Introduction

1.1 What is numerical analysis

Numerical analysis is concerned with the mathematical derivation, description and analysis of methods of obtaining solutions of mathematical problems which are difficult to be solved by the analytical methods. For example when we solve the algebraic equations as

$$x^9 - 5x^4 + x^2 - 2x + 5,$$

the transcendental equation such as

$$x^2 \sin x + \ln x = \tan x + 10,$$

or when evaluating the integral

$$\int_1^5 \frac{\sin x}{x} dx,$$

also when we solve the differential equations as

$$\frac{dy}{dt} = y^2 + t^2, \quad y(0) = 1,$$

and some partial differential equations.

Another useful concept of numerical analysis is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of $y = p(x)$ passes through a finite set of given data points, which is named **interpolation**. For these problems we try to get algorithms that have a finite number of steps to obtain from the given data a numerical solution that is an approximation to the exact solution. The numerical solution (or solutions) to the problem can be one number (as an approximation to the finite integral or a solution of the transcendental equation).

This note is supposed to be as an introductory text on numerical analysis for undergraduate mathematicians, and computer scientists. Some calculus is included in the early chapter both to refresh the reader's memory and to provide a foundation on which we may build. Matlab will be used for technical computing. Since it is a powerful, comprehensive and easy-to-use.

1.2 Review

The following theorems are useful in our study.

Theorem 1.1 Rolle's Theorem

Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a)=f(b)=0$ then a number c in (a,b) exists with $f'(c)=0$.

Theorem 1.2 Mean Value Theorem

If $f \in C[a,b]$ and f is differentiable on (a,b), then a number c in (a,b) exists with $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Theorem 1.3 Extreme Value Theorem

If $f \in C[a,b]$, then $c_1, c_2 \in [a,b]$ exist with $f(c_1) \le f(x) \le f(c_2)$ for each $x \in [a,b]$. If, in addition, f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ or where f' is zero.

Theorem 1.4 Generalised Rolle's Theorem

Suppose $f \in C[a,b]$ is n times differentiable on (a,b). If $f(x)$ is zero at the $n+1$ distinct numbers $x_0, \ldots, x_n$ in $[a,b]$, then a number c in (a,b) exists with $f^{(n)}(c)=0$.

Theorem 1.5 Intermediate Value Theorem

If $f \in C[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there exists a number c in (a,b) for which $f(c)=k$.

Theorem 1.6 Taylor's Theorem

Suppose $f \in C^n[a,b]$, and $f^{(n+1)}$ exists on $[a,b]$, and $x_0 \in [a,b]$. For every $x \in [a,b]$ there exists a number $\xi(x)$ between x_0 and x with

$$f(x)=P_n(x)+R_n(x),$$

where

$$P_n(x)=f(x_0)+(x-x_0)f'(x_0)+\frac{(x-x_0)^2}{2}f''(x_0)+\cdots+\frac{(x-x_0)^n}{n!}f^{(n)}(x_0)$$

and

$$R_n=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi(x))$$

In the special case $x_0=0$, the series is called a **Maclaurin series**.

1.3 Errors

Most real numbers cannot be represented exactly by the floating-point representation previously given, and thus they must be approximated. Here are three types of errors that occur in a computation:

1. Initial data errors

When the equations of the mathematical model are formed, these errors arise because of idealistic assumption made to simplify the model, inaccurate measurements of data miscopying of figures, or the inaccurate representation of mathematical constants for example, if the constant π occur in an equation we must replace π by 3.1416, or 3.141593, etc.

2. Truncation errors

This error occurs when we are forced to use mathematical techniques that give approximate rather than exact answers. Truncation errors in numerical analysis usually occurs because many numerical methods are iterative in nature, with the approximations theoretically becoming more accurate as we take more iterations. As a practical matter, we must stop the iteration after a finite number of steps, and thus introduce a truncation error.

3. Rounding and Chopping errors

We say that a number x is **chopped** to n digits or figures when all digits that follow the nth digit are discarded and none of the remaining n digits is changed. Conversely, x is rounded to n digits or figures when x is replaced by an n-digit number that approximates x with minimum error.

We recommend the following rules for correct rounding to n decimal digits

- If the digits beyond the nth digit are greater than or equal to 5000... then ``round up'': drop all digits beyond the nth and add 1 to the nth.
- If they are less than 5000... then ``round down'': discard all digits beyond the nth.

The following definition describes two methods for measuring approximation error.

Definition 1 If x^* is an approximation to x the **absolute error** is $a.e. = |x - x^*|$ and the **relative error** is $r.e. = \frac{|x-x^*|}{|x|}$, provided that $x \neq 0$.

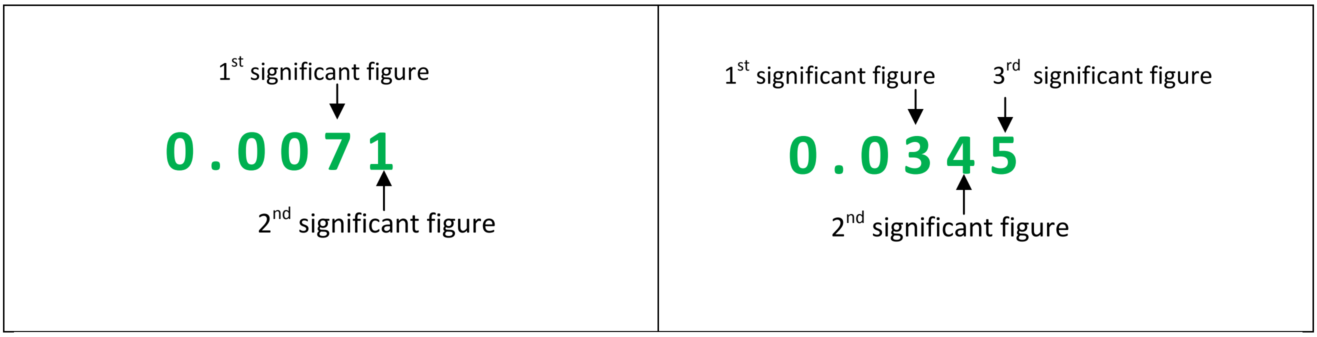
Example 1 Consider the following table

x	x^*	$\lvert x - x^*\rvert$	$\frac{\lvert x - x^*\rvert}{x^*}$
0.3000×10^1	0.3100×10^1	0.1	0.333×10^{-1}
0.3000×10^{-3}	0.3100×10^{-3}	0.1×10^{-4}	0.333×10^{-1}
0.3000×10^4	0.3100×10^4	0.1×10^3	0.333×10^{-1}

This example shows that the same relative error, 0.333×10^{-1}, occurs for widely varying absolute errors. As a measure of accuracy, the absolute error may be misleading and the relative error more meaningful.

Counting significant figures

Significant figures start at the first non-zero number, so ignore the zeros at the front, but not the ones in between. Look at the following examples:



Definition 2 The number x^* is said to approximate x to t **significant digits** (or figures) if t is the largest nonnegative integer for which $r.e. = \frac{|x-x^*|}{x^*} \le 5 \times 10^{-t}$.

Example 2 Given a relative error $r.e. =\ 0.5$, how many significant digits do we have?

Solution

$$r.e. = 0.5 \le 5 \times 10^{-t}$$

Taking $\log_{10}$ of both sides results,

$$\log_{10} r.e. = \log_{10}(0.5) \le \log_{10} 5 - t$$

$$t \le \log_{10} \frac{5}{5 \times 10^{-1}} = 1$$

There is one significant digit.

Example 3 Consider x = 3.29 and $x^* = 3.2$. How accurate is the approximation x^*?

Solution The relative error is

$$r.e. = \left|\frac{3.29 - 3.2}{3.29}\right| = \frac{9 \times 10^{-2}}{3.29} \approx 3 \times 10^{-2}$$

$$t \le \log_{10} \frac{5}{3 \times 10^{-2}} = 2 + \log_{10} \frac{5}{3}$$

which implies that t lies between 2 and 3. There are 2 significant digits.