PHYS 701

Ch. 3

Coherence and Interference

Chapter 3

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Coherence and Interference

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The Prin. of Linear Superposition

The theory of optical interference is based essentially on the principle of linear superposition of electromagnetic fields. According to this principle, the electric field $\mathbf{E}$ produced at a point in empty space jointly by several different sources is equal to the vector sum

$$\mathbf{E} = \mathbf{E}_{(1)} + \mathbf{E}_{(2)} + \mathbf{E}_{(3)} + \cdots$$

where $\mathbf{E}_{(1)}, \mathbf{E}_{(2)}, \mathbf{E}_{(3)}, \ldots$ are the fields produced at the point in question separately by the different sources. The same is true for magnetic fields. The principle is a consequence of the fact that Maxwell's equation for the vacuum are linear differential equations.

Let us consider two plane harmonic linearly polarized waves of the same frequency ω. The electric fields are then

$$\mathbf{E}_{(1)} = \mathbf{E}_1 \exp i(\mathbf{k}_1 \cdot \mathbf{r} - \omega t + \phi_1)$$

$$\mathbf{E}_{(2)} = \mathbf{E}_2 \exp i(\mathbf{k}_2 \cdot \mathbf{r} - \omega t + \phi_2)$$

The Prin. of Linear Superposition

Here the quantities ϕ_1 and ϕ_2 have been introduced to allow for any phase difference between the sources of the two waves. If the phase difference $\phi_1 - \phi_2$ is constant, the two sources are said to be *mutually coherent*. The resulting waves are also mutually coherent in this case.

We know from that the irradiance at a point is proportional to the square of the amplitude of the light field at the point in question. Thus the superposition of our two monochromatic plane waves, aside from a constant proportionality factor, results in an irradiance function

$$
\begin{aligned}
I = |\mathbf{E}|^2 = \mathbf{E} \cdot \mathbf{E}^* &= (\mathbf{E}_{(1)} + \mathbf{E}_{(2)}) \cdot (\mathbf{E}^*_{(1)} + \mathbf{E}^*_{(2)}) \\
&= |\mathbf{E}_1|^2 + |\mathbf{E}_2|^2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2 \cos\theta \\
&= I_1 + I_2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2 \cos\theta
\end{aligned}
$$

where

$$\theta = \mathbf{k}_1 \cdot \mathbf{r} - \mathbf{k}_2 \cdot \mathbf{r} + \phi_1 - \phi_2$$

The Prin. of Linear Superposition

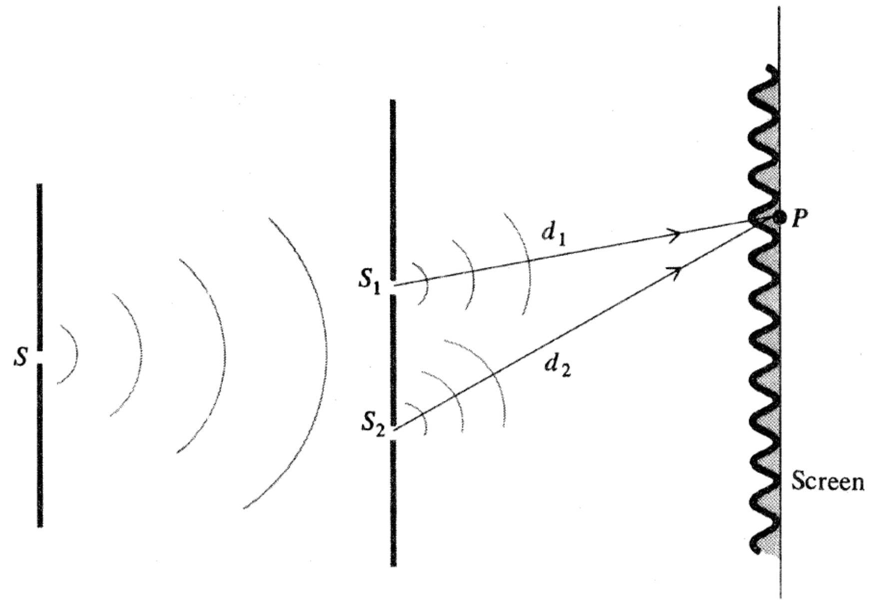
The term $2\mathbf{E}_1 \cdot \mathbf{E}_2 \cos\theta$ is called the *interference term*. This term indicates I can be greater than or less than the sum $I_1 + I_2$, depending on the value of θ. Since θ depends on $\mathbf{r}$, periodic spatial variations in the intensity occur. These variations are the familiar interference fringes that are seen when two mutually coherent beams of light are combined.

If the sources of the two waves are mutually incoherent, then the quantity $\phi_1 - \phi_2$ varies with time in a random fashion. The result is that the mean value of $\cos\theta$ is zero, and there is no interference. This is the reason interference fringes are not observed with two separate (ordinary) light sources.

In the event that the two waves are polarized, then the interference term also depends on the polarization. In particular, if the polarizations are mutually orthogonal, then $\mathbf{E}_1 \cdot \mathbf{E}_2 = 0$.

Young's Experiment

The classic experiment that demonstrates interference of light was first performed by Thomas Young in 1802. In the original experiment sunlight was used as the source, but any bright source such as a tungsten filament lamp or an arc would be satisfactory. Light is passed through a pinhole S so as to illuminate an aperture consisting of two pinholes or narrow slits S_1 and S_2 as shown in Figure.



Young's Experiment

If a white screen is placed in the region beyond the slits, a pattern of bright and dark interference bands can be seen. The key to the experiment is the use of a single pinhole S to illuminate the aperture. This provides the necessary mutual coherence between the light coming from the two slits S_1 and S_2.

The usual elementary analysis of the Young experiment involves finding the difference in phase between the two waves arriving at a given point P over the distances d_1 and d_2 as shown. Assuming that we have spherical waves whose phase factors are of the form $e^{i(kr-\omega t)}$ the phase difference at P will be $k(d_2 - d_1)$.

Bright fringes occur when this difference is 0, $\pm 2\pi, \pm 4\pi, \ldots \pm 2n\pi$, where n is a integer. These values give maxima for the interference term in Equation. Since $k = 2\pi/\lambda$, we see that the equation

$$k(d_2 - d_1) = \pm 2n\pi$$

Young's Experiment

is equivalent to

$$|d_2 - d_1| = n\lambda$$

that is, the path difference is equal to an integral number of wavelengths.

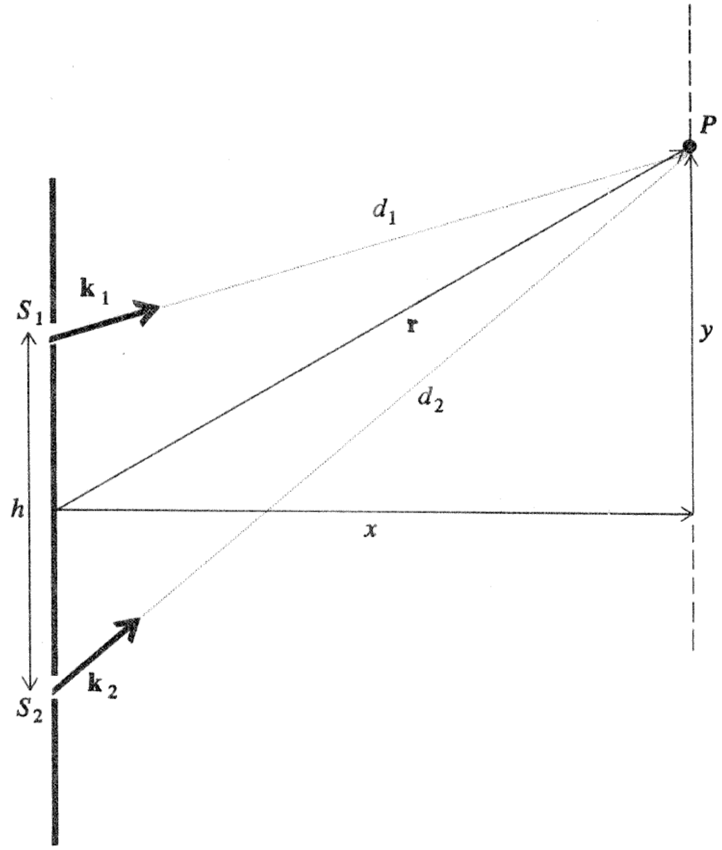
The result above can be related to the physical parameters of the experimental arrangement by labeling the geometry as shown in Figure. Here h is the slit separation and x is the distance from the slit aperture to the screen. The distance y is measured on the screen from the central axis as shown. Equation is then equivalent to

$$\left[x^2 + \left(y + \frac{h}{2}\right)^2\right]^{1/2} - \left[x^2 + \left(y - \frac{h}{2}\right)^2\right]^{1/2} = n\lambda$$

By use of the binomial expansion we find that an approximate equivalent expression is given by

Young's Experiment

$$\frac{yh}{x} = n\lambda$$



Young's Experiment

The approximation is valid when y and h are both small compared to x. Bright fringes occur at the points

$$y = 0, \pm \frac{\lambda x}{h}, \pm \frac{2\lambda x}{h}, \quad \cdot \ \cdot \ \cdot$$

If the slits are covered by optical devices such as phase retarders, polarizers, and so forth, the fringe pattern will change. For example, if a relative phase difference of π is introduced by placing a thin plate of glass over one slit, then the whole interference pattern would shift by one half the fringe separation, so that a bright fringe would occur where a dark fringe was located before.

Young's Experiment

Example 1:

In a two-slit interference experiment of the Young type, the aperture-to-screen distance is 2 m and the wavelength is 600 nm. If it is desired to have a fringe spacing of 1 mm, what is the required slit separation?

Solution:

Young's Experiment

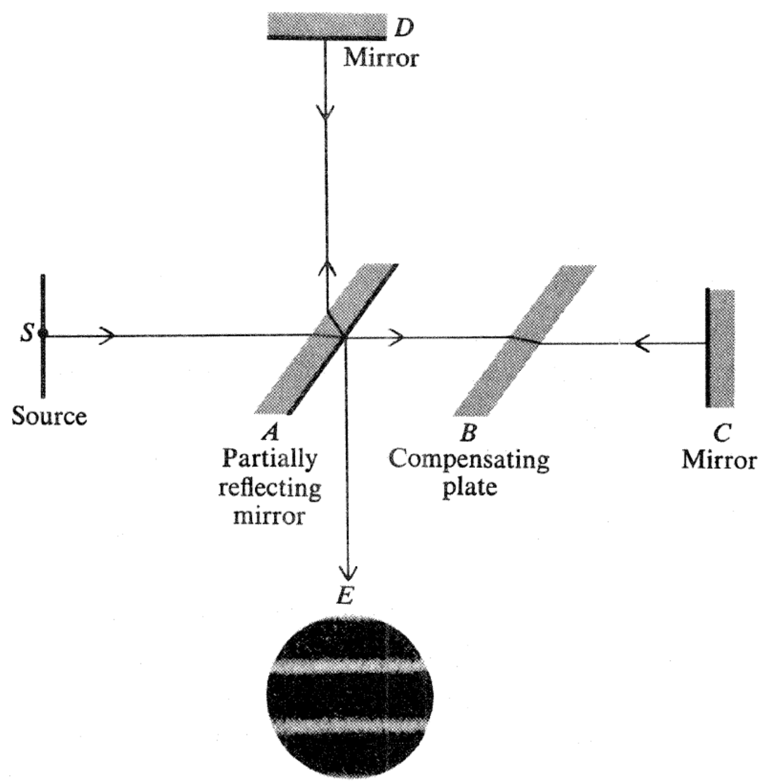
Example 2:

In the experiment of Example 1 a thin plate of glass ($n = 1.5$) of thickness 0.05 mm is placed over one of the slits. What is the resulting lateral fringe displacement at the screen?

Solution:

The Michelson Interferometer

Perhaps the best known and most versatile interferometric device is the interferometer developed by Michelson in 1880. The basic design is shown in Figure. Light from the source S falls on a lightly silvered



The Michelson Interferometer

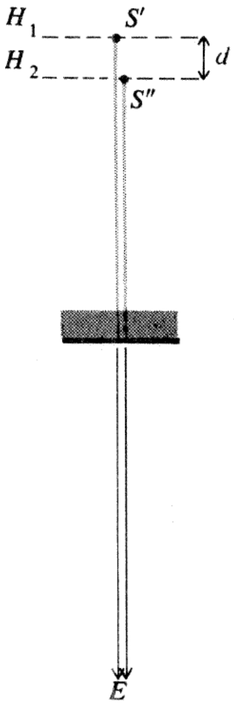
glass plate A, which divides the beam into two parts. These separated beams are reflected back to A by mirrors C and D as shown. Usually a compensating plate B is inserted in one beam in order that the two optical paths include the same thickness of glass. The compensating plate is necessary when observing fringes with white light.

The interference pattern is observed at E. Here the light appears to come from two virtual source planes H_1 and H_2 as indicated in Figure. The corresponding virtual point sources S' and S'' in these planes are mutually coherent. If d is the path difference between the two rays reaching E, that is, the distance between S_1' and S_2'', then from Equations

$$I = |\mathbf{E}|^2 = I_1 + I_2 + 2\mathbf{E}_1 \cdot \mathbf{E}_2 \cos\theta$$

$$\theta = \mathbf{k}_1 \cdot \mathbf{r} - \mathbf{k}_2 \cdot \mathbf{r} + \phi_1 - \phi_2$$

The Michelson Interferometer



the irradiance is proportional to

$$1 + \cos\theta = 1 + \cos kd = 1 + \cos\frac{2\pi d}{\lambda}$$

The Michelson Interferometer

Now if the mirrors are slightly tilted so that the virtual source planes H_1 and H_2 are not quite parallel, then alternate bright and dark fringes appear across the field of view when the eye is placed at E. These fringes, called *localized fringes*, appear to come from the region of H_1 and H_2. On the other hand if H_1 and H_2 are parallel, then the fringes are seen as circular and appear to come from infinity.

Several localized colored fringes can be observed with white light if H_1 and H_2 intersect at some point in the field of view. In this case the central fringe is dark owing to the fact that one ray is internally reflected in plate A, whereas the other ray is externally reflected in A and, accordingly, the two rays reaching E are 180 degrees out of phase for $d = 0$.

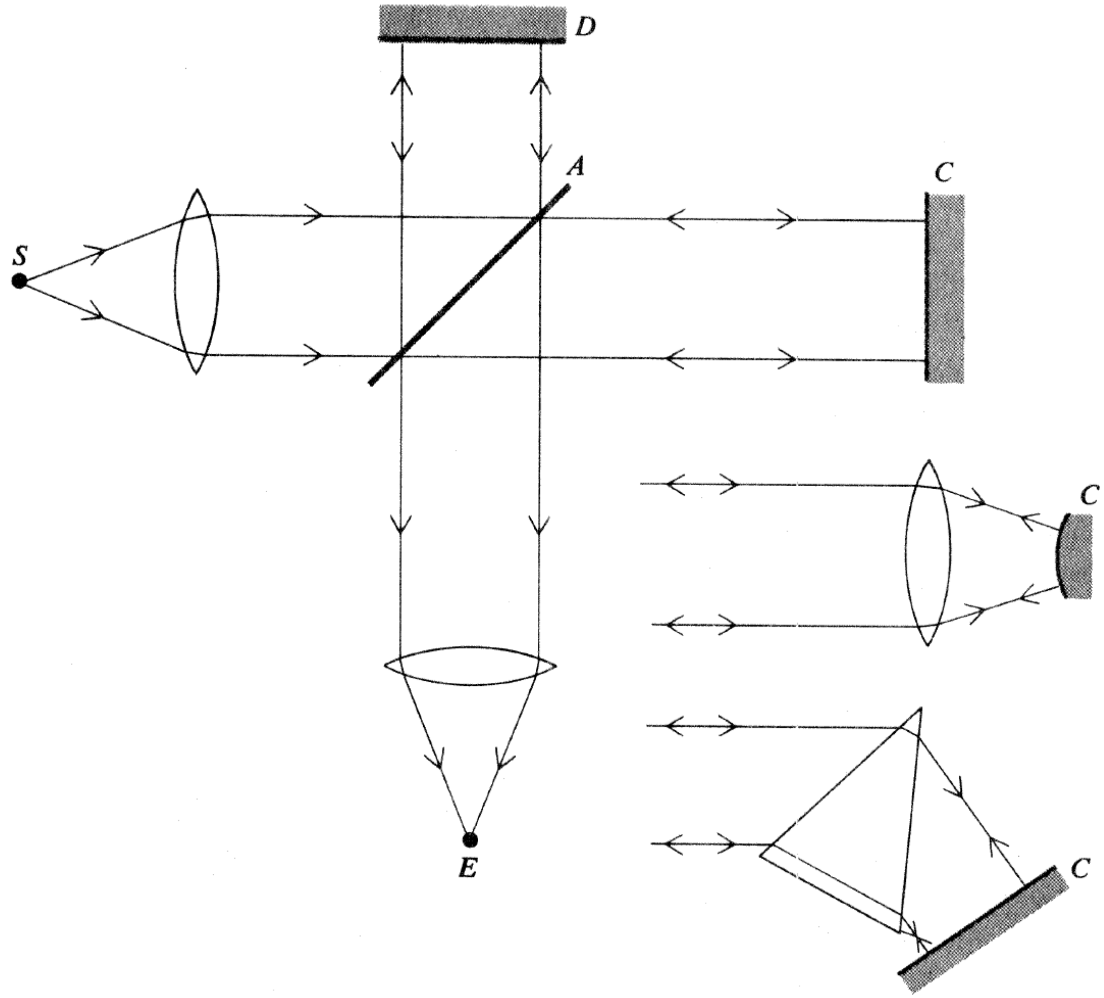
One of the many uses of the Michelson interferometer is the determination of the index of refraction of gases. An evacuated optical cell is placed in one of the optical paths of the interferometer. The gas whose index is to be measured is then allowed to flow into the cell.

The Michelson Interferometer

This is equivalent to changing the lengths of the optical path and causes the interference fringes to move across the field of view. The number of such fringes gives the effective change in optical path from which the index of refraction of the gas can be calculated.

A modification of the Michelson interferometer known as the Twyman-Green interferometer is shown in Figure. This inteferometer is used for testing optical elements such as lenses, mirrors, and prisms. Collimated light is used in this case. The optical element to be tested is placed in one of the paths as shown. Imperfections are rendered visible by distortions in the fringe pattern.

The Michelson Interferometer



The Michelson Interferometer

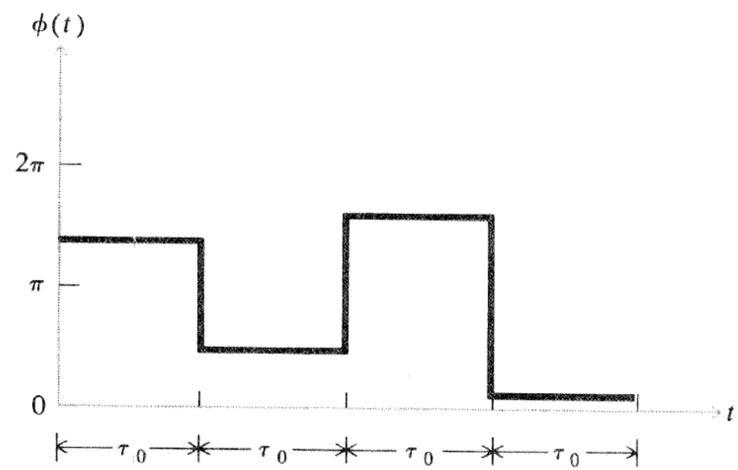
Example 3:

A Michelson interferometer can be used to determine the index of refraction of a gas. The gas is made to flow into an evacuated glass cell of length l placed in one arm of the interferometer. The interference fringes are counted as they move across the view aperture when the gas flows into the cell. Show that the effective optical path difference of the light beam for the full cell versus the evacuated cell is $2l(n-1)$, where n is the index of refraction of the gas, and hence that a number $N = 4l(n-1)/\lambda$ fringes move across the field of view as the cell is filled. How many fringes would be counted if the gas were air ($n = 1.0003$) for a 10-cm cell using yellow sodium light $\lambda = 590$ nm?

Solution:

Coher. Time and Coher. Length

In order to see how the degree of partial coherence is related to the characteristics of the source, let us consider the case of a hypothetical "quasimonochromatic" source having the following property: The oscillation and the subsequent field vary sinusoidally for a certain time τ_0 and then change phase abruptly. This sequence keeps repeating indefinitely. A graph is shown in Figure. We shall call τ_0 the *coherence time*. The phase change that occurs after each coherence time is considered to be randomly distributed between 0 and 2π.



Coher. Time and Coher. Length

The time dependence of this quasimonochromatic field can be expressed as

$$E(t) = E_0 e^{-i\omega t}\ e^{i\phi(t)}$$

where the phase angle $\phi(t)$ is a random step function, illustrated in Figure. One can regard the above kind of a field as an approximation to that of a radiating atom, the abrupt changes of phase being the result of collisions.

Suppose a beam of light, whose field is represented by Equation is divided into two beams that are subsequently brought together to produce interference. The degree of partial coherence can be evaluated as follows: It is assumed that

$$|E_1| = |E_2| = |E|$$

Coher. Time and Coher. Length

Then, since we are concerned here with *self-coherence*, we delete the subscripts and write

$$\gamma(\tau) = \frac{\langle E(t)\, E^*(t+\tau)\rangle}{\langle |E|^2\rangle}$$

From Equation

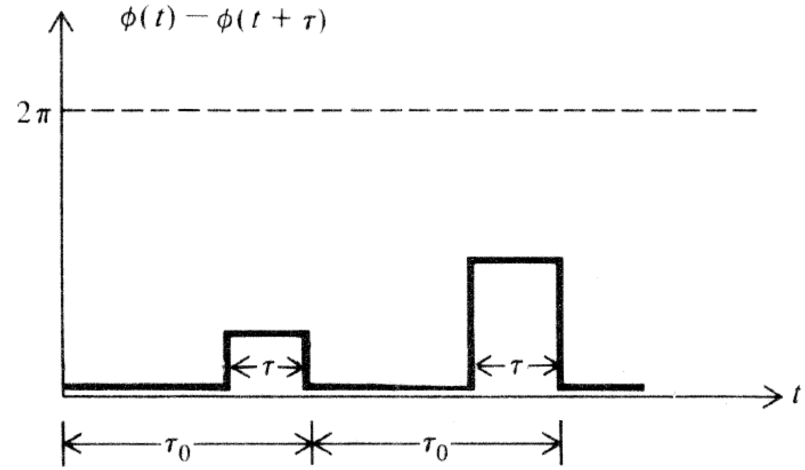
$$E(t) = E_0 e^{-i\omega t}\, e^{i\phi(t)}$$

we have

$$\gamma(\tau) = \langle e^{i\omega\tau}\, e^{i[\phi(t)-\phi(t+\tau)]}\rangle$$
$$= e^{i\omega\tau} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{i[\phi(t)-\phi(t+\tau)]}\, dt$$

Coher. Time and Coher. Length

Consider the quantity $\phi(t) - \phi(t + \tau)$. This is plotted in Figure



Now for the first coherence time interval, $0 < t < \tau_0$, we observe that $\phi(t) - \phi(t + \tau) = 0$ for $0 < t < \tau_0 - \tau$. On the other hand for $\tau_0 - \tau < t < \tau_0$ it assumes some random value between 0 and 2π. The same is true for each succeeding coherence time interval.

Coher. Time and Coher. Length

The integral in Equation is easily evaluated as follows: For the first interval we have

$$\frac{1}{\tau_0}\int_0^{\tau_0} e^{i[\phi(t)-\phi(t+\tau)]}\,dt = \frac{1}{\tau_0}\int_0^{\tau_0-\tau} dt + \frac{1}{\tau_0}\int_{\tau_0-\tau}^{\tau_0} e^{i\Delta}\,dt$$

$$= \frac{\tau_0-\tau}{\tau_0} + \frac{\tau}{\tau_0}\,e^{i\Delta}$$

where Δ is the random phase difference.

The same result is obtained for all subsequent intervals, except that Δ is different for each interval. Since Δ is random, the terms involving $e^{i\Delta}$ will average to zero. The other term, $(\tau_0-\tau)/\tau_0$, is the same for all intervals, hence it is equal to average value of the integral in question. Of course if $\tau > \tau_0$, then the phase difference $\phi(t)-\phi(t+\tau)$ is always random and, consequently, the whole integral averages to zero.

Coher. Time and Coher. Length

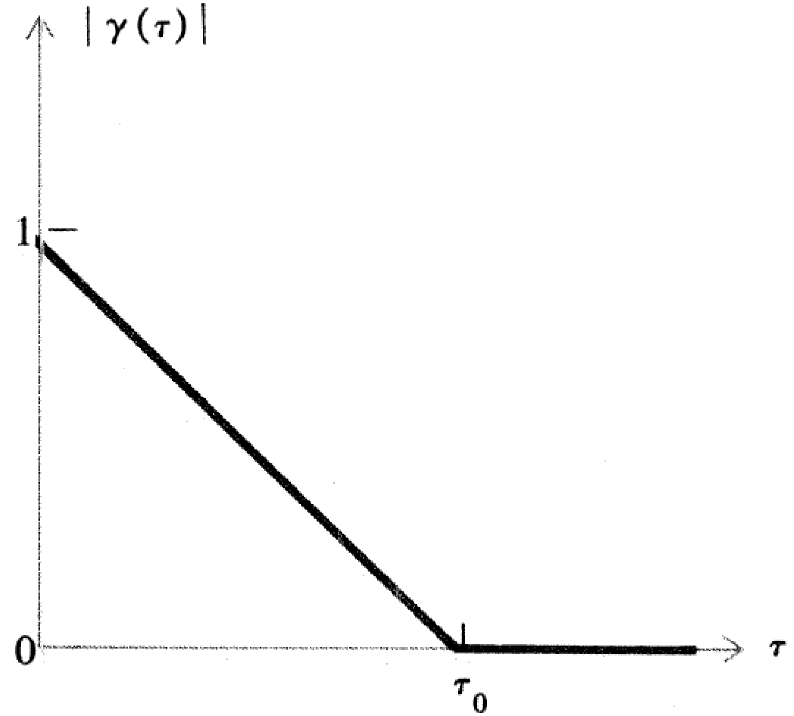
From the above result we find that the normalized autocorrelation function for a quasimonochromatic source is given by

$$\gamma(\tau) = \left(1 - \frac{\tau}{\tau_0}\right) e^{i\omega\tau} \qquad \tau < \tau_0$$
$$= 0 \qquad \tau \geqslant \tau_0$$

$$|\gamma(\tau)| = 1 - \frac{\tau}{\tau_0} \qquad \tau < \tau_0$$
$$= 0 \qquad \tau \geqslant \tau_0$$

A graph of $|\gamma|$ is shown in Figure. We found in the previous section that this quantity is

Coher. Time and Coher. Length



equal to the fringe visibility $\mathscr{V}$ for the case of equal amplitudes in a two-beam interference arrangement. Evidently the fringe visibility drops to zero if τ exceeds the coherence time τ_0. This means that the path difference between the two beams must not exceed the value

Coher. Time and Coher. Length

$$c\tau_0 = l_c$$

in order to obtain interference fringes. The quantity l_c is called the *coherence length*. It is essentially the length of an uninterrupted wave train.

In the actual case of radiating atoms, the time between collisions is not constant but varies randomly from one collision to the next. Consequently, the wave trains vary in length in a similar random fashion. In this more realistic situation we can define the coherence time as the *average* value of the individual coherence times and similarly for the coherence length. The actual mathematical form of the degree of coherence and of the fringe visibility will then depend on the precise statistical distribution of the lengths of the wave trains.

Coher. Time and Coher. Length

In any case the fringe visibility will be large (of the order of unity) for path differences that are small compared to the average coherence length. Conversely, the fringe visibility will become small and approach zero as the path difference becomes larger than the average coherence length.

Coherence and Line Width

According to the theorem, stated here without proof, a function $f(t)$ can be expressed as an integral over the variable ω in the following way:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) e^{-i\omega t} \, d\omega$$

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{i\omega t} \, dt$$

The functions $f(t)$ and $g(\omega)$ are called *Fourier transforms* of each other and are said to constitute a Fourier transform pair. In our present application the variables t and ω are time and frequency, respectively. The function $g(\omega)$ then constitutes a frequency resolution of the time dependent function $f(t)$ or stated in another way, $g(\omega)$ represents the function in the frequency domain.

Coherence and Line Width

Let us consider now the particular case in which the function $f(t)$ represents a single wave train of finite duration τ_0. The time variation of this wave train is given by the function

$$f(t) = e^{-i\omega_0 t} \quad \text{for } -\frac{\tau_0}{2} < t < \frac{\tau_0}{2}$$

$$f(t) = 0 \quad \text{otherwise}$$

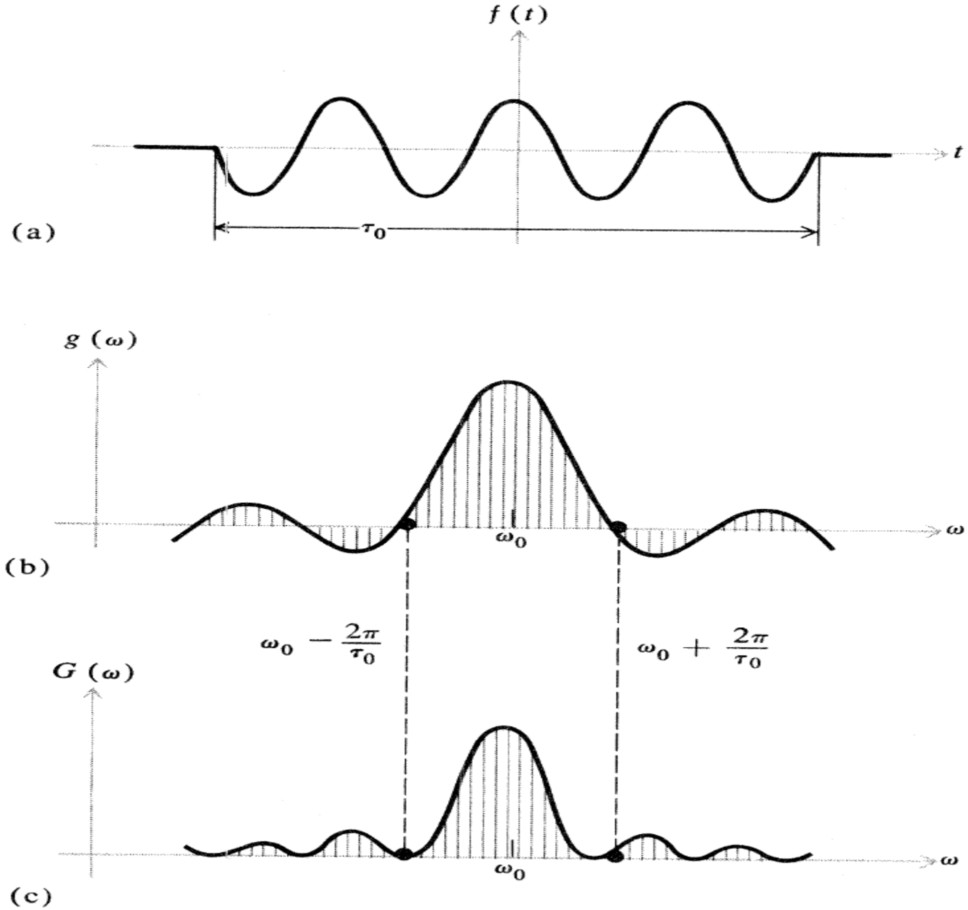
Taking the Fourier transform, we have

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\tau_0/2}^{+\tau_0/2} e^{i(\omega-\omega_0)t}\, dt$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin\,[(\omega-\omega_0)\tau_0/2]}{\omega-\omega_0}$$

Coherence and Line Width

A plot of the real part of the function $f(t)$ is shown in Figure



Coherence and Line Width

Also plotted is a graph of the *power spectrum:*

$$G(\omega) = |g(\omega)|^2$$

This function, in the case of a finite wave train, is given by

$$G(\omega) = |g(\omega)|^2 = \frac{2 \sin^2 [(\omega - \omega_0)\tau_0/2]}{\pi(\omega - \omega_0)^2}$$

We see that the spectral distribution is maximum for $\omega = \omega_0$ and drops to zero for $\omega = \omega_0 \pm 2\pi/\tau_0$. Secondary maxima and minima also occur as shown in the diagram. Most of the energy is contained in the region between the first two minima on either side of the central maximum at ω_0. The "width" $\Delta\omega$ of the frequency distribution is therefore given by

Coherence and Line Width

$$\Delta\omega = \frac{2\pi}{\tau_0}$$

or

$$\Delta\nu = \frac{1}{\tau_0}$$

and the coherence length l_c is

$$l_c = \frac{c}{\Delta\nu}$$

We can also express the coherence length in terms of wavelength. Using the fact that $\Delta\nu/\nu = |\Delta\lambda|/\lambda$, we obtain

$$l_c = \frac{\lambda^2}{\Delta\lambda}$$

where $\Delta\lambda$ is the width of the spectrum line on the wavelength scale.

Coherence and Line Width

Example 4:

A filter is used to obtain approximately monochromatic light from a white source. If the pass band of the filter is 10 nm, what is the coherence length and coherence time of the filtered light? The mean wavelength is 600 nm.

Solution:

Coherence and Line Width

Example 5:

What is the line width in hertz and in nanometers of the light from a helium-neon laser whose coherence length is 5 km? The wavelength is 633 nm.

Solution: