CALCULUS 110

(2.2) The Limit of a Function

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(2.2) The Limit of a Functions

Definition

Suppose $f(x)$ is defined when x is near the number a (this means that f is defined on some open interval that contains a, except possibly at a itself.) then we write:

$$\lim_{x\to a} f(x) = L$$

And say " the limit of $f(x)$ as x approaches a, equals L"

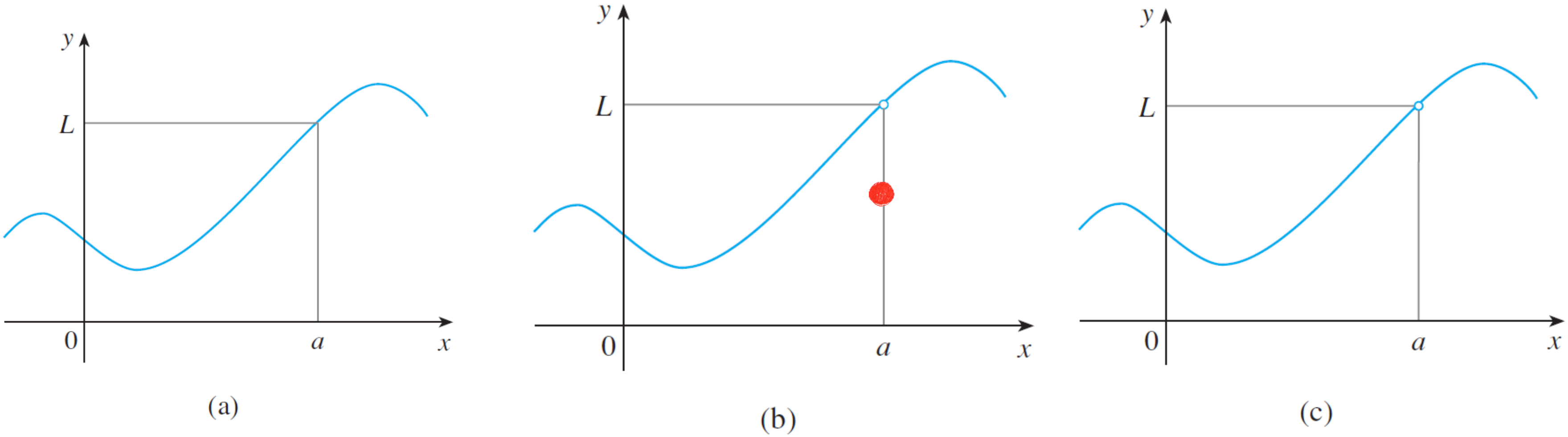
An alternative notation for the limits: $f(x) \to L$ as $x \to a$

remark

1. $x \to a$ doesn't mean that $x = a$.

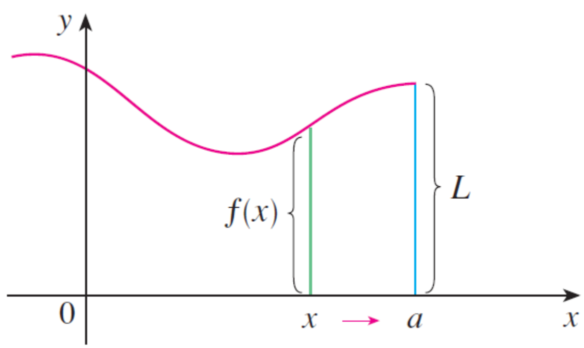
2. In fact a doesn't have to be in the domain.
i.e. $f(x)$ need not even be defined when $x = a$.

We have the following cases



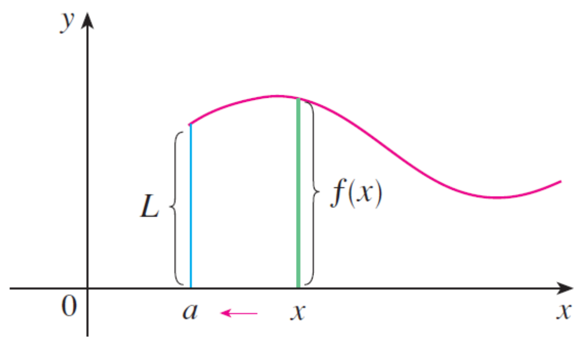
One - Sided Limits

$\lim_{x \to a^-} f(x) = L$ is called left hand limit or limit from the left. Here x is close to a and less than $(x < a)$.



