# CALCULUS 110

## (2.5) Continuity

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# 2.5 Continuity

A continuous process is one that takes place gradually, without interruption or abrupt change.

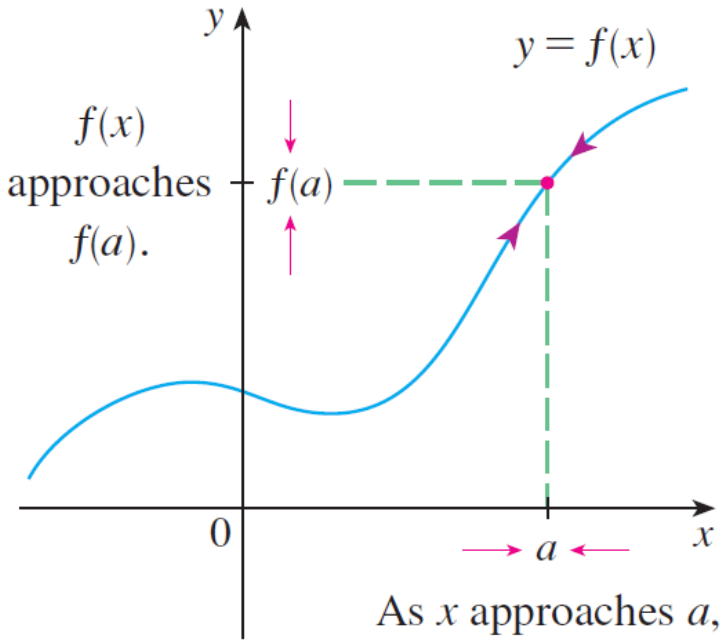
## Continuous at a number $a$

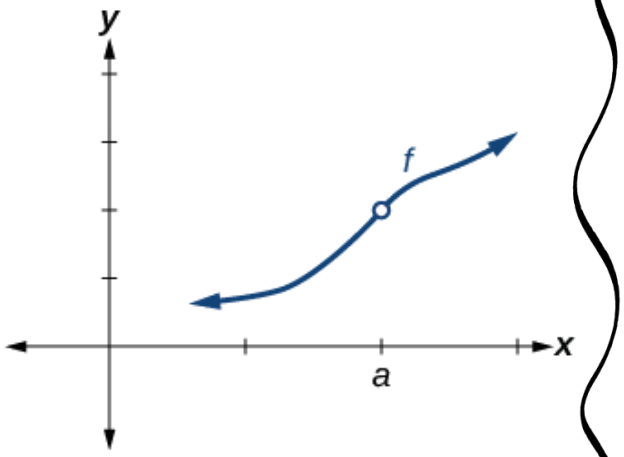
A function $f$ is continuous at a number $a$ if:

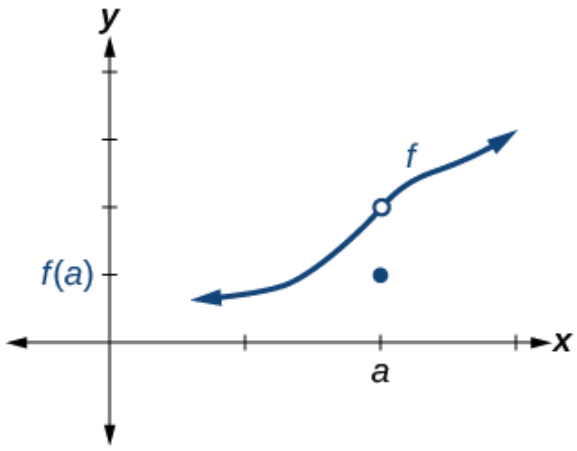
$$\lim_{x \to a} f(x) = f(a)$$

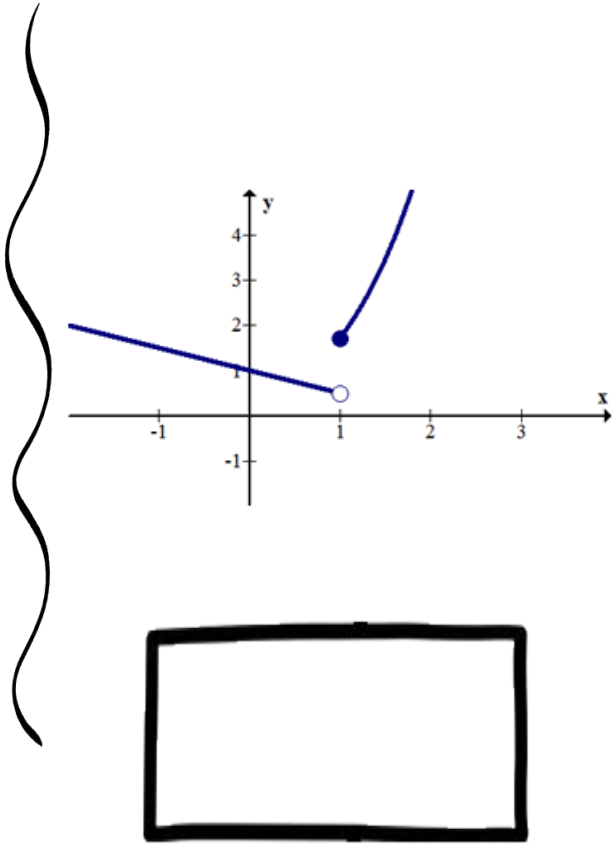
So $f(x)$ is continuous at $a$ if:

1. $f(a)$ is defined $(a \in D_f)$
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$




y
f
x
a

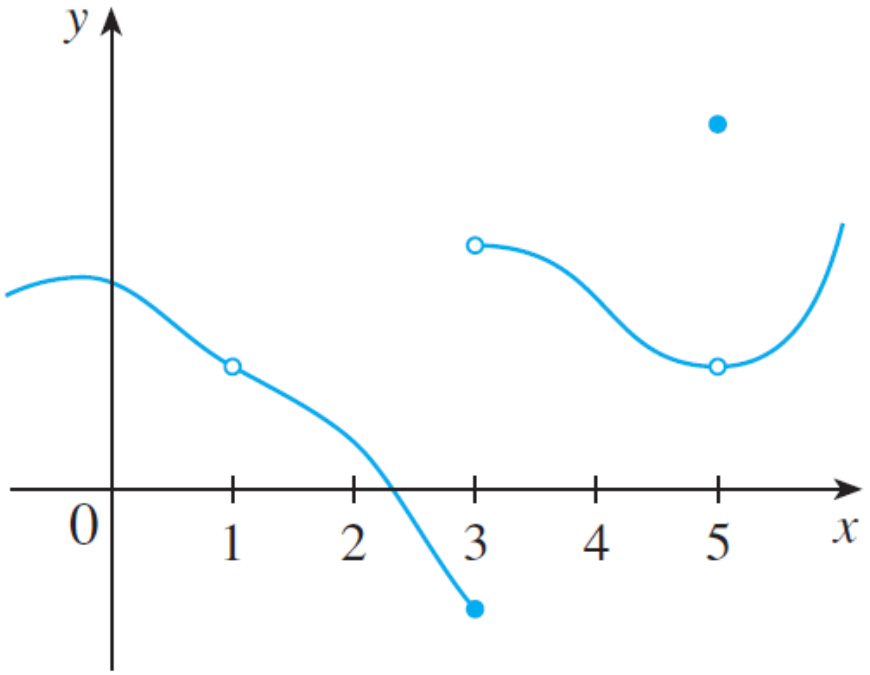

y
f
f(a)
x
a


y
4
3
2
1
-1
1
2
3
-1
x

## Example 1

The following figure shows the graph of a function $f$. At which numbers is $f$ discontinuous? Why?

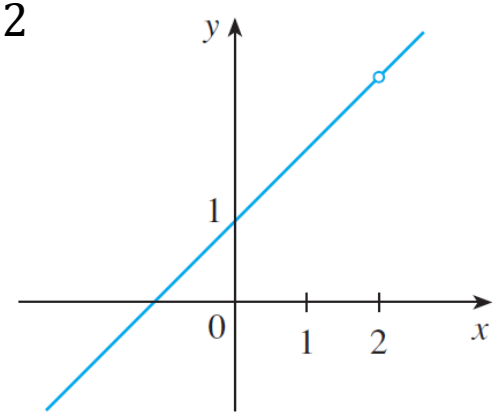
## solution



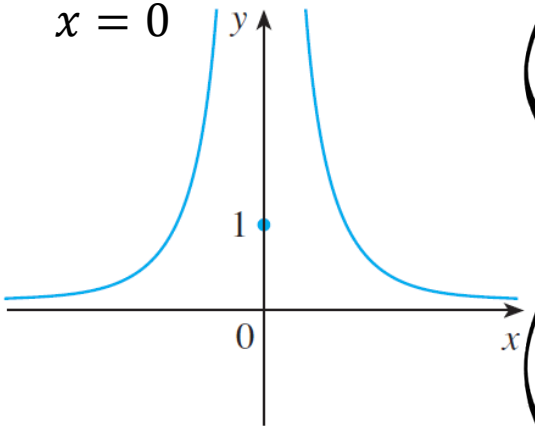
## Example 2

Where are each of the following functions discontinuous?

