^{\infty} \lambda^k p_k \right) \right]_{\lambda = 0}
\]

\[; k \geq 0. \tag{2.11}\]

Eqs.(2.11) are easy to set computer code to get many polynomials as we need in the calculation of the numerical as well as explicit solutions. The nonlinear system eqs.(2.5)-(2.8) is constructed in the a form of the recursive relations given by:

\[
\begin{align*}
A_k &= \sum_{j=0}^{k} u_j \cdot v_{x(k-j)}, & B_k &= \sum_{j=0}^{k} v_j \cdot u_{x(k-j)}, \\
C_k &= \sum_{j=0}^{k} w_j \cdot p_{x(k-j)}, & D_k &= \sum_{j=0}^{k} p_j \cdot w_{x(k-j)}, \\
E_k &= \sum_{j=0}^{k} u_j \cdot w_{x(k-j)}, & F_k &= \sum_{j=0}^{k} u_j \cdot p_{x(k-j)},
\end{align*} \tag{2.12}\]

where the functions \(f_1(x), f_2(x), f_3(x)\) and \(f_4(x)\) are getting from the initial conditions. We construct the solution \(u(x, t), v(x, t), w(x, t)\) and \(p(x, t)\) as:

\[
\begin{align*}
\lim_{k \to \infty} \phi_k &= u(x, t), & \lim_{k \to \infty} \psi_k &= v(x, t), \\
\lim_{k \to \infty} \tau_k &= w(x, t), & \lim_{k \to \infty} \Omega_k &= p(x, t)
\end{align*} \tag{2.14}\]
where

\[
\phi_k(x, t) = \sum_{n=0}^{k} u_k(x, t), \quad \psi_k(x, t) = \sum_{n=0}^{k} v_k(x, t), \\
\tau_k(x, t) = \sum_{n=0}^{k} w_k(x, t), \quad \Omega_k(x, t) = \sum_{n=0}^{k} p_k(x, t); \quad k \geq 0
\] (2.15)

and the recurrence relation is given as in (2.13). The decomposition series (2.15) solutions are generally converged very rapidly in real physical problem [1]. The convergence of the decomposition series have been investigated by several authors. The theoretical treatment of convergence of the decomposition method has been considered in the Literature [29]. For example, in [20] the authors proposed a new convergence proof of Adomian’s technique based on properties of convergent series. To demonstrate the convergence of the decomposition method, the results of numerical example are presented in the following section, and presented a few terms are required to obtain solutions.

Consider the initial conditions

\[
u(x, t) = u(x, 0) = r_1 - 2\mu^2 \tanh^2(\mu x) = f_1(x), \\
v(x, t) = v(x, 0) = r_2 + b_2 \tanh^2(\mu x) = f_2(x), \\
w(x, t) = w(x, 0) = r_3 + f_1 \tanh(\mu x) = f_3(x), \\
p(x, t) = p(x, 0) = t_0 + t_1 \tanh(\mu x) = f_4(x).
\] (2.16)

where \( r_1 = \frac{-b_2 + 4\mu^4}{6\mu^2}, r_2 = \frac{-b_2^2 + 4f_1t_1\mu^2 - 8b_2\mu^4}{8\mu^4}, r_3 = \frac{-f_1t_0}{t_1}, \mu, b_2, t_0, t_1 \) and \( f_1 \) are arbitrary constants, substitute the initial conditions Eq. (2.16) into Eq. (2.13) gives some terms:

\[
u_1(x, t) = -6f_1t_0 + 6r_3t_1)\mu \sec h^2(\mu x) \\
-12b_2r_1 + 12f_1t_1 - 32b_2\mu^2 - 24r_2\mu^2)\mu \sec h^2(\mu x) \tanh(\mu x), \\
-3(3f_1t_0 + r_3t_1)(3r_1 - 2\mu^2) + (3r_1 + 2\mu^2) \cosh(2\mu x))t^2 \sec h^4(\mu x) \tanh(\mu x)
\]

\[
u_2(x, t) = -6(-6b_2^2 + b_2r_2 - 3f_1r_1t_1 + 2(8b_2r_1 + 6r_1r_2 - f_1t_1))\mu \sec h^2(\mu x) \\
+32(-3b_2r_1 - 3f_1t_1 + 8b_2\mu^2 + 6r_2\mu^2)\mu^2 \sec h^6(\mu x) \\
-3(3f_1t_0 + r_3t_1)(3r_1 - 2\mu^2) + (3r_1 + 2\mu^2) \cosh(2\mu x))t^2 \sec h^4(\mu x) \tanh(\mu x)
\]
\begin{align*}
  w_1(x,t) &= (3r_1 - 2\mu^2) f_1 \mu \sec h^2(\mu x), \\
  w_2(x,t) &= -\frac{1}{2} f_1 (9r_1^2 - 48r_1^2 \mu^2 + 28\mu^4 \\
 &\quad + (3r_1 - 2\mu^2)^2 \cosh^2(2\mu x)) \mu^2 \sec h^4(\mu x) \tanh(x),
\end{align*}

\begin{align*}
  p_1(x,t) &= (3r_1 - 2\mu^2)t_1 \mu \sec h^2(\mu x), \\
  p_2(x,t) &= -\frac{1}{2} t_1 (9r_1^2 - 48r_1^2 \mu^2 + 28\mu^4 \\
 &\quad + (3r_1 - 2\mu^2)^2 \cosh^2(2\mu x)) \mu^2 \sec h^4(\mu x) \tanh(x) \\
  \text{and soon}
\end{align*}