$$\lim_{x \to a^-} f(x) = L$$

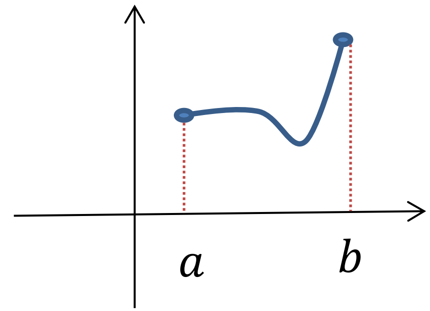
$\lim_{x \to a^+} f(x) = L$ is called right hand limit or limit from the right. Here x is close to a and greater than a. $(x > a)$



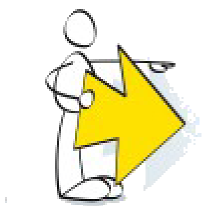
$$\lim_{x \to a^+} f(x) = L$$

$$\boxed{\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)}$$

When do we have to study the limit from the left and from the right?



1. If we are studying the limit at the endpoint
2. If the definition of the function change at this point. See example 7.



When do we say that limit does not exist?

1. If $\lim\limits_{x\to a^+} f(x)$ or $\lim\limits_{x\to a^-} f(x)$ does not exist. (D.N.E).
2. If both limits exist but $\lim\limits_{x\to a^+} f(x) \neq \lim\limits_{x\to a^-} f(x)$

Examples

1. $\lim_{x\to 5}\sqrt{x^2-25} = \begin{cases} \lim_{x\to 5^+}\sqrt{x^2-25} = \_\_\_\_\_ \\ \lim_{x\to 5^-}\sqrt{x^2-25} = \_\_\_\_\_ \end{cases}$

$\lim_{x\to 5}\sqrt{x^2-25}$ _____

2. $\lim_{x\to 2}\sqrt{x-2} = \begin{cases} \lim_{x\to 2^+}\sqrt{x-2} = \_\_\_\_\_ \\ \lim_{x\to 2^-}\sqrt{x-2} = \_\_\_\_\_ \end{cases}$

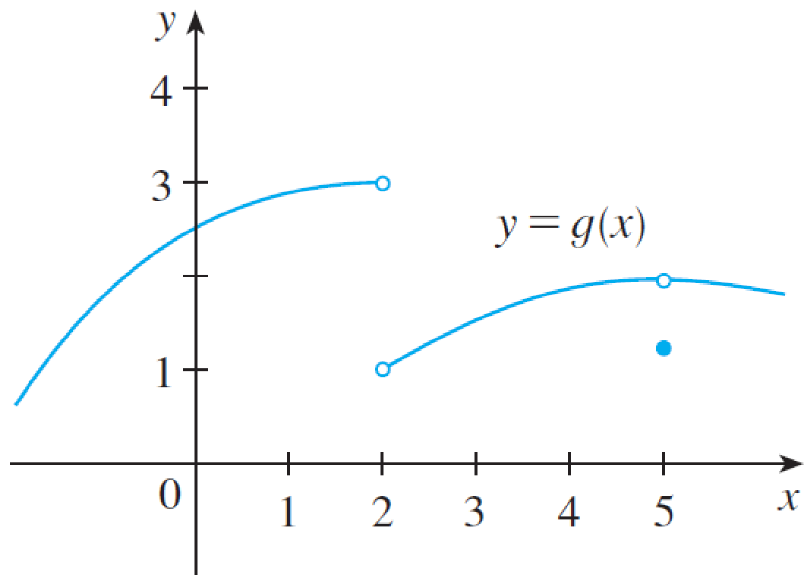
$\lim_{x\to 2}\sqrt{x-2} = \_\_\_\_\_$

3. $\lim_{x\to 2}\frac{-3x}{x-2} = \begin{cases} \lim_{x\to 2^+}\frac{-3x}{x-2} = \_\_\_\_\_ \\ \lim_{x\to 2^-}\frac{-3x}{x-2} = \_\_\_\_\_ \end{cases}$

