PHYS 701

Ch. 4

Diffraction

Chapter 4

Chapter Four

Diffraction

- ***General Theory of Diffraction***
- ***Frauhofer and Fresnel Diffraction***
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General Theory of Diffraction

The essential features of diffraction phenomena can be explained qualitatively by Huygens' principle. This principle in its original form states that the propagation of a light wave can be predicted by assuming that each point of the wave front acts as the source of a secondary wave that spreads out in all directions. The envelope of all the secondary waves is the new wave front.

We shall not attempt to treat diffraction by a direct application of Huygens' principle. We want a more quantitative approach. Our strategy will be to cast Huygens' principle into a precise mathematical form known as the *Fresnel-Kirchhoff formula*.

Let us recall Green's theorem that states that if U and V are any two scalar-point functions that satisfy the usual conditions of continuity and integrability,

General Theory of Diffraction

Green's theorem can be proved from the divergence theorem

$$\int\int \text{grad}_{\mathbf{n}}\, \mathbf{F}\, d\mathscr{A} = \int\int\int \nabla \cdot \mathbf{F}\, d\mathscr{V}$$

by setting

$$\mathbf{F} = U\, \nabla\, V - V\, \nabla\, U$$

and using the vector identity

$$\nabla \cdot (U\, \nabla\, V) = U\, \nabla^2 V + (\nabla U) \cdot (\nabla V).$$

then the following equality holds:

$$\iint (V\, \text{grad}_{\mathbf{n}}\, U - U\, \text{grad}_{\mathbf{n}}\, V)\, d\mathscr{A} = \iiint (V\, \nabla^2 U - U\, \nabla^2 V)\, d\mathscr{V}$$

General Theory of Diffraction

The left-hand integral extends over any closed surface $\mathscr{A}$, and the right-hand integral includes the volume $\mathscr{V}$ within that surface. By "grad_n" is meant the normal component of the gradient at the surface of integration.

In particular, if both U and V are wave functions; that is, if they satisfy the regular wave equations

$$\nabla^2 U = \frac{1}{u^2}\frac{\partial^2 U}{\partial t^2} \qquad \nabla^2 V = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$$

and if they both have a harmonic time dependence of the form $e^{\pm i\omega t}$, then it is straightforward to show that the volume integral in Green's theorem is identically zero. The theorem then reduces to

$$\iint (V \,\text{grad}_n U - U \,\text{grad}_n V)\, d\mathscr{A} = 0$$

General Theory of Diffraction

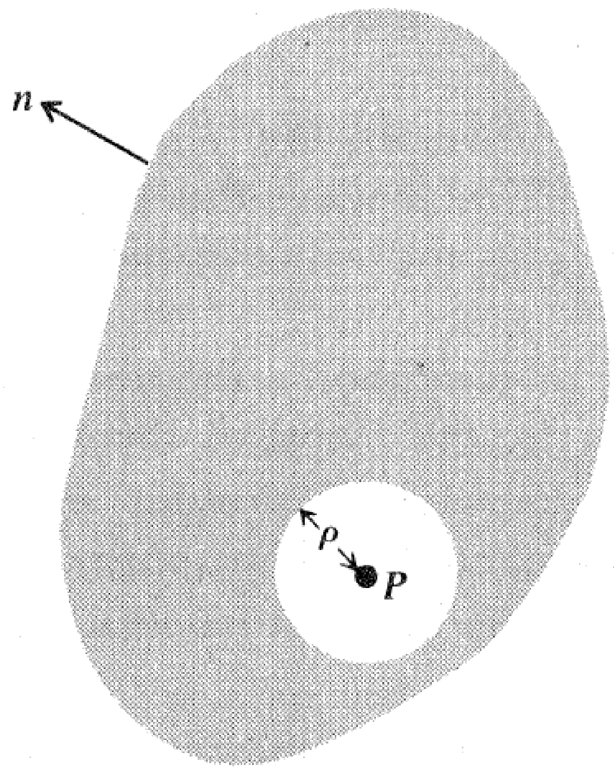
Now suppose that we take V to be the wave function

$$V = V_0 \frac{e^{i(kr+\omega t)}}{r}$$

This particular function represents spherical waves converging to the point P ($r = 0$). We let the volume enclosed by the surface of integration include the point P. Since V becomes infinite at P, we must exclude that point from the integration. This is accomplished by the standard method of subtracting an integral over a small sphere of radius ρ centered at P, as indicated in Figure. Over this small sphere, $r = \rho$ and $\text{grad}_{\mathbf{n}} = -\partial/\partial r$. Hence we can write

$$\iint \left(\frac{e^{ikr}}{r} \text{grad}_{\mathbf{n}}\, U - U \,\text{grad}_{\mathbf{n}} \frac{e^{ikr}}{r} \right) d\mathscr{A} - \iint \left(\frac{e^{ikr}}{r} \frac{\partial U}{\partial r} - U \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \right)_{r=\rho} \rho^2 \, d\Omega = 0$$

General Theory of Diffraction



where $d\Omega$ is the element of solid angle on the sphere centered at P, and $\rho^2\, d\Omega$ is the corresponding element of area. The common factor $V_0 e^{i\omega t}$ has been canceled out.

General Theory of Diffraction

We now let ρ shrink to zero. Then, in the limit as ρ approaches zero the integrand of the second integral approaches the value that U has at the point P, namely U_p. This is easily verified by performing the indicated operations. Consequently, the second integral itself, including the sign, approaches the value

$$\iint U_p \, d\Omega = 4\pi U_p$$

This equation

$$\iint \left(\frac{e^{ikr}}{r} \operatorname{grad}_n U - U \operatorname{grad}_n \frac{e^{ikr}}{r} \right) d\mathscr{A} - \iint \left(\frac{e^{ikr}}{r} \frac{\partial U}{\partial r} - U \frac{\partial}{\partial r} \frac{e^{ikr}}{r} \right)_{r=\rho} \rho^2 \, d\Omega = 0$$

then becomes, on rearranging terms,

General Theory of Diffraction

$$U_p = -\frac{1}{4\pi} \iint \left(U \,\text{grad}_n \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \text{grad}_n\, U \right) d\mathscr{A}$$