To find the other components of \( u(x,t), v(x,t), w(x,t) \) and \( p(x,t) \) but we see it is difficult to continuum in this manner to find the components any terms, so we see it is write computer code for generating the adomian polynomials by using algorithm was obtained using Mathematica software, so we only consider five terms in evaluating the approximate solutions of a generalized Ito system. It achieves a high level of accuracy.

\begin{align*}
  u_{\text{app}}(x,t) &= \sum_{k=0}^{5} u_k(x,t), \\
  v_{\text{app}}(x,t) &= \sum_{k=0}^{5} v_k(x,t), \\
  w_{\text{app}}(x,t) &= \sum_{k=0}^{5} w_k(x,t), \\
  p_{\text{app}}(x,t) &= \sum_{k=0}^{5} p_k(x,t),
\end{align*}

we can obtain the expression of \( u(x,t), v(x,t), w(x,t) \) and \( p(x,t) \) which is in a Taylor series, then the closed form solutions yields as follows:

\begin{align*}
  u(x,t) &= r_1 - 2\mu^2 \tanh^2[\mu(x - \frac{b_2 t}{2\mu^2})], \\
  v(x,t) &= r_2 + b_2 \tanh^2[\mu(x - \frac{b_2 t}{2\mu^2})], \\
  w(x,t) &= r_3 + f_1 \tanh[\mu(x - \frac{b_2 t}{2\mu^2})], \\
  p(x,t) &= t_0 + t_1 \tanh[\mu(x - \frac{b_2 t}{2\mu^2})],
\end{align*}

\begin{equation}
  (2.17)
\end{equation}

where \( r_1 = \frac{-b_2 + 4\mu^2}{6\mu^2}, r_2 = \frac{-b_2^2 + 4f_1 t_1 \mu^2 - 8b_2 \mu^4}{8\mu^4}, r_3 = -\frac{f_1 t_0}{t_1}, \mu, b_2, t_0, t_1 \) and \( f_1 \) are arbitrary constants. This result can be verified through substitution. Thus, we obtain the solutions of a generalized Ito system.

In tables (1), (2), (3) and (4) show the absolute errors between the exact solution and the numerical solution of a generalized Ito system (2.16) with initial conditions for \( u(x,t), v(x,t), w(x,t) \) and \( p(x,t) \), respectively. The graphs of the analytic and numerical solutions for \( u(x,t), v(x,t), w(x,t) \) and \( p(x,t) \) are depicted in Fig. 1, 2, 3 and 4, respectively.
The absolute errors at some points for \( u(x, t) \) with initial conditions Eq.(2.16) when 
\[
\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6.
\]

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The absolute errors at some points for \( v(x, t) \) with initial conditions Eq.(2.16) when 
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\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6.
\]

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The absolute errors at some points for \( w(x, t) \) with initial conditions Eq.(2.16) when 
\[
\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6.
\]

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The absolute errors at some points for \( p(x, t) \) with initial conditions Eq.(2.16) when 
\[
\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6.
\]

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<th>( x_i )</th>
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<td>( 3.99125 \times 10^{-14} )</td>
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<td>( 5.55112 \times 10^{-17} )</td>
<td>( 4.996 \times 10^{-16} )</td>
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<td>( 1.57652 \times 10^{-14} )</td>
<td>( 5.68989 \times 10^{-14} )</td>
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<td>0.4</td>
<td>( 4.996 \times 10^{-16} )</td>
<td>( 1.66533 \times 10^{-16} )</td>
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<td>( 1.80966 \times 10^{-14} )</td>
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<tr>
<td>0.5</td>
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<td>( 4.44089 \times 10^{-16} )</td>
<td>( 2.94209 \times 10^{-15} )</td>
<td>( 1.99285 \times 10^{-14} )</td>
<td>( 7.81042 \times 10^{-14} )</td>
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Fig. 1. Comparison between the exact solution and the numerical solution for $u(x, t)$ with initial conditions Eq. (2.16) when $\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6$.

Fig. 2. Comparison between the exact solution and the numerical solution for $v(x, t)$ with initial conditions Eq. (2.16) when $\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6$. 
Fig. 3. Comparison between the exact solution and the numerical solution for $w(x, t)$ with initial conditions Eq. (2.16) when $\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6$.

Fig. 4. Comparison between the exact solution and the numerical solution for $p(x, t)$ with initial conditions Eq. (2.16) when $\mu = 0.5, b_2 = 0.03, t_0 = -0.4, t_1 = -0.1, f_1 = 0.6$. 
References