$\lim_{x\to 2}\frac{-3x}{x-2} = \_\_\_\_\_$

Example 7

The graph of a function g is shown in the following figure. Use it to state the values (if they exist) of the following:



solution

a $\lim_{x \to 2^-} g(x) =$ _

b $\lim_{x \to 2^+} g(x) =$ _

c $\lim_{x \to 2} g(x)$ _

d $\lim_{x \to 5^-} g(x) =$ _

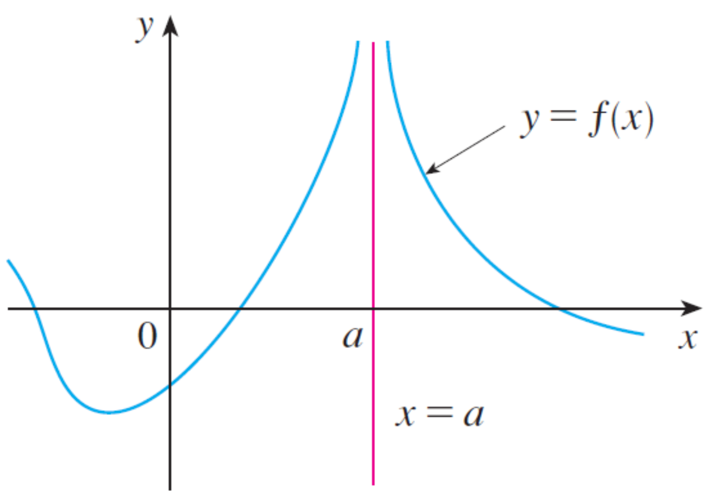
e $\lim_{x \to 5^+} g(x) =$ _

f $\lim_{x \to 5} g(x) =$

Infinite Limits

$\lim_{x \to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x close to a, but not equal to a.

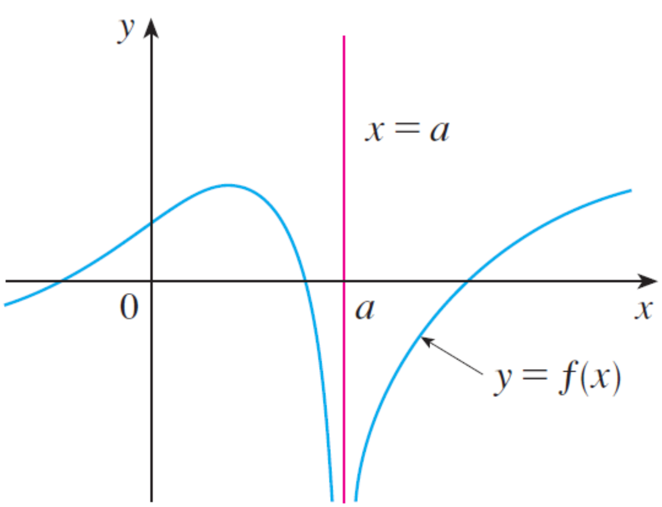
Another notation:
$f(x) \to \infty$ as $x \to a$



$$\lim_{x \to a} f(x) = \infty$$

$\lim_{x \to a} f(x) = -\infty$ means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

Another notation:
$f(x) \to -\infty$ as $x \to a$



$$\lim_{x \to a} f(x) = -\infty$$

remark

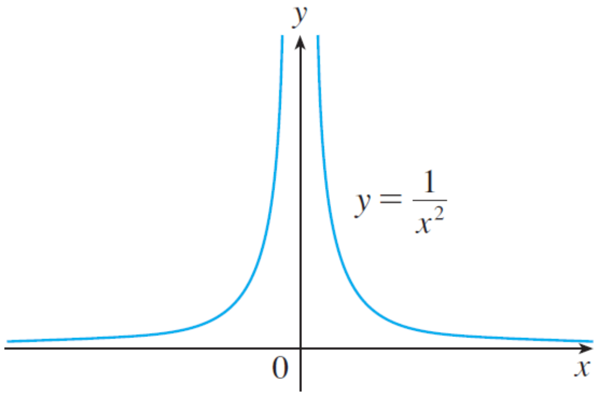
If $\lim_{x \to a} f(x) = \pm\infty$, we can say that limit does not exist.
The converse is not true.

$\lim_{x \to a} f(x) = \pm\infty$ are called infinite limits.