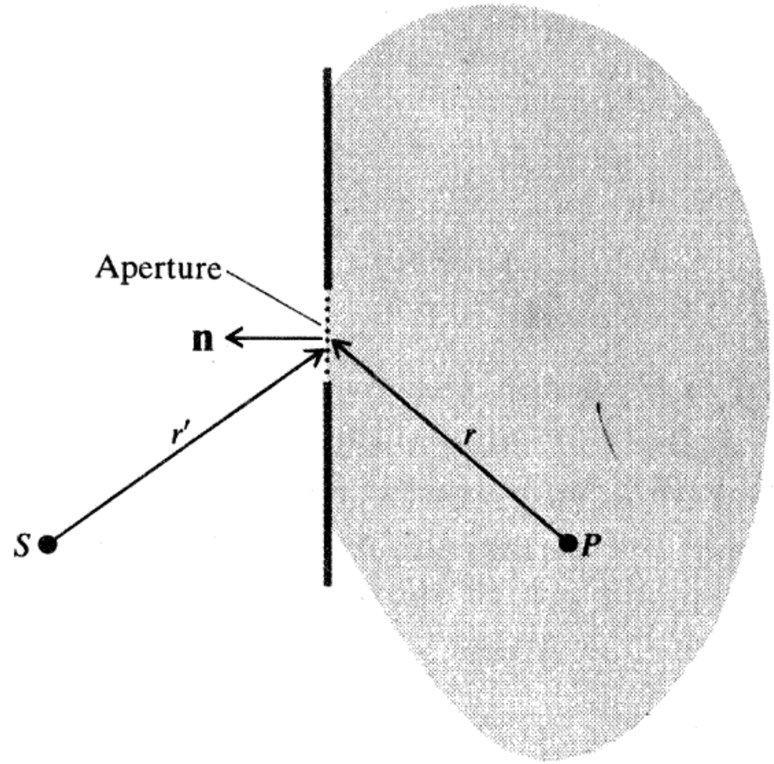
This equation is known as the *Kirchhoff integral theorem*. It relates the value of any scalar wave function at any point *P inside* an arbitrary closed surface to the value of the wave function *at* the surface.

The Fresnel-Kirchhoff Formula

We now proceed to apply the Kirchhoff integral theorem to the general problem of diffraction of light.

The diffraction is produced by an aperture of arbitrary shape in an otherwise opaque partition. This partition separates a light source from a receiving point (Figure).

General Theory of Diffraction



If r' denotes the position of a point on the aperture relative to the source S, then the wave function at the aperture is given by the expression

General Theory of Diffraction

$$U = U_0 \frac{e^{i(kr' - \omega t)}}{r'}$$

which represents spherical monochromatic waves traveling outward from S. The Kirchhoff integral theorem then yields

$$U_p = \frac{U_0 e^{-i\omega t}}{4\pi} \iint \left(\frac{e^{ikr}}{r} \operatorname{grad}_n \frac{e^{ikr'}}{r'} - \frac{e^{ikr'}}{r'} \operatorname{grad}_n \frac{e^{ikr}}{r} \right) d\mathscr{A}$$

where the integration extends only over the aperture opening.

The operations indicated in the integrand are carried out as follows:

$$\operatorname{grad}_n \frac{e^{ikr}}{r} = \cos(\mathbf{n},\mathbf{r}) \frac{\partial}{\partial r} \frac{e^{ikr}}{r} = \cos(\mathbf{n},\mathbf{r}) \left(\frac{ike^{ikr}}{r} - \frac{e^{ikr}}{r^2} \right)$$

$$\operatorname{grad}_n \frac{e^{ikr'}}{r'} = \cos(\mathbf{n},\mathbf{r}') \frac{\partial}{\partial r'} \frac{e^{ikr'}}{r'} = \cos(\mathbf{n},\mathbf{r}') \left(\frac{ike^{ikr'}}{r'} - \frac{e^{ikr'}}{r'^2} \right)$$

General Theory of Diffraction

where (**n**,**r**′) and (**n**,**r**′) denote the angles between the vectors and the normal to the surface of integration. Consequently Equation gives

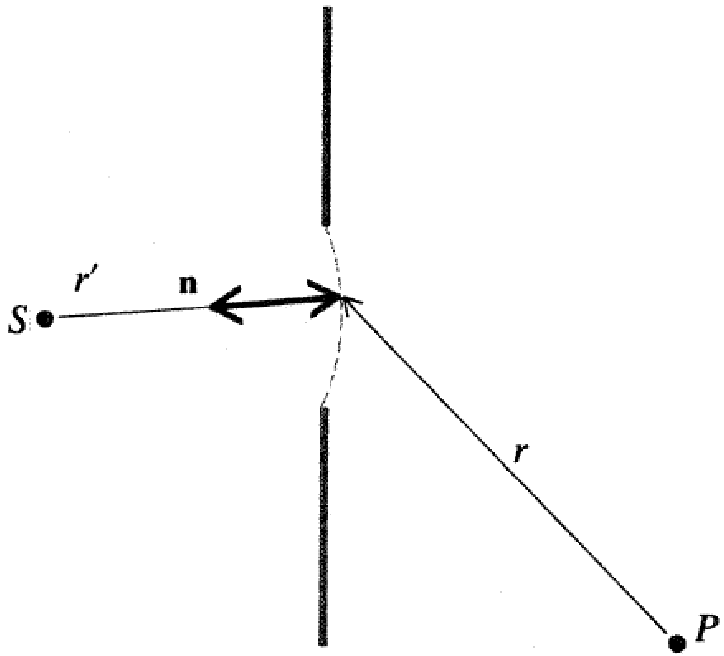
$$U_p = -\frac{ikU_0 e^{-i\omega t}}{4\pi} \iint \frac{e^{ik(r+r')}}{rr'} [\cos(\mathbf{n},\mathbf{r}) - \cos(\mathbf{n},\mathbf{r}')]\, d\mathscr{A}$$