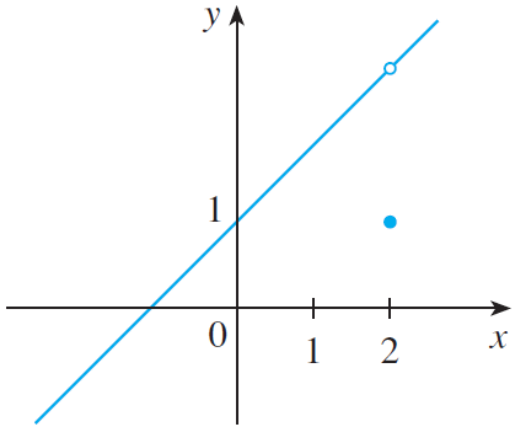
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$




(b) $f(x) = \begin{cases} \dfrac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

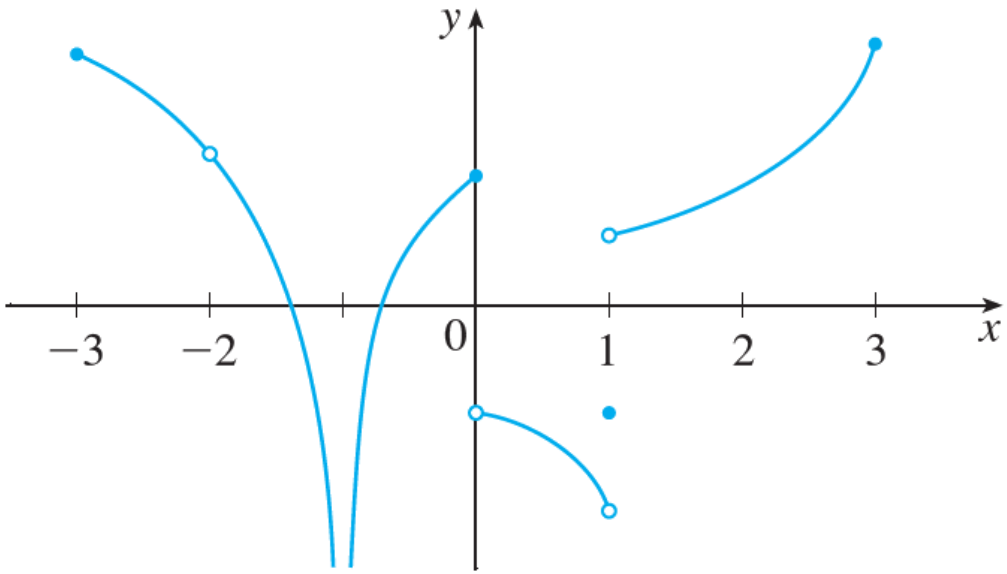



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$



# Exercise 2

From the graph of t, state the intervals on which t is continuous.



# Continuous from the right (left)

## continuous from the right

A function $f$ is continuous from the right at a number $a$ if:

$$\lim_{x \to a^+} f(x) = f(a)$$

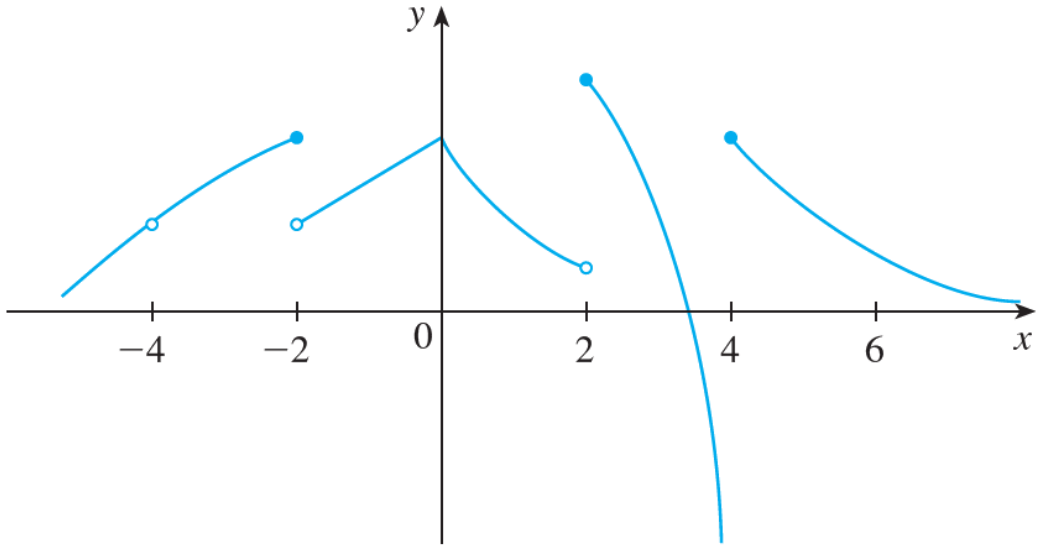
## continuous from the left

And $f$ is continuous from the left at $a$ if:

$$\lim_{x \to a^-} f(x) = f(a)$$

# Exercise 3

(a) From the graph of $f$, state the numbers at which $f$ is discontinuous and explain why.




(b) For each of the numbers stated in part (a), determine whether $f$ is continuous from the right, or from the left, or neither.

## Exercise 45

For what value of the constant $c$ is the function $f$ continuous everywhere.

$$f(x)=\begin{cases} cx^2+2x, & x<2 \\ x^3-cx, & x\geq 2 \end{cases}$$

## solution

## Examples

1 The function $f(x)=\begin{cases} x^2 \ \ if \ x\geq 0 \\ 0 \ \ if \ \ x<0 \end{cases}$ is

(a) continuous at $x=0$
(b) continuous on $\mathbb{R}$.
(c) continuous only on $(-\infty,0)\cup(0,\infty)$
(d) continuous only on $(0,\infty)$.

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2 The function $f(x)=\begin{cases} x-1 & if & x<-1 \\ 2 & if & -1\leq x\leq 1 \\ x+1 & if & x>1 \end{cases}$

is

(a) $\{-1,1\}$ (b) $\{1\}$

(c) $\{-1\}$ (d) $[-1,1]$

# Examples on continuity from one side:

**3** The function $\sqrt{3-x}$ is continuous from the right at $x = 3$.

(a) True (b) False

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**4** The function $\sqrt{x-3}$ is

(a) continuous at $x = 3$.
(b) continuous from the right at $x = 3$.
(c) continuous from the left at $x = 3$.
(d) continuous at $x = 0$.

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**5** The function $f(x) = \begin{cases} x+1 & if\ x \geq 2 \\ x^2 & if\ x < 2 \end{cases}$ is

(a) continuous from the right at $x = 2$.
(b) continuous from the left at $x = 2$.
(c) continuous on $\mathbb{R}$.
(d) continuous at $x = 2$.

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## Theorem 4

If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

1. $f + g$
2. $f \cdot g$
3. $cf$
4. $f - g$
5. $\frac{f}{g}$ (if $g(a) \neq 0$)

## Theorem 6

- $\lim_{\theta \to 0} \cos\theta = 1$
- $\lim_{\theta \to 0} \sin\theta = 0$

## Theorem 5

- Any polynomial is continuous everywhere, that is , it is continuous on $\mathbb{R} = (-\infty, \infty)$.
- Any rational function is continuous wherever it is defined, that is, it is continuous on its domain.

## Theorem 7

The following types of functions are continuous at every number in their domains:

- Polynomials
- Trigonometric functions
- Exponential functions
- Rational functions
- Root functions
- Inverse trigonometric functions
- Logarithmic functions

## Theorem 8

If $f$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$,

then $\lim_{x\to a} f\big(g(x)\big) = f(b)$

i.e. $\lim_{x\to a} f\big(g(x)\big) = f\left(\lim_{x\to a} g(x)\right)$

## Theorem 9

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)x = f\big(g(x)\big)$ is continuous at $a$.

## Example 7

Evaluate $\lim_{x\to\pi} \dfrac{\sin x}{2 + \cos x}$

## solution

## Example 5

Find $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

solution

## Example 6

Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

solution

## Example8

Evaluate $\lim_{x\to 1} arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$

## solution

## Example 9

Where are the following functions continuous?

(a) $h(x) = \sin x^2$

solution

(b) $F(x) = \ln(1 + \cos x)$

solution


17, 20, 21, 25, 38, 43