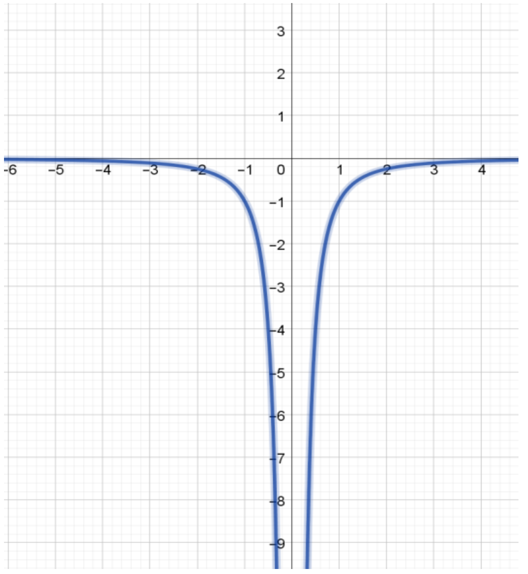
Example 8

a Find $\lim_{x \to 0} \frac{1}{x^2}$ if limit exists.

Solution

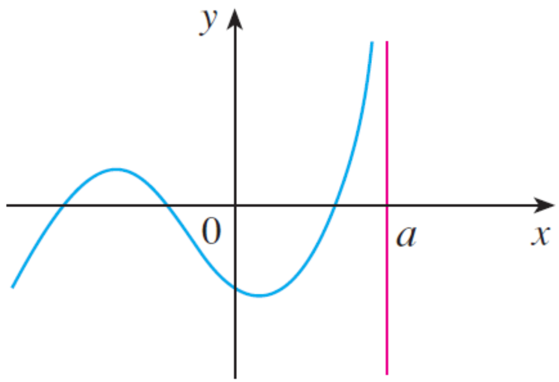


b Find $\lim_{x \to 0} \left(-\frac{1}{x^2}\right)$

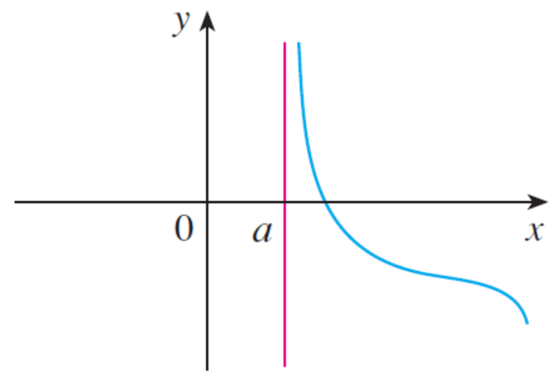


One - sided infinite limits

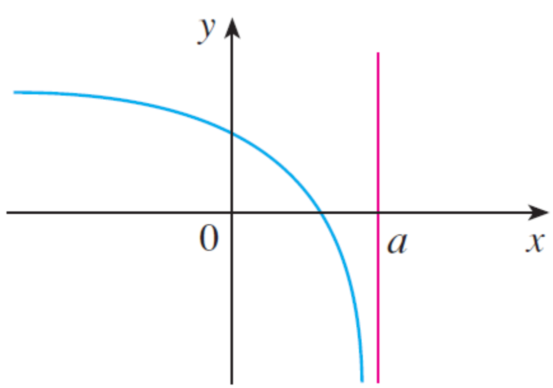
a



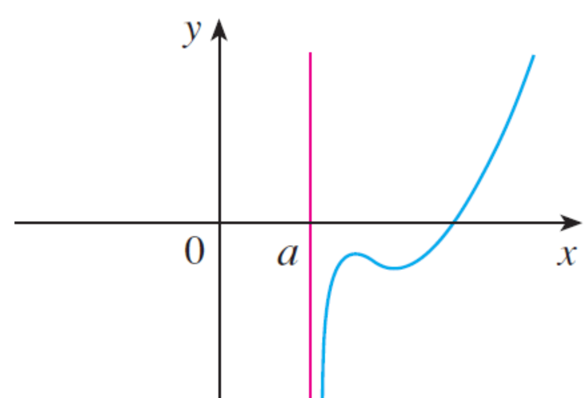
b



c



d



Vertical asymptote

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

a $\lim_{x \to a} f(x) = \infty$

b $\lim_{x \to a} f(x) = -\infty$

c $\lim_{x \to a^-} f(x) = \infty$,

d $\lim_{x \to a^-} f(x) = -\infty$,

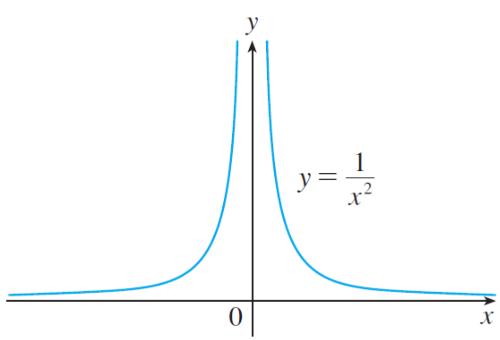
e $\lim_{x \to a^+} f(x) = \infty$

f $\lim_{x \to a^+} f(x) = -\infty$

How to find vertical asymptotes

1 From the graph of $\frac{1}{x^2}$

$x = 0$, is a vertical asymptote.