This equation is known as the *Fresnel Kirchhoff integral formula.*

It is, in effect, a mathematical statement of Huygens' principle. This is most easily seen by applying the formula to a specific case, namely, that of a circular aperture with the source symmetrically located as shown in Figure. The surface of integration is taken to be a spherical cap bounded by the aperture opening. In this case r' is constant and $\cos(\mathbf{n},\mathbf{r}') = -1$. The Fresnel-Kirchhoff formula then reduces to

$$U_p = -\frac{ik}{4\pi} \iint \frac{U_{\mathscr{A}} e^{i(kr-\omega t)}}{r} [\cos(\mathbf{n},\mathbf{r}) + 1]\, d\mathscr{A}$$

General Theory of Diffraction



where

$$U_{\mathscr{A}} = \frac{U_o e^{ikr'}}{r'}$$

General Theory of Diffraction

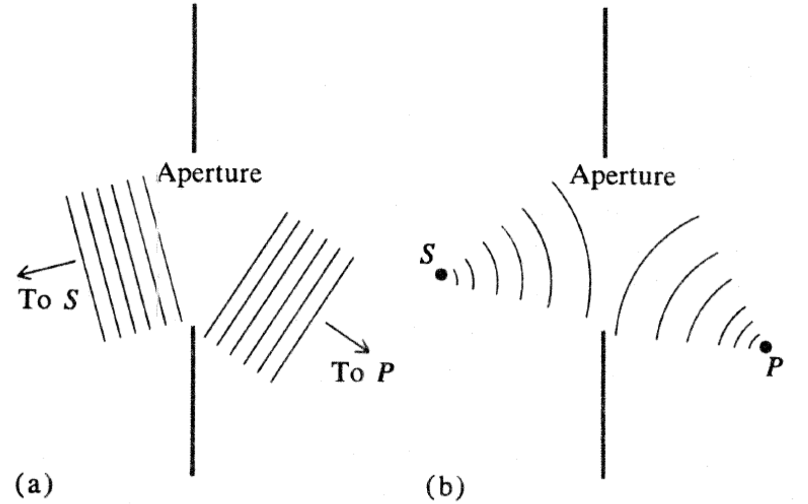
Equation can be given the following simple interpretation:

$U_{\mathscr{A}}$ is the complex amplitude of the incident primary wave at the aperture. From this primary wave each element $d\mathscr{A}$ of the aperture gives rise to a secondary spherical wave

$$\frac{U_{\mathscr{A}} e^{i(kr-\omega t)}}{r} d\mathscr{A}$$

Frauhofer and Fresnel Diffraction

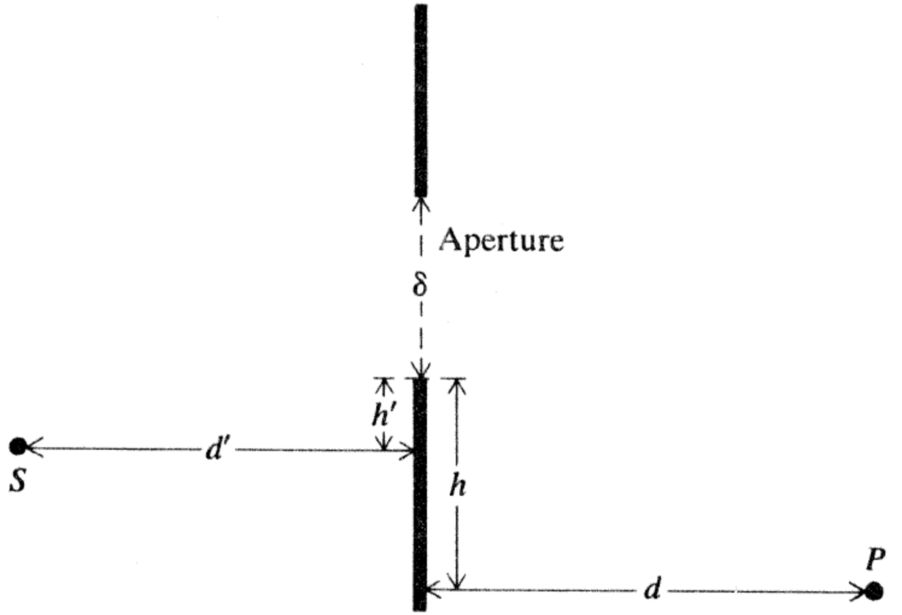
In the detailed treatment of diffraction it is customary to distinguish between two general cases. These are known as *Fraunhofer diffraction* and *Fresnel diffraction*. Qualitatively speaking, Fraunhofer diffraction occurs when both the incident and diffracted waves are effectively plane. This will be the case when the distances from the source to the diffracting aperture and from the aperture to the receiving point are both large enough for the curvatures of the incident and diffracted waves to be neglected [Figure].



Diffraction by an aperture.
(a) Fraunhofer case; (b) Fresnel case.

Frauhofer and Fresnel Diffraction

If either the source or the receiving point is close enough to the diffracting aperture so that the curvature of the wave front is significant, then one has Fresnel diffraction [Figure (b)]. There is, of course, no sharp line of distinction between the two cases. However, a quantitative criterion can be obtained as follows. Consider Figure, which shows the general geometry of the diffraction problem.



Frauhofer and Fresnel Diffraction

The receiving point P is located a distance d from the plane of the diffracting aperture, and the source S is a distance d' from this plane. One edge of the aperture is located a distance h from the foot of the perpendicular drawn from P to the plane of the aperture. The corresponding distance for the source is h' as shown. The size of the aperture opening is δ. From the figure it is seen that the variation Δ of the quantity $r + r'$ from one edge of the aperture to the other is given by

$$\Delta = \sqrt{d'^2 + (h' + \delta)^2} + \sqrt{d^2 + (h + \delta)^2} - \sqrt{d'^2 + h'^2} - \sqrt{d^2 + h^2}$$
$$= \left(\frac{h'}{d'} + \frac{h}{d}\right)\delta + \frac{1}{2}\left(\frac{1}{d'} + \frac{1}{d}\right)\delta^2 + \cdots$$

The quadratic term in the expansion above is essentially a measure of the curvature of the wave front. The wave is effectively plane over the aperture if this term is negligibly small compared to the wavelength of the light, that is, if

Frauhofer and Fresnel Diffraction

$$\frac{1}{2}\left(\frac{1}{d'}+\frac{1}{d}\right)\delta^2 \ll \lambda$$

This is the criterion for Fraunhofer diffraction. If this condition does not obtain, the curvature of the wave front becomes important and the diffraction is of the Fresnel type. Similar considerations apply in the case of diffraction by an opaque object or obstacle. Then δ is the linear size of the object.

Frauhofer and Fresnel Diffraction

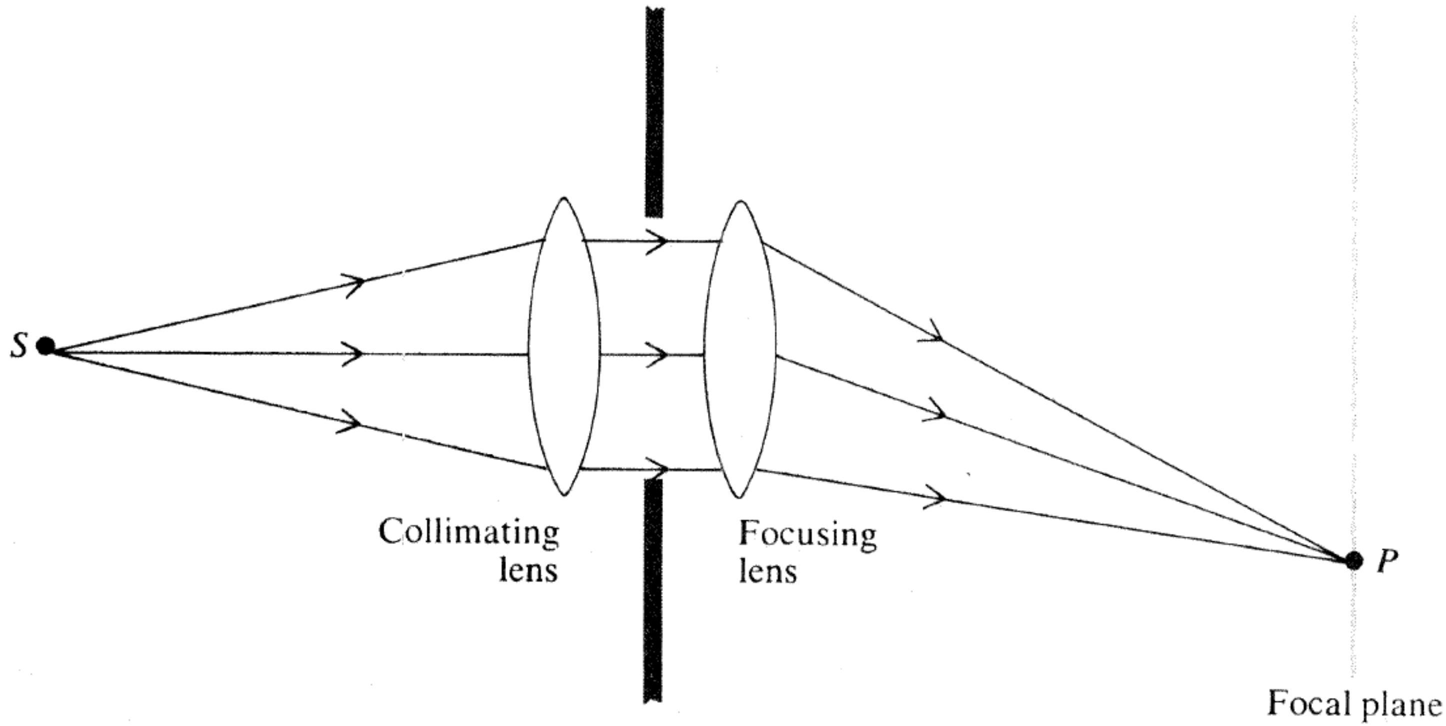
Example 1:

In a diffraction experiment a point (pinhole) source of wavelength 600 nm is to be used. The distance from the source to the diffracting aperture is 10 m, and the aperture is a hole of 1-mm diameter. Determine whether Fresnel or Fraunhofer diffraction applies when the screen-to-aperture distance is (a) 1 cm (b) 2 m.

Solution:

Fraunhofer Diffraction Patterns

The usual experimental arrangement for observing Fraunhofer diffraction is shown in Figure. Here the aperture is *coherently illuminated* by means of a point monochromatic source and a collimating lens. A second lens is placed behind the aperture as shown.



Fraunhofer Diffraction Patterns

The incident and diffracted wave fronts are therefore strictly plane, and the Fraunhofer case is rigorously valid. In applying the Fresnel-Kirchhoff formula [Equation] to the calculation of the diffraction patterns, the following simplifying approximations are taken to be valid:

$$U_p = -\frac{ikU_0e^{-i\omega t}}{4\pi}\iint \frac{e^{ik(r+r')}}{rr'}\left[\cos(\mathbf{n},\mathbf{r}) - \cos(\mathbf{n},\mathbf{r}')\right] d\mathscr{A}$$

(1) The angular spread of the diffracted light is small enough for the obliquity factor $[\cos(\mathbf{n},\mathbf{r}) - \cos(\mathbf{n},\mathbf{r}')]$ not to vary appreciably over the aperture and to be taken outside the integral.

(2) The quantity $e^{ikr'}/r'$ is very nearly constant and can be taken outside the integral.

(3) The variation of the remaining factor e^{ikr}/r over the aperture comes principally from the exponential part, so the factor $1/r$ can be replaced by its mean value and taken outside the integral.

Fraunhofer Diffraction Patterns

Consequently, the Fresnel-Kirchhoff formula reduces to the very simple equation

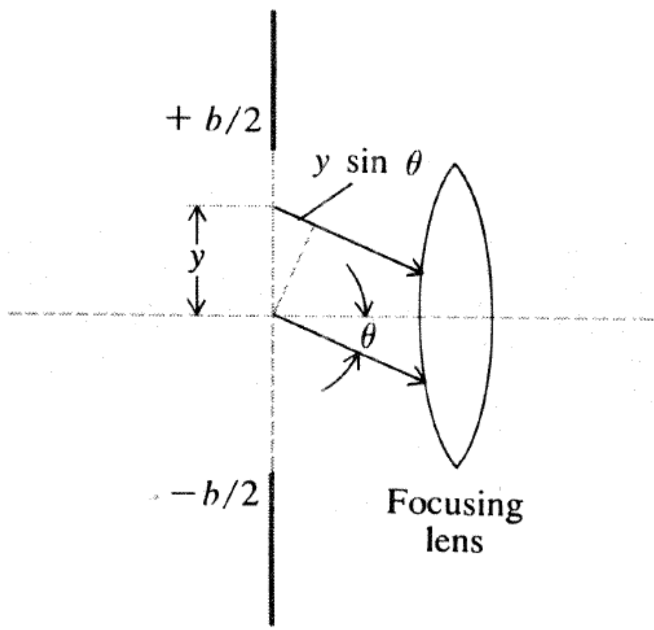
$$U_p = C \iint e^{ikr} \, d\mathscr{A}$$

where all constant factors have been lumped into one constant C. The formula above states that the distribution of the diffracted light is obtained simply by integrating the phase factor e^{ikr} over the aperture.

The Single Slit

The case of diffraction by a single narrow slit is treated here as a one-dimensional problem. Let the slit be of length L and of width b. The element of area is then $d\mathscr{A} = L\,dy$ as indicated in Figure.

Fraunhofer Diffraction Patterns



Furthermore, we can express r as

$$r = r_0 + y \sin \theta$$

where r_0 is the value of r for $y = 0$, and where θ is the angle shown. The diffraction formula then yields

Fraunhofer Diffraction Patterns

$$U = Ce^{ikr_0} \int_{-b/2}^{+b/2} e^{iky \sin\theta} L \, dy$$

$$= 2\,Ce^{ikr_0} L \frac{\sin\left(\frac{1}{2}kb \sin\theta\right)}{k \sin\theta} = C' \left(\frac{\sin\beta}{\beta}\right)$$

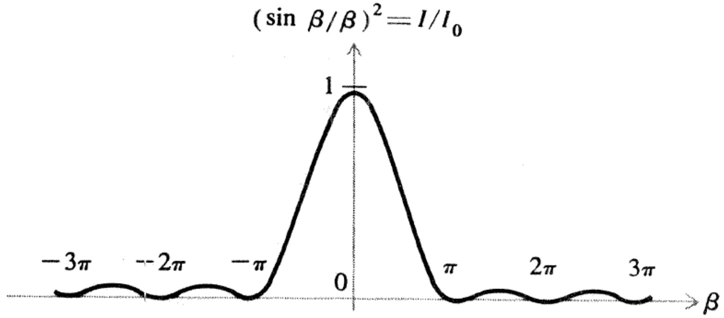
where $\beta = \frac{1}{2}kb \sin\theta$, and $C' = e^{ikr_0} CbL$ is merely another constant.

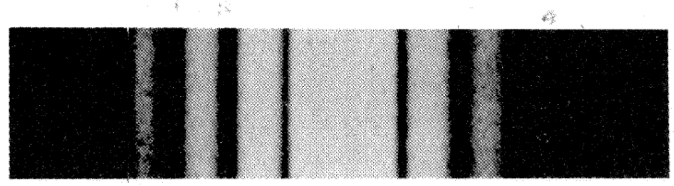
Thus $C' (\sin\beta/\beta)$ is the total amplitude of the light diffracted in a given direction defined by β. This light is brought to a focus by the second lens, and the corresponding irradiance distribution in the focal plane is given by the expression

$$I = |U|^2 = I_0 \left(\frac{\sin\beta}{\beta}\right)^2$$

Fraunhofer Diffraction Patterns

where $I_0 = |CLb|^2$, which is the irradiance for $\theta = 0$. The distribution is plotted in Figure. The maximum value occurs at $\theta = 0$, and zero values occur for $\beta = \pm\pi, \pm 2\pi, \ldots$, and so forth. Secondary maxima of rapidly diminishing value occur between these zero values. Thus the diffraction pattern at the focal plane consists of a central bright band. On either side there are alternating bright and dark bands.





Fraunhofer Diffraction Patterns

Table gives the relative values of I of the first three secondary maxima.

RELATIVE VALUES OF THE MAXIMA OF DIFFRACTION PATTERNS OF RECTANGULAR AND CIRCULAR APERTURES

	Rectangular	*Circular*
Central Max	1	1
1st Max	0.0496	0.0174
2d Max	0.0168	0.0042
3rd Max	0.0083	0.0016

The first minimum, $\beta = \pi$, corresponds to

$$\sin\theta = \frac{2\pi}{kb} = \frac{\lambda}{b}$$

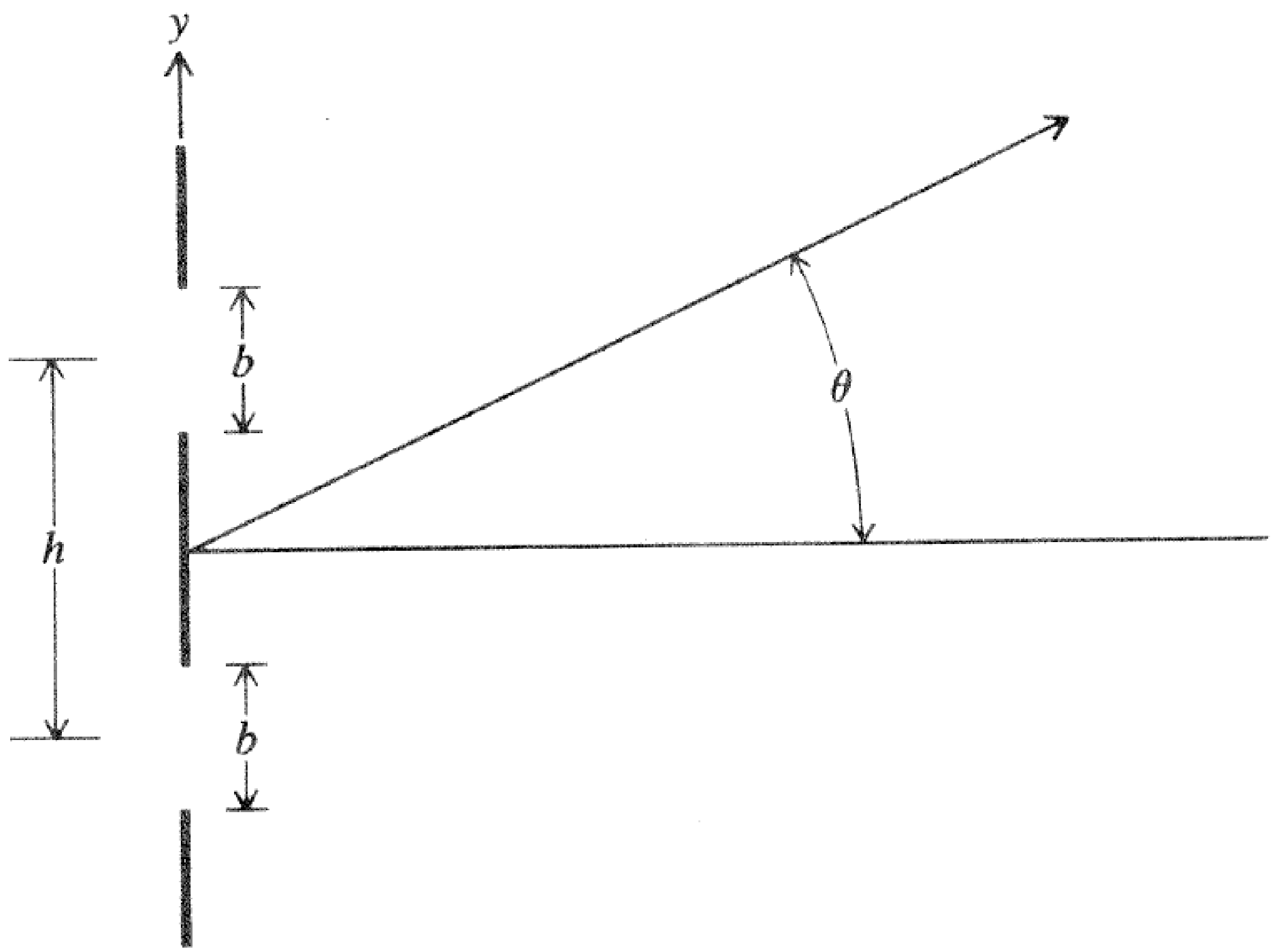
Fraunhofer Diffraction Patterns

Thus, for a given wavelength, the angular width of the diffraction pattern varies inversely with the slit width, and the amplitude of the central maximum is proportional to the area of the slit. For very narrow slits the pattern is dim but wide. It shrinks and becomes brighter as the slit is widened.

The Double Slit

Let us consider a diffracting aperture consisting of two parallel slits, each of width b and separated by a distance h (Figure). As with the single slit, we treat this case as a one-dimensional problem. The relevant diffraction integral is evaluated as follows:

Fraunhofer Diffraction Patterns



Fraunhofer Diffraction Patterns

$$\int_{\mathscr{A}} e^{iky\sin\theta}\,dy = \int_0^b e^{iky\sin\theta}\,dy + \int_h^{h+b} e^{iky\sin\theta}\,dy$$

$$= \frac{1}{ik\sin\theta}\left(e^{ikb\sin\theta} - 1 + e^{ik(h+b)\sin\theta} - e^{ikh\sin\theta}\right)$$

$$= \left(\frac{e^{ikb\sin\theta} - 1}{ik\sin\theta}\right)\left(1 + e^{ikh\sin\theta}\right)$$

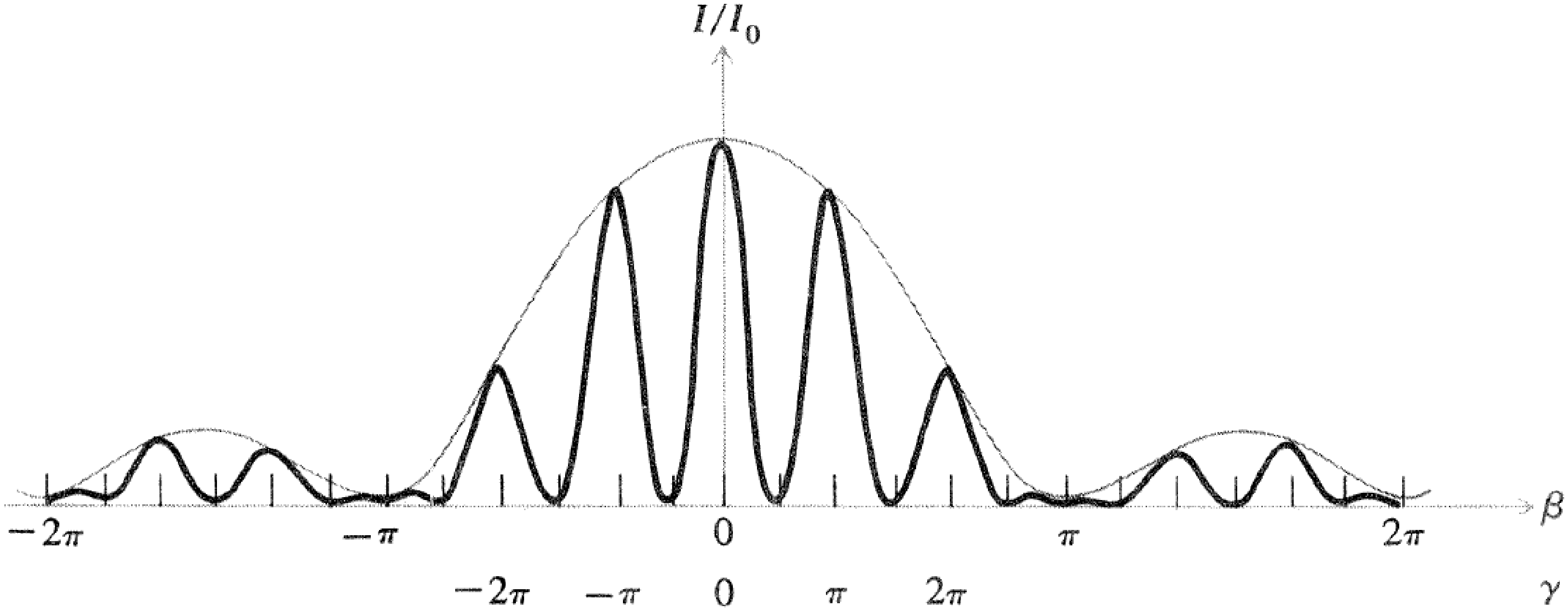
$$= 2b\,e^{i\beta}\,e^{i\gamma}\,\frac{\sin\beta}{\beta}\cos\gamma$$