2 $f(x) = \dfrac{p(x)}{q(x)}$ $\dfrac{\text{عدد}}{0}$

(i) Find x such that $q(x) = 0$

(ii) Substitute those x in $p(x)$ and then find if $p(x) = 0 \Rightarrow x$ isn't vertical asymptote. And if $p(x) = any\ number \neq 0$, then x is a vertical asymptote.

Example

Find the vertical asymptote of

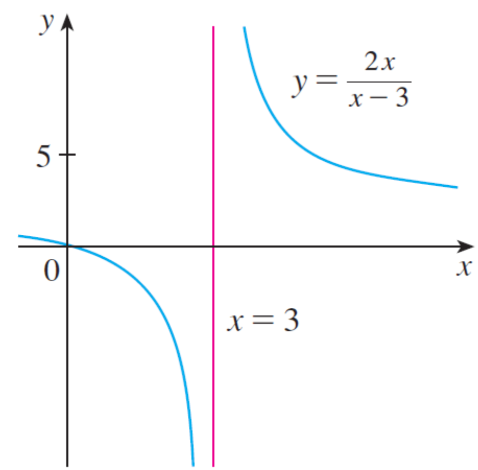
$$f(x) = \frac{x^2 - 4}{x^2 + x - 6} \quad = \frac{p(x)}{q(x)}$$

Solution

Example 9

Find $\lim\limits_{x \to 3^+} \frac{2x}{x-3}$ and $\lim\limits_{x \to 3^-} \frac{2x}{x-3}$

Solution

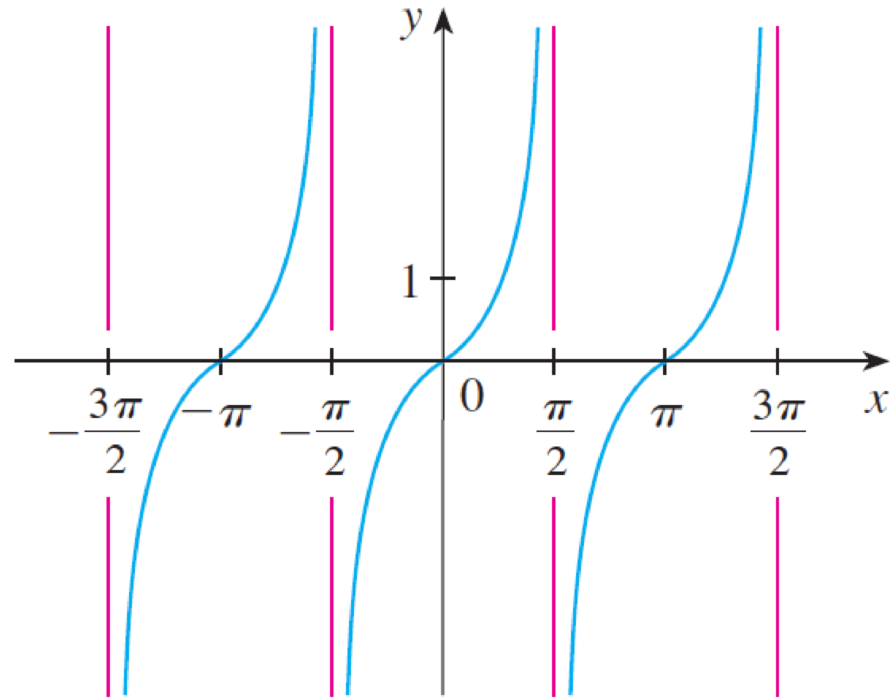


Example 10

Find the vertical asymptotes of $f(x) = tan\, x$.

Solution

1



2 Find $\lim_{x \to \frac{\pi}{2}^+} tan\, x =$

$\lim_{x \to \frac{\pi}{2}^-} tan\, x =$

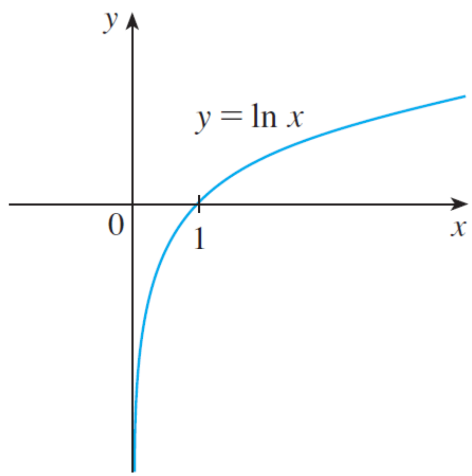
$\lim_{x \to (2n+1)\frac{\pi}{2}^+} tan\, x =$

$\lim_{x \to (2n+1)\frac{\pi}{2}^-} tan\, x =$

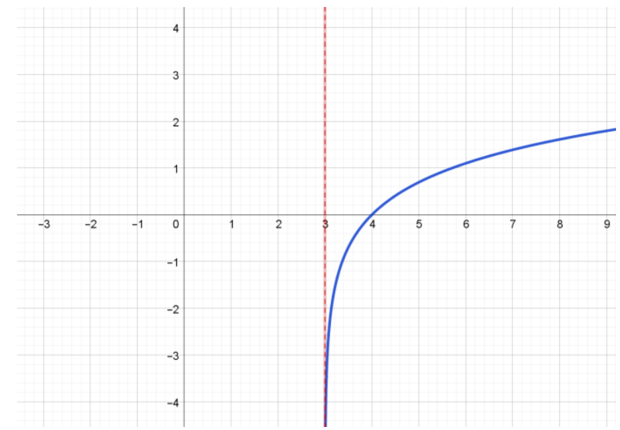
Example

Find the vertical asymptotes of $f(x) = ln\, x$

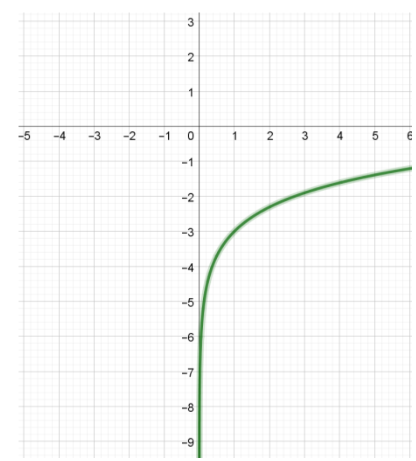
Solution



$f(x) = ln(x - 3)$

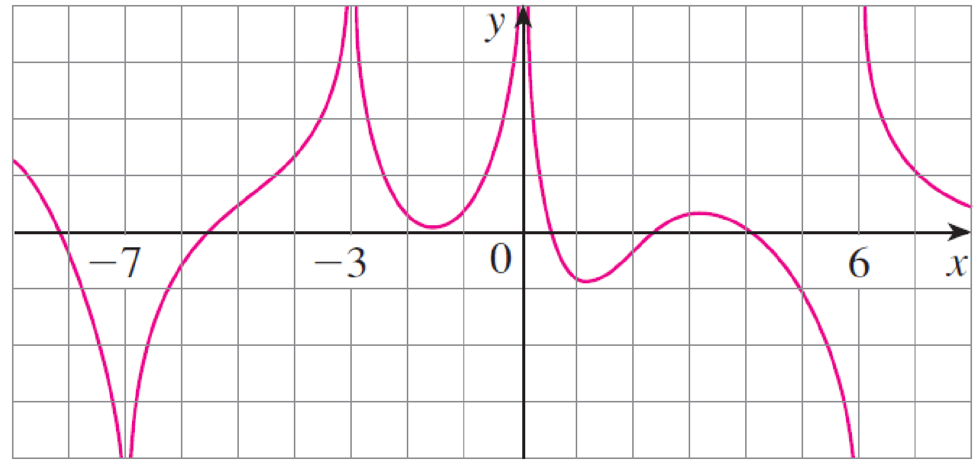


$f(x) = ln\, x - 3$



Exercise 9

For the function f whose graph is shown, state the following



(a) $\lim\limits_{x \to -7} f(x) =$

(b) $\lim\limits_{x \to -3} f(x) =$

(c) $\lim\limits_{x \to 0} f(x) =$

(d) $\lim\limits_{x \to 6^-} f(x) =$

(e) $\lim\limits_{x \to 6^+} f(x) =$

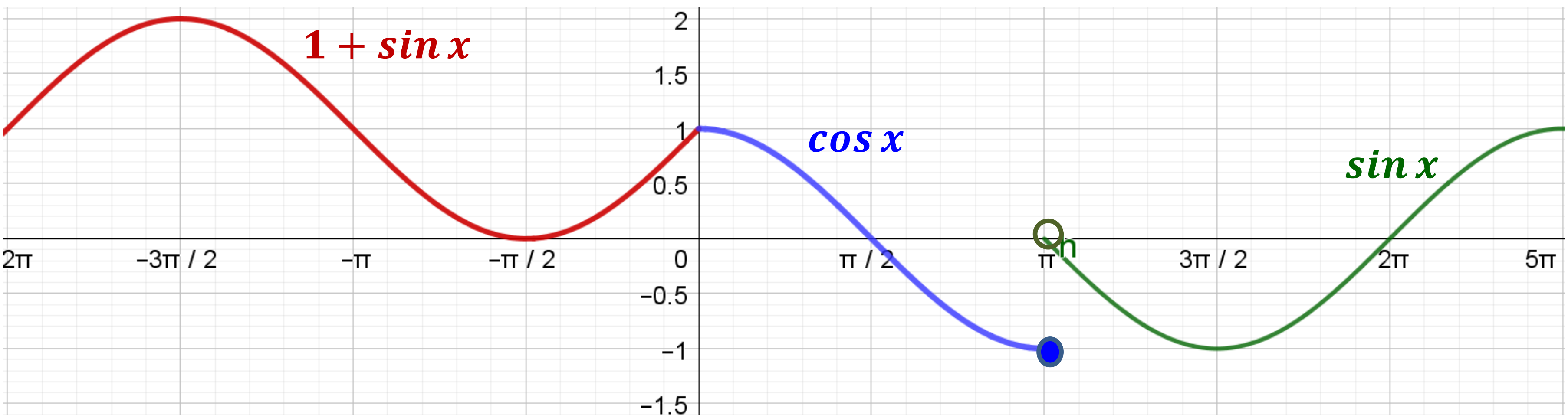
(f) The equations of the vertical asymptotes.

Solution

Exercise 12

Sketch the graph of the function and use it to determine the value of a for which $\lim_{x \to a} f(x)$ exists

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \le x \le \pi \\ \sin x & \text{if } x > \pi \end{cases}$$



$$\lim_{x \to 0} f(x) =$$

$$\lim_{x \to \pi} f(x)$$

Exercise 39

Determine the infinite limit.

$$\lim_{x \to 2\pi^-} x \csc x$$

Solution

$$\lim_{x \to 2\pi^-} x \csc x =$$

$$\lim_{x \to 2\pi^+} x \csc x =$$

Exercise 44

Find the vertical asymptotes

$$y = \frac{x^2 + 1}{3x - 2x^2} = \frac{P(x)}{Q(x)}$$

Solution

note

a. Some functions have no vertical asymptote since their domain is $\mathbb{R}$ such as: polynomials, $\sin x$, $\cos x$, e^x.

b. $$\frac{1}{0} = \begin{cases} +\infty \\ -\infty \end{cases}$$

c. **Vertical Asymptotes of Trigonometric Functions**

$\tan x$, $\sec x$, have $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

$\cot x$, $\csc x$ have $x = 0, \pm\pi, \pm 2\pi, \ldots$

note

$$\lim_{x\to a}\frac{1}{(x-a)^n} = \begin{cases} n \text{ is even} & \displaystyle\lim_{x\to a}\frac{1}{(x-a)^n}=\infty \\ n \text{ is } odd & \begin{cases} \displaystyle\lim_{x\to a^+}\frac{1}{(x-a)^n}=\infty \\ \displaystyle\lim_{x\to a^-}\frac{1}{(x-a)^n}=-\infty \end{cases} \end{cases}$$

Example

$$\lim_{x\to 3}\frac{1}{(x-3)^4} =$$

$$\lim_{x\to 4}\frac{1}{(x-4)^3} = \Bigg\{$$

homework

4 , 7 , 8 , 11