where $\beta = \frac{1}{2}kb\sin\theta$ and $\gamma = \frac{1}{2}kh\sin\theta$. The corresponding irradiance distribution function is

$$I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\gamma$$

Fraunhofer Diffraction Patterns

The factor $(\sin \beta/\beta)^2$ is the previously found distribution function for a single slit. Here this factor constitutes an envelope for the interference fringes given by the term $\cos^2 \gamma$. A plot is shown in Figure.



Fraunhofer Diffraction Patterns

Bright fringes occur for $\gamma = 0, \pm\pi, \pm 2\pi$, and so forth. The angular separation between fringes is given by $\Delta\gamma = \pi$, or, approximately, in terms of the angle θ

$$\Delta\theta \approx \frac{2\pi}{kh} = \frac{\lambda}{h}$$

It is interesting to note that this is equivalent to the result of the analysis of Young's experiment [Equation].

$$y = 0, \pm\frac{\lambda x}{h}, \pm\frac{2\lambda x}{h}, \ . \ . \ .$$

Fraunhofer Diffraction Patterns

Example 2:

A collimated beam of light from a helium–neon laser ($\lambda = 633$ nm) falls normally on a slit 0.5 mm wide. A lens of 50 cm focal length placed just behind the slit focuses the diffracted light on a screen located at the focal distance. Calculate the distance from the center of the diffraction pattern (central maximum) to the first minimum and to the first secondary maximum.

Solution:

Fraunhofer Diffraction Patterns

Example 3:

If white light were used in the above diffraction experiment, for what wavelength would the fourth maximum coincide with the third maximum for red light ($\lambda = 650$ nm)?

Solution:

Fraunhofer Diffraction Patterns

Example 4:

In a single-slit diffraction pattern the intensity of the successive bright fringes falls off as we go out from the central maximum. Approximately which fringe number has a peak intensity that is $\frac{1}{2}$ percent of the central fringe intensity? (Assume Fraunhofer diffraction applies.)

Solution:

Fraunhofer Diffraction Patterns

Example 5:

In the Fraunhofer diffraction pattern of a double slit, it is found that the fourth secondary maximum is missing. What is the ratio of slit width b to slit separation h?

Solution:

Fraunhofer Diffraction Patterns

Example 6:

Show that the Fraunhofer diffraction pattern of a double slit reduces to that of a single slit of width $2b$ when the slit width is equal to the separation, that is, when $h = b$.

Solution:

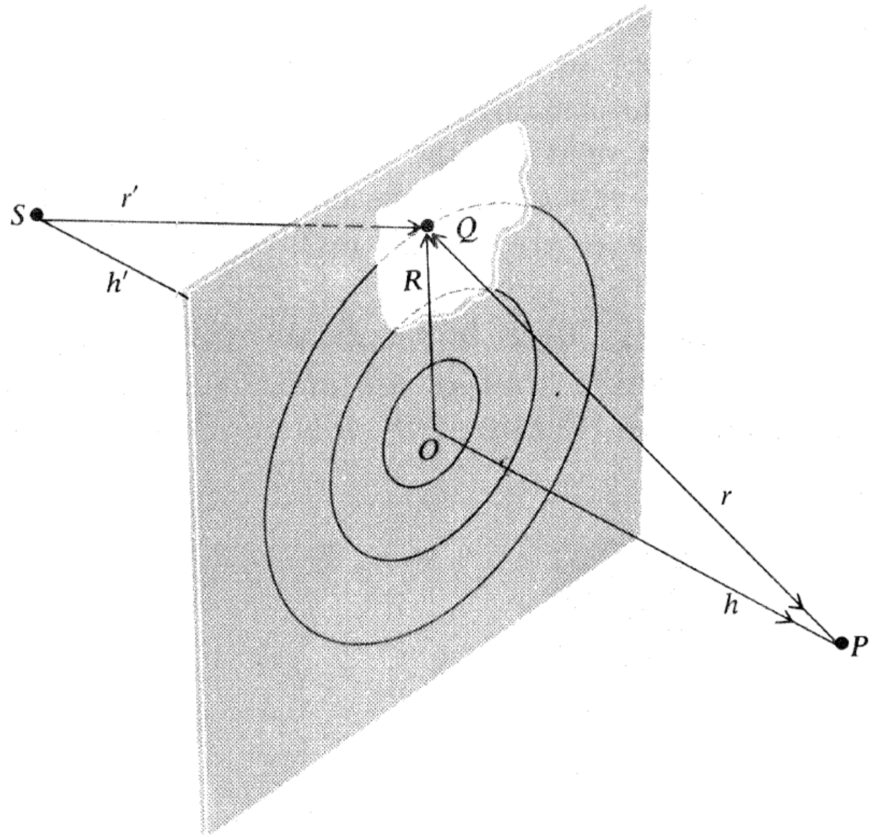
Fresnel Diffraction Patterns

According to the criteria discussed, diffraction is of the Fresnel type when either the light source or the observing screen, or both, are so close to the diffracting aperture that the curvature of the wave front becomes significant. Since one is no longer dealing with plane waves, Fresnel diffraction is mathematically more difficult to treat than Fraunhofer diffraction but is actually simpler to observe experimentally because all that is needed is a source of light, an observing screen, and the diffracting aperture. The previously mentioned fringe effects seen around shadows are examples of Fresnel diffraction.

Fresnel Zones

Consider a plane aperture illuminated by a point source S (Figure) such that a straight line connecting S to the receiving point P is perpendicular to the plane of the aperture.

Fresnel Diffraction Patterns



Let O be the point of intersection of the line SP with the aperture plane, and call R the distance from O to any point Q in the aperture. Then the distance $PQS = r + r'$ can be expressed in terms of R as follows:

Fresnel Diffraction Patterns

$$r + r' = (h^2 + R^2)^{1/2} + (h'^2 + R^2)^{1/2}$$

$$= h + h' + \frac{1}{2} R^2 \left(\frac{1}{h} + \frac{1}{h'}\right) + \cdots$$

where h and h' are the distances OP and OS, respectively. Now suppose that the aperture is divided up into regions bounded by concentric circles, $R = constant$, defined such that $r + r'$ differs by $\frac{1}{2}$ wavelength from one boundary to the next. These regions are called *Fresnel zones*. From Equation the successive radii are $R_1 = \sqrt{\lambda L}$, $R_2 = \sqrt{2\lambda L}$, . . . $R_n = \sqrt{n\lambda L}$, where λ is the wavelength, and

$$L = \left(\frac{1}{h} + \frac{1}{h'}\right)^{-1}$$

If R_n and R_{n+1} are the inner and outer radii of the $n+1$st zone, then the area is $\pi R_{n+1}^2 - \pi R_n{}^2 = \pi R_1{}^2$. This is independent of n. The areas of the complete zones are therefore all equal.

Fresnel Diffraction Patterns

Typically, the radii of the low-order Fresnel zones are very small. For example, if $h = h' = 50$ cm and $\lambda = 600$ nm, then we find $R_1 = (\lambda L)^{1/2} = 0.4$ mm, approximately. Also, since R_n is proportional to $n^{1/2}$, we see that the radius of the hundredth zone is only about 4 mm.

Zone Plate

If an aperture is constructed so as to obstruct alternate Fresnel zones, say the even-numbered ones, then the remaining terms in the summation are all of the same sign. Thus

$$|U_p| = |U_1| + |U_3| + |U_5| + \cdots$$

Such an aperture is called a *zone plate*. It acts very much like a lens, because $|U_p|$, and hence the irradiance at P, is now much larger than if there were no aperture. The equivalent focal length is L in Equation

Fresnel Diffraction Patterns

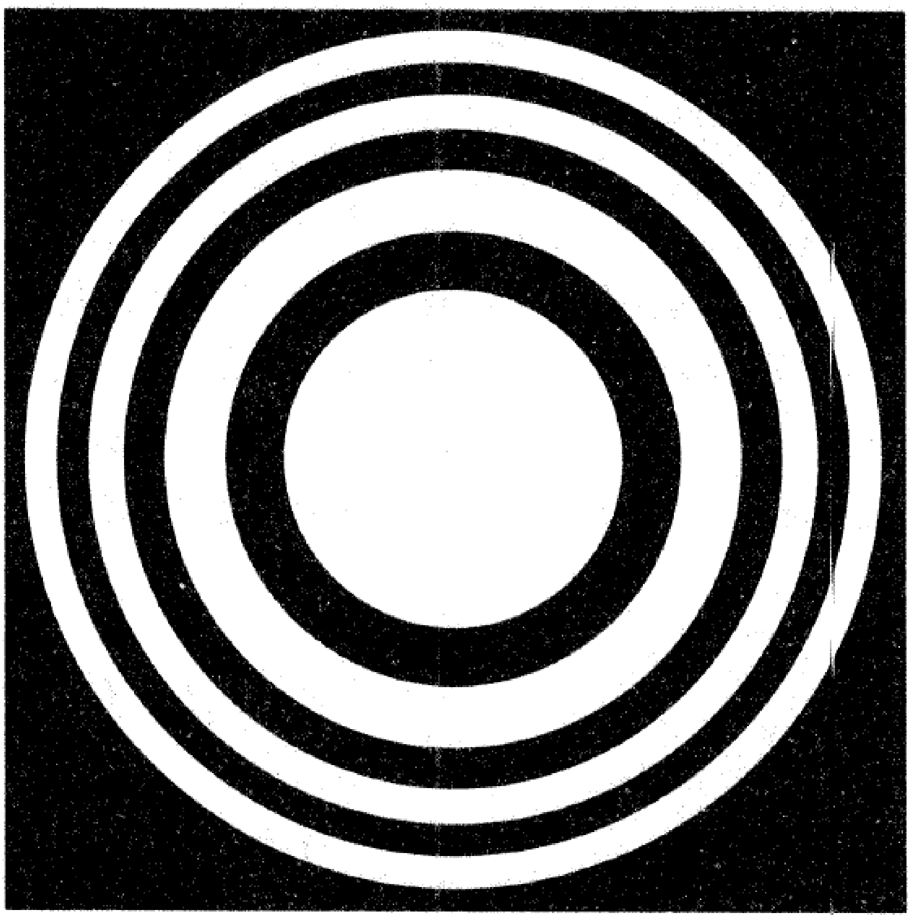
$$L = \left(\frac{1}{h} + \frac{1}{h'}\right)^{-1}$$

It is given by

$$L = \frac{R_1{}^2}{\lambda}$$

Zone plates can be made by photographing a drawing similar to that of Figure. The resulting photographic transparency can focus light and form images of distant objects. It is a very chromatic lens, however, since the focal length is inversely proportional to the wavelength.

Fresnel Diffraction Patterns



Fresnel Diffraction Patterns

Example 7:

Apply Equation
$$U_p = -\frac{ik}{4\pi}\iint \frac{U_{\mathscr{A}}\, e^{i(kr-\omega t)}}{r}\,[\cos(\mathbf{n},\mathbf{r}) + 1]\, d\mathscr{A}$$

directly to show that the value of U_p, contributed by the first Fresnel zone alone, is twice the value with no aperture at all.

Solution: