Part 4

- Regression/Curve Fitting (Chapter 17)
  - Statistical Analysis
  - Linear Regression
  - Nonlinear Regression

- Interpolation (Chapter 18)
  - Methods

Statistics

- Basic Statistics with numerical solutions

  -Mean

  -Standard deviation

  -Variance

  -Normal Distribution

Simple Statistics

- In course of engineering study, if several measurements are made of a particular quantity, additional insight can be gained by summarizing the data in one or more well chosen statistics that convey as much information as possible about specific characteristics of the data set.
- These descriptive statistics are most often selected to represent
  - The location of the center of the distribution of the data,
  - The degree of spread of the data.

- *Arithmetic mean*. The sum of the individual data points (yi) divided by the number of points (n).

$$\bar{y} = \frac{\sum y_i}{n}$$

$$i = 1, \ldots, n$$

- *Standard deviation*. The most common measure of a spread for a sample.

$$S_y = \sqrt{\frac{S_t}{n-1}}$$

$$S_t = \sum (y_i - \bar{y})^2$$

or

$$S_y^2 = \frac{\sum y_i^2 - \left(\sum y_i\right)^2 / n}{n-1}$$

- *Variance*. Representation of spread by the square of the standard deviation.

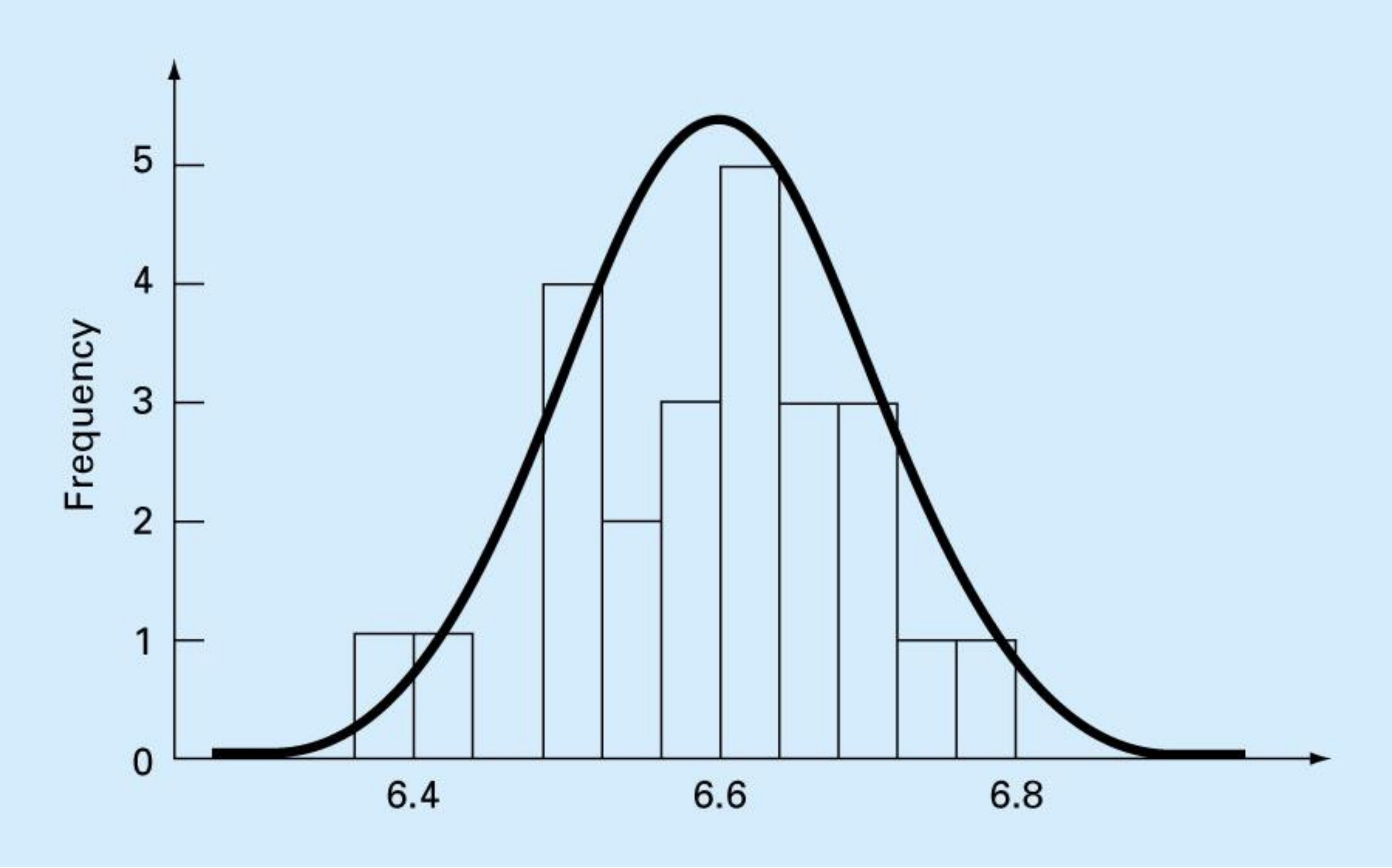
$$S_y^2 = \frac{\sum (y_i - \bar{y})^2}{n-1}$$

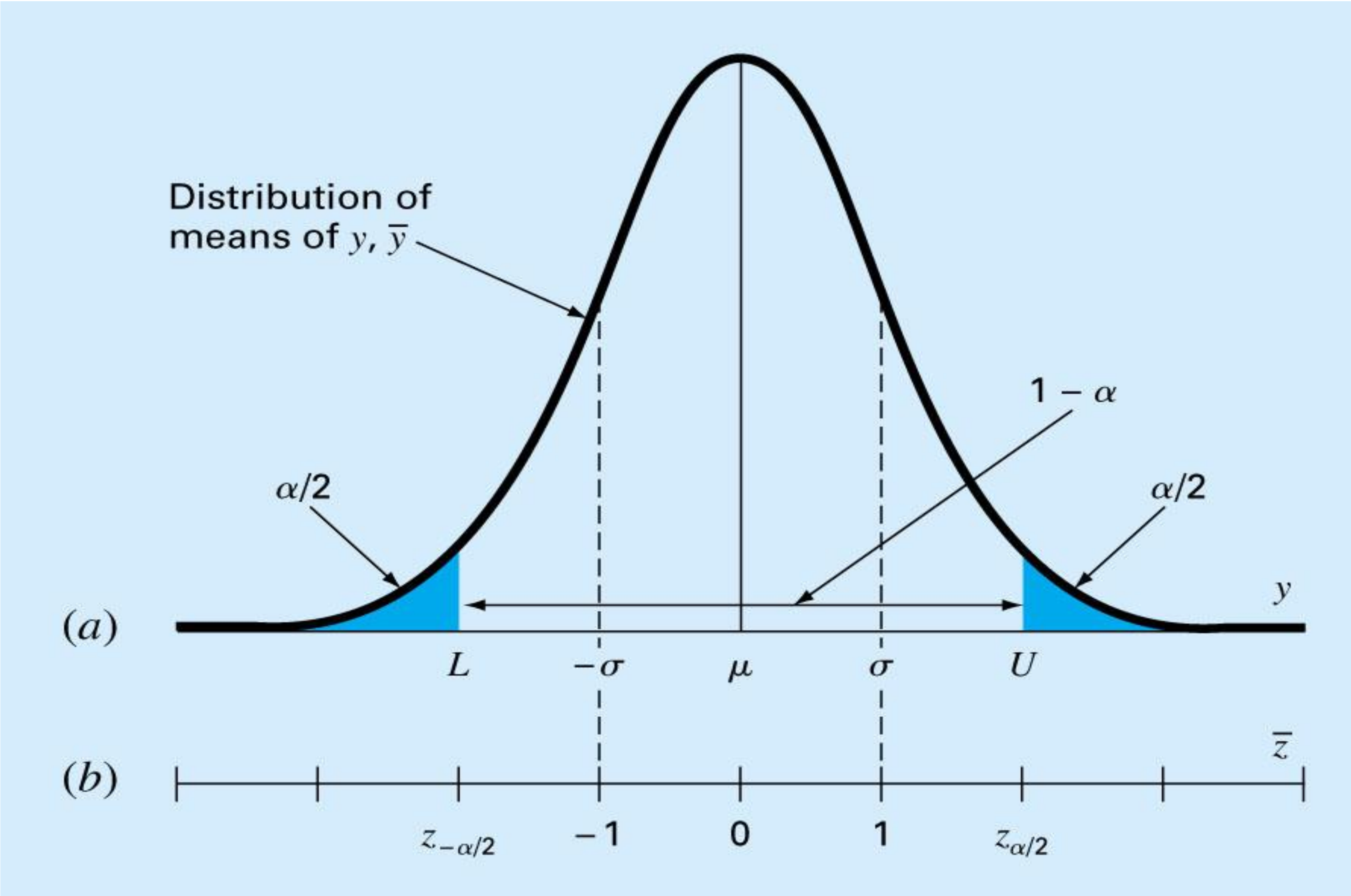
Degrees of freedom

- *Coefficient of variation*. Has the utility to quantify the spread of data.

$$c.v. = \frac{S_y}{\bar{y}} 100\%$$

Normal Distribution




Distribution of means of y, $\bar{y}$
$1 - \alpha$
$\alpha/2$
$\alpha/2$
y
(a)
L
$-\sigma$
μ
σ
U
$\bar{z}$
(b)
$z_{-\alpha/2}$
-1
0
1
$z_{\alpha/2}$

Random Data

Different statistical analysis

- Pdf (probability density function)
- Mean
- Standard deviation
- Variance

Example

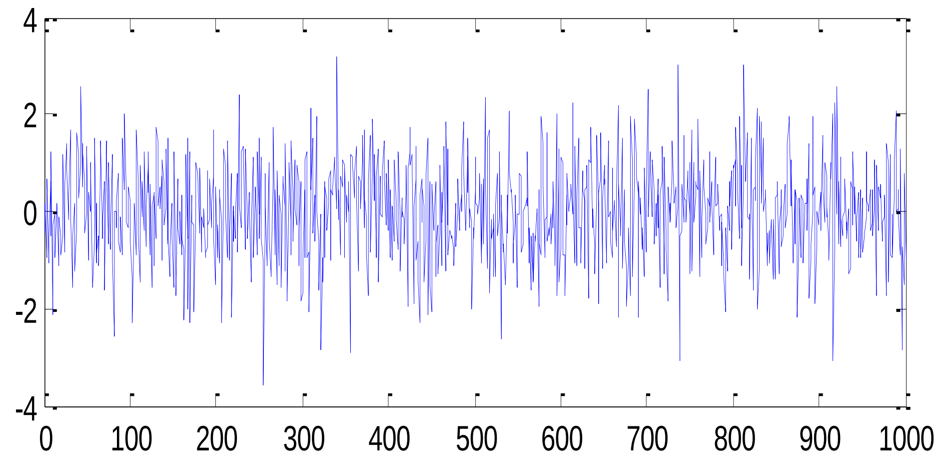
Computing using MATLAB the normal distribution of a random data with length of N=1000 samples.

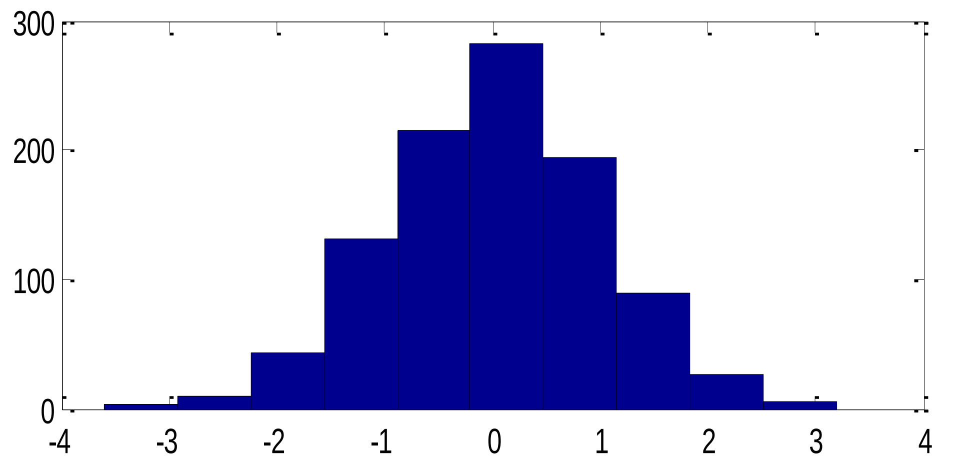
```
x=randn(1,1000);
N=length(x);
for i=1:N
    m(i)=x(i)/N;   %mean
end

mm=mean(x);   %mean
s=std(x);  %standard deviation
h=his(x);  % normal distribution

subplot(211),plot(x)
subplot(212),hist(x)

Y=[m mm s]
```





Y =

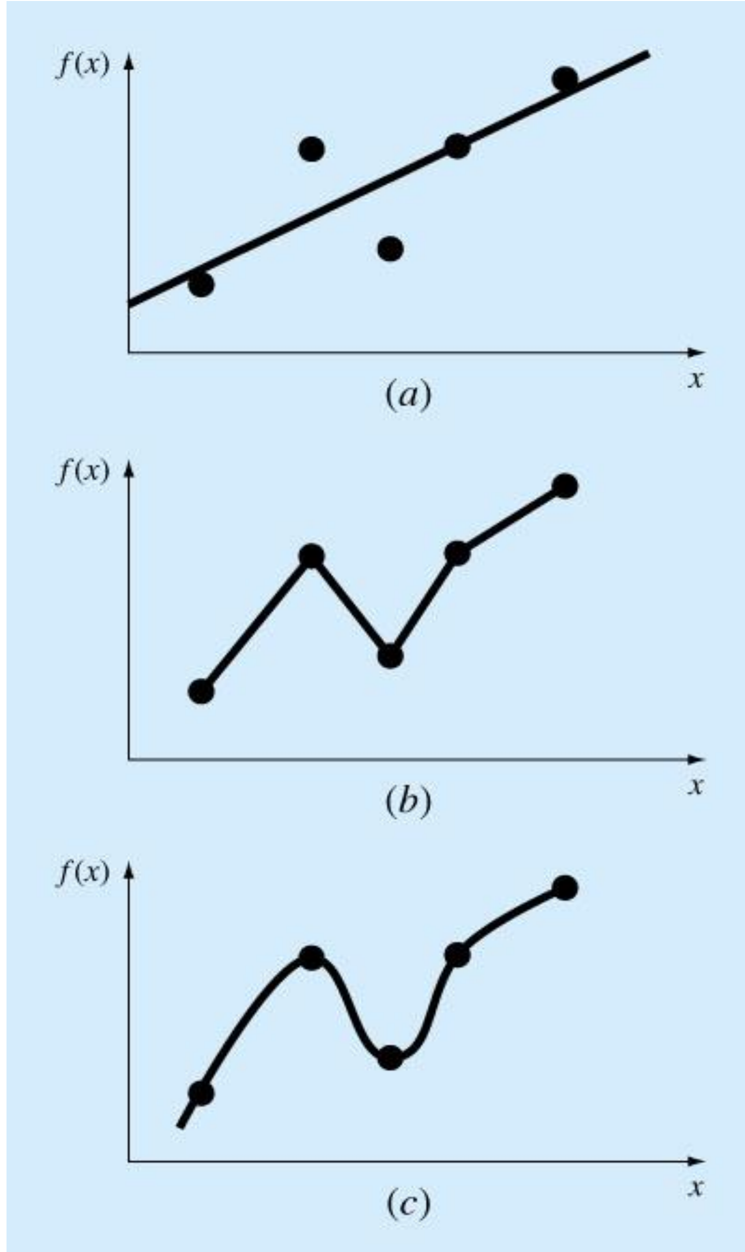
-0.0234   -0.0234   1.0287

CURVE FITTING

- Describes techniques to fit curves (*curve fitting*) to discrete data to obtain intermediate estimates.

- There are two general approaches two curve fitting:
  - *Data exhibit a significant degree of scatter*. The strategy is to derive a single curve that represents the general trend of the data.
  - *Data is very precise.* The strategy is to pass a curve or a series of curves through each of the points.

- In engineering two types of applications are encountered:
  - Trend analysis. Predicting values of dependent variable, may include extrapolation beyond data points or interpolation between data points.
  - Hypothesis testing. Comparing existing mathematical model with measured data.


f(x)
x
(a)
f(x)
x
(b)
f(x)
x
(c)

Least Squares Criteria

- Fitting a straight line to a set of paired observations: $(x_1, y_1), (x_2, y_2),\ldots,(x_n, y_n)$.

  $y=a_0+a_1x+e$

  a_1- slope

  a_0- intercept

  e- error, or residual, between the model and the observations

Criteria for a "Best" Fit/

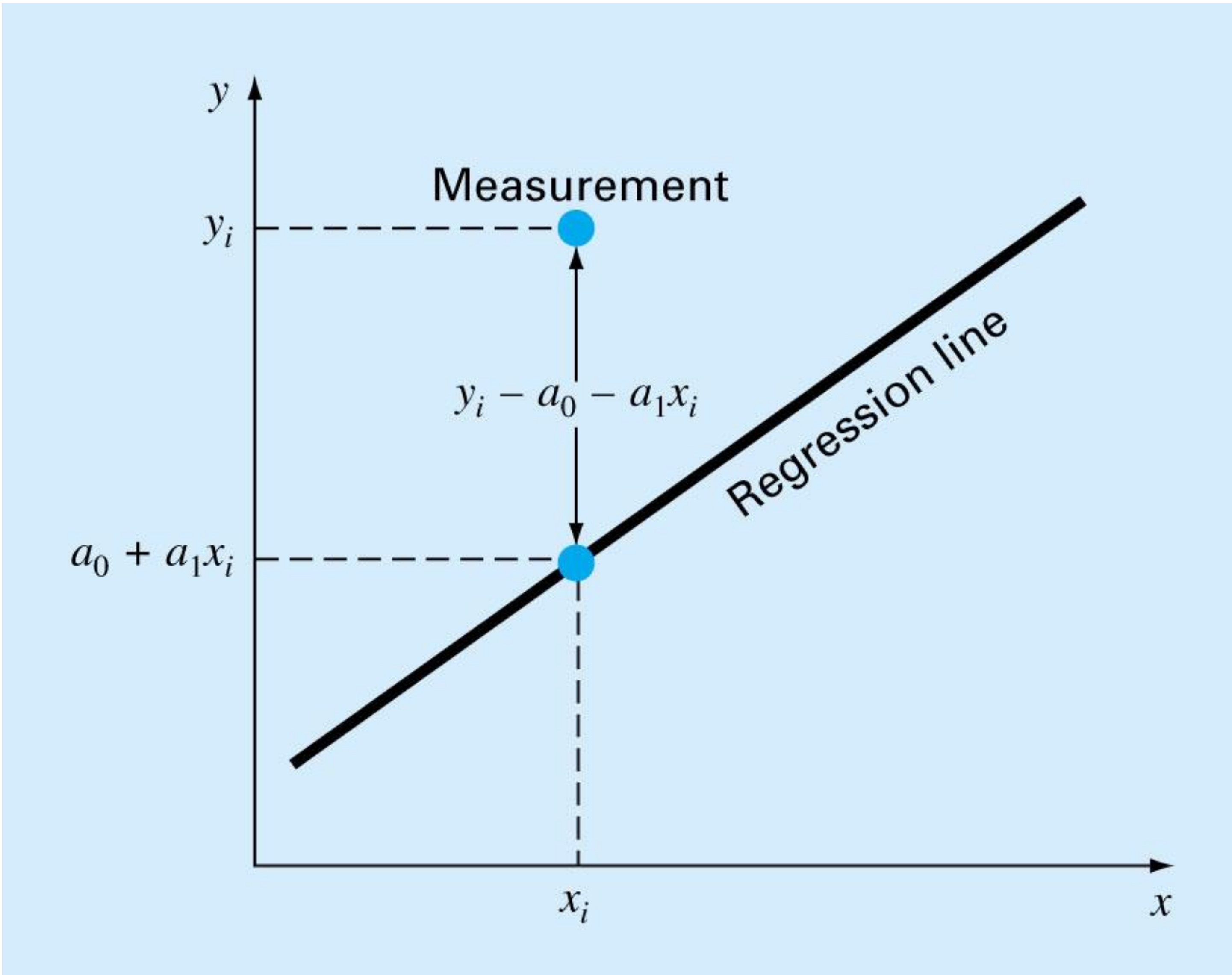
- Minimize the sum of the residual errors for all available data:

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_o - a_1 x_i)$$

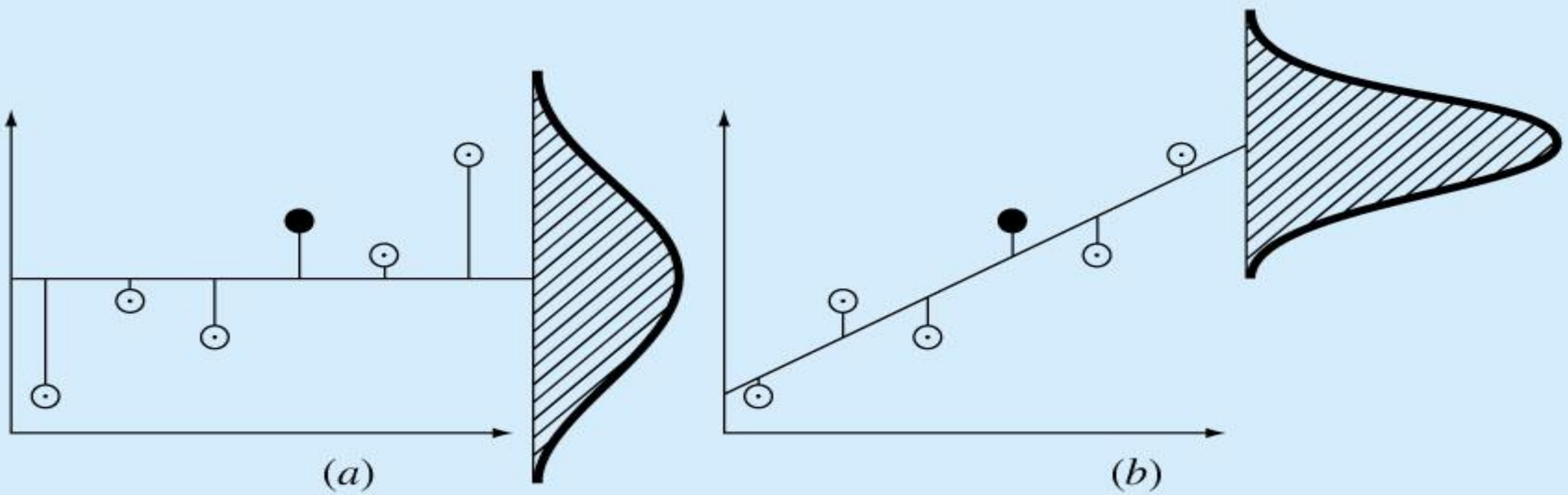
n = total number of points

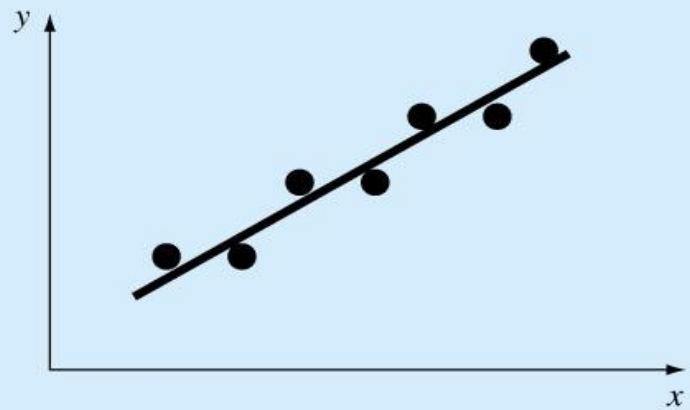
- However, this is an inadequate criterion, so is the sum of the absolute values

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1 x_i|$$

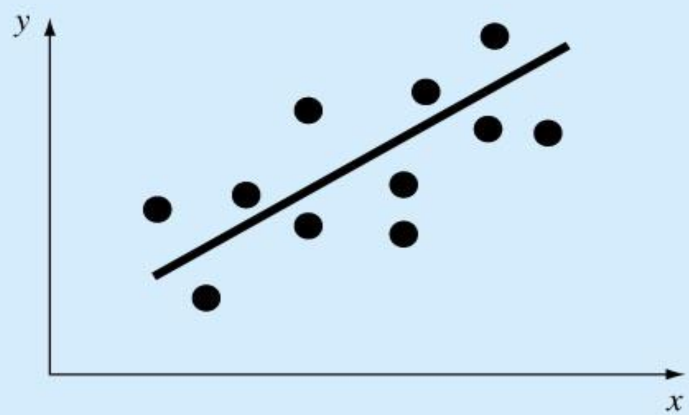


y
Measurement
y_i
$y_i - a_0 - a_1x_i$
Regression line
$a_0 + a_1x_i$
x_i
x


(a)
(b)



(*a*)



(*b*)

- Best strategy is to minimize the sum of the squares of the residuals between the measured y and the y calculated with the linear model:

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i, \text{measured} - y_i, \text{model})^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

- Yields a unique line for a given set of data.

Linear Least Squares

$$y = a_0 z_0 + a_1 z_1 + a_2 z_2 + \cdots + a_m z_m + e$$

$z_0, z_1, \ldots, z_m$ are $m+1$ basis functions

$$\{Y\} = [Z]\{A\} + \{E\}$$

$[Z]$ – matrix of the calculated values of the basis functions at the measured values of the independent variable

$\{\mathrm{Y}\}$ – observed valued of the dependent variable

$\{\mathrm{A}\}$ – unknown coefficients

$\{\mathrm{E}\}$ – residuals or errors

$$S_r = \sum_{i=1}^{n} \left(y_i - \sum_{j=0}^{m} a_j z_{ji} \right)^2$$

Minimized by taking its partial derivative w.r.t. each of the coefficients and setting the resulting equation equal to zero

Least-Squares Fit of a Straight Line/

$$\frac{\partial S_r}{\partial a_o} = -2\sum (y_i - a_o - a_1 x_i) = 0$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum \left[(y_i - a_o - a_1 x_i) x_i\right] = 0$$

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i$$

$$0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2$$

$$\left.\begin{aligned} &\sum a_0 = n a_0 \\ &n a_0 + \left(\sum x_i\right) a_1 = \sum y_i \end{aligned}\right\}$$

Normal equations, can be solved simultaneously

$$a_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - \left(\sum x_i\right)^2}$$

$$a_0 = \bar{y} - a_1 \bar{x}$$

Mean values

Least Squares Approximation

Least Squares Linear Fitting: If we are given a set of data points,

$$(x_i, y_i) \quad i = 1, 2, \ldots, n$$

can we use a line to fit these data points? The answer is positive.

If the line is expressed as

$$y = a_0 + a_1 x$$

where a_0 and a_1 are the two best values to be determined. Obviously, the error e_i of each point (x_i, y_i) with respect to $y = a_0 + a_1 x$ will be

$$e_i = y_i - y\,|_{x=x_i} = y_i - (a_0 + a_1 x_i)$$

The least squares criterion requires that

$$S = e_1^2 + e_2^2 + \ldots + e_n^2 = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_1 x_i - a_0)^2$$

be a minimum.

At a minimum for S, the two derivatives $\partial S / \partial a_0$ and $\partial S / \partial a_1$ will both be zero:

$$\left.\begin{aligned} \frac{\partial S}{\partial a_1} &= \sum_{i=1}^{n} 2(y_i - a_1 x_i - a_0)(-x_i) = 0, \\ \frac{\partial S}{\partial a_0} &= \sum_{i=1}^{n} 2(y_i - a_1 x_i - a_0)(-1) = 0, \end{aligned}\right\} \quad \begin{aligned} a_1 \sum_{i=1}^{n} x_i^2 + a_0 \sum_{i=1}^{n} x_i &= \sum_{i=1}^{n} x_i y_i, \\ a_1 \sum_{i=1}^{n} x_i + a_0 n &= \sum_{i=1}^{n} y_i \end{aligned}$$

Thus, a_0 and a_1 can be obtained so that the data points are linearly fitted.

In fact, we can write the above equations into a linear system:

$$\begin{bmatrix} \sum_{i=1}^{n} (x_i)^0 & \sum_{i=1}^{n} (x_i)^1 \\ \sum_{i=1}^{n} (x_i)^1 & \sum_{i=1}^{n} (x_i)^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \end{pmatrix}$$

Least Squares Polynomials:

Instead of matching the data in every node, the least square method is trying to fit n pairs of data by a polynomial of a pre-determined degree, say m,

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x = \sum_{j=0}^{m} a_j x^j$$

We define the fitting errors

$$e_i = y_i - \sum_{j=0}^{m} a_j x_i^j \qquad\qquad S = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - \sum_{j=0}^{m} a_j x_i^j \right)^2$$

In order to achieve minimal error S (least square error), all the partial derivatives

$$\partial S / \partial a_0 , \partial S / \partial a_1 , \ldots , \partial S / \partial a_m$$

must equal to 0. Writing the equations for these given $m + 1$ equations:

$$
\begin{aligned}
\frac{\partial S}{\partial a_0} &= \sum_{i=1}^{n} 2\left(y_i - \sum_{j=0}^{m} a_j x_i^j \right)(-1) = 0 \\
\frac{\partial S}{\partial a_1} &= \sum_{i=1}^{n} 2\left(y_i - \sum_{j=0}^{m} a_j x_i^j \right)(-x_i) = 0 \\
&\vdots \\
\frac{\partial S}{\partial a_m} &= \sum_{i=1}^{n} 2\left(y_i - \sum_{j=0}^{m} a_j x_i^j \right)(-x_i^m) = 0
\end{aligned}
\quad \Bigg\}
$$

$$
\begin{aligned}
a_0 n + a_1 \sum x_i + \cdots + a_n \sum x_i^m &= \sum y_i \\
a_0 \sum x_i + a_1 \sum x_i^2 + \cdots + a_n \sum x_i^{m+1} &= \sum x_i y_i \\
a_0 \sum x_i^2 + a_1 \sum x_i^3 + \cdots + a_n \sum x_i^{m+2} &= \sum x_i^2 y_i \\
&\vdots \\
a_0 \sum x_i^n + a_1 \sum x_i^{n+1} + \cdots + a_n \sum x_i^{2m} &= \sum x_i^m y_i
\end{aligned}
$$

Or solving the following system

$$
\begin{pmatrix}
n & \Sigma x_i & \Sigma x_i^2 & \cdots & \Sigma x_i^m \\
\Sigma x_i & \Sigma x_i^2 & \Sigma x_i^3 & \cdots & \Sigma x_i^{m+1} \\
\Sigma x_i^2 & \Sigma x_i^3 & \Sigma x_i^4 & \cdots & \Sigma x_i^{m+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\Sigma x_i^m & \Sigma x_i^{m+1} & \Sigma x_i^{m+2} & \cdots & \Sigma x_i^{2m}
\end{pmatrix}
\begin{pmatrix}
a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m
\end{pmatrix}
=
\begin{bmatrix}
\Sigma y_i \\ \Sigma x_i y_i \\ \Sigma x_i^2 y_i \\ \vdots \\ \Sigma x_i^m y_i
\end{bmatrix}
$$

Example:

To demonstrate how the method is used, we would fit a quadratic to the following data:

x_i:	0.05	0.11	0.15	0.31	0.46	0.52	0.70	0.74	0.82	0.98	1.17
y_i:	0.956	0.890	0.832	0.717	0.571	0.539	0.378	0.370	0.306	0.242	0.104

These data are actually a perturbation of the relation

$$y = 1 - x + 0.2x^2$$

Obviously we have

$$\Sigma x_i = 6.01 \ \Sigma x_i^2 = 4.6545 \ \Sigma x_i^3 = 4.1150 \ \Sigma x_i^4 = 3.9161$$

$$n = 11 \qquad \Sigma y_i = 5.9050 \ \Sigma x_i y_i = 2.1839 \ \Sigma x_i^2 y_i = 1.3357$$

Thus the equation system to be solved is:

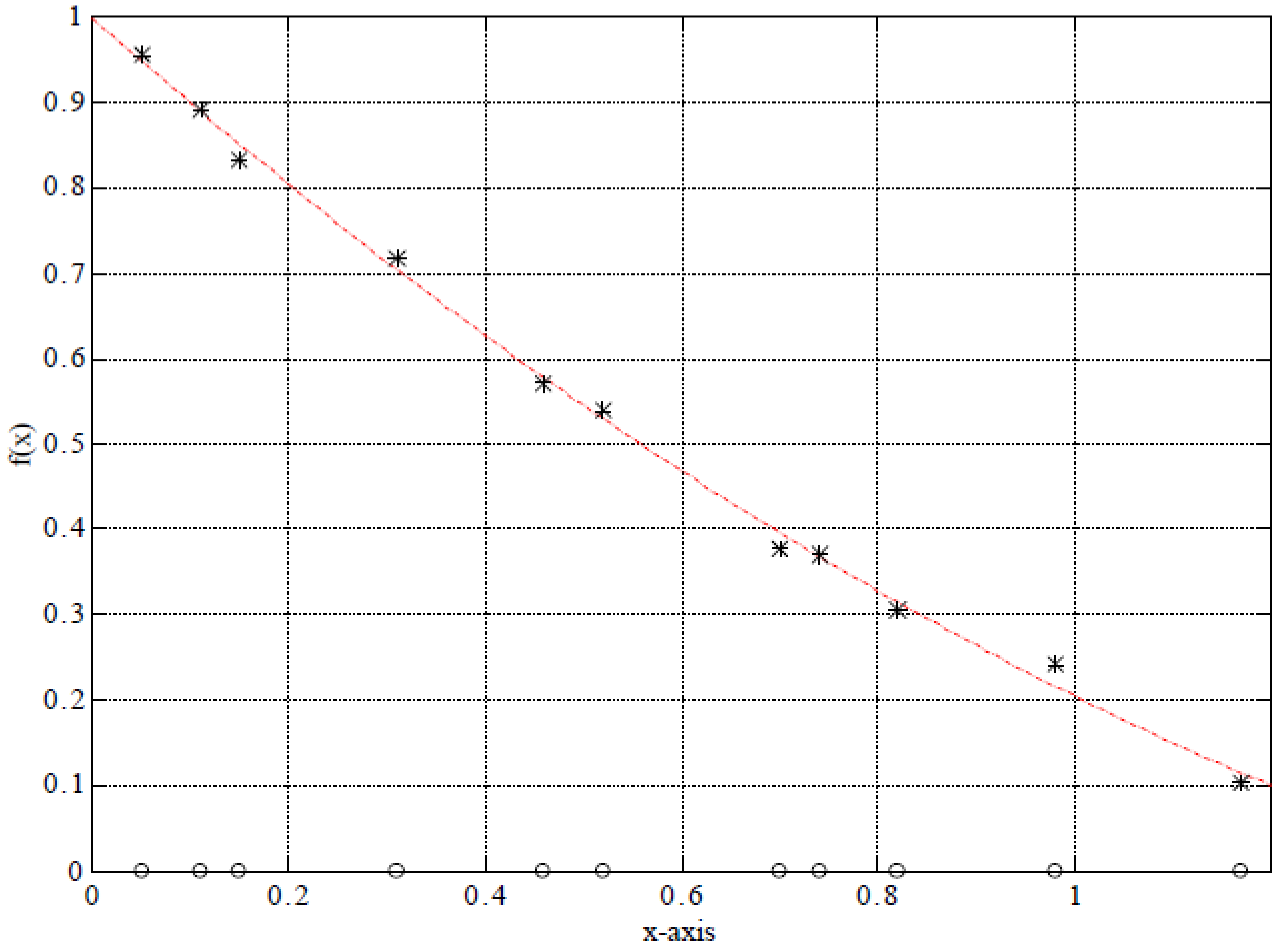
$$
\left.\begin{aligned}
11.0000a_0 + 6.0100a_1 + 4.6545a_2 &= 5.9050 \\
6.0100a_0 + 4.6545a_1 + 4.1150a_2 &= 2.1839 \\
4.6545a_0 + 4.1150a_1 + 3.9161a_2 &= 1.3357
\end{aligned}\right\}
\qquad
\begin{aligned}
a_0 &= 0.998 \\
a_1 &= -1.018 \\
a_2 &= 0.225
\end{aligned}
$$

The above linear system can be solved using methods given in Part I of of this course or using MATLAB software package.

Thus, the least square quadratic fit is given by

$$y = 0.998 - 1.018x + 0.225x^2$$

Compare this to $y = 1 - x + 0.2x^2$. We do not expect to reproduce the coefficients exactly because of the error in the data. Figure of next page shows a plot of the data and its fitting-curve.



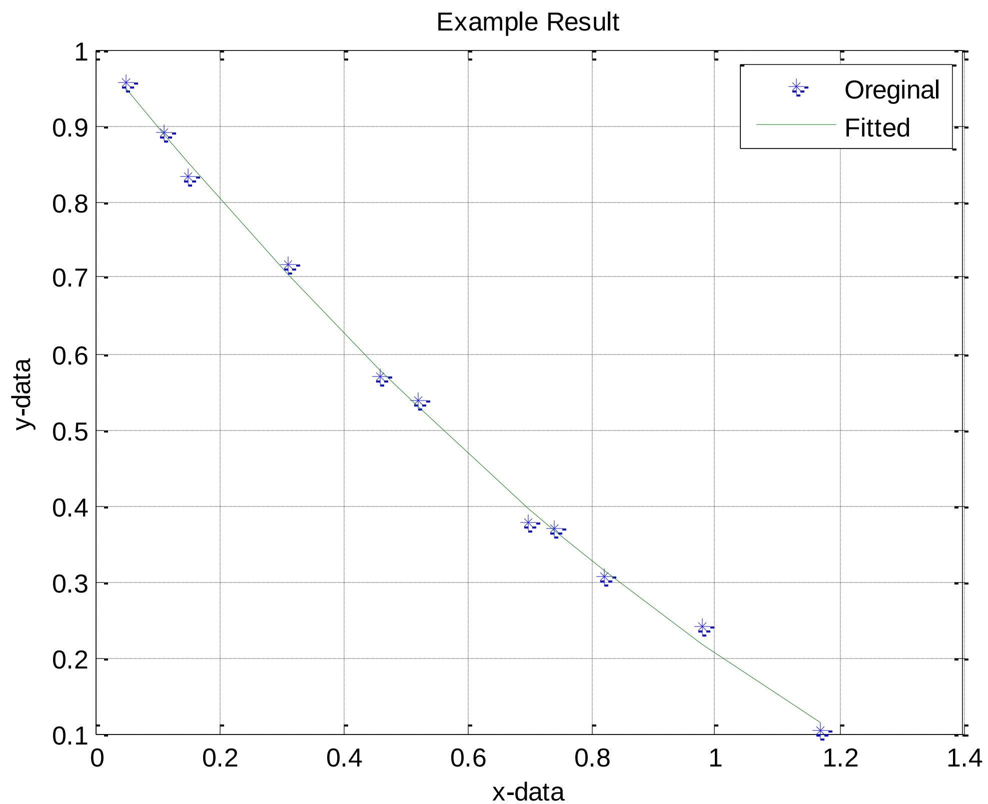
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
f(x)
0
0.2
0.4
0.6
0.8
1
x-axis

Obtaining Polynomials using MATLAB

```
clear all
x=[.05 .11 .15 .31 .46 .52 .70 .74 .82 .98 1.17];
y=[.956 .890 .832 .717 .571 .539 .378 .370 .306 .242 .104];

[P]=polyfit(x,y,2);
Pe=fliplr(P);
ye=Pe(1)+Pe(2)*x+Pe(3)*x.^2;

plot(x,y,'*',x,ye,'-')
grid
```



Example 1

The torque, T needed to turn the torsion spring of a mousetrap through an angle, is given in the table below.

Find the constants for the model given by:

$$T = k_1 + k_2\theta$$

Table: Torque vs Angle for a torsional spring

Angle, θ	**Torque, T**
Radians	*N-m*
0.698132	0.188224
0.959931	0.209138
1.134464	0.230052
1.570796	0.250965
1.919862	0.313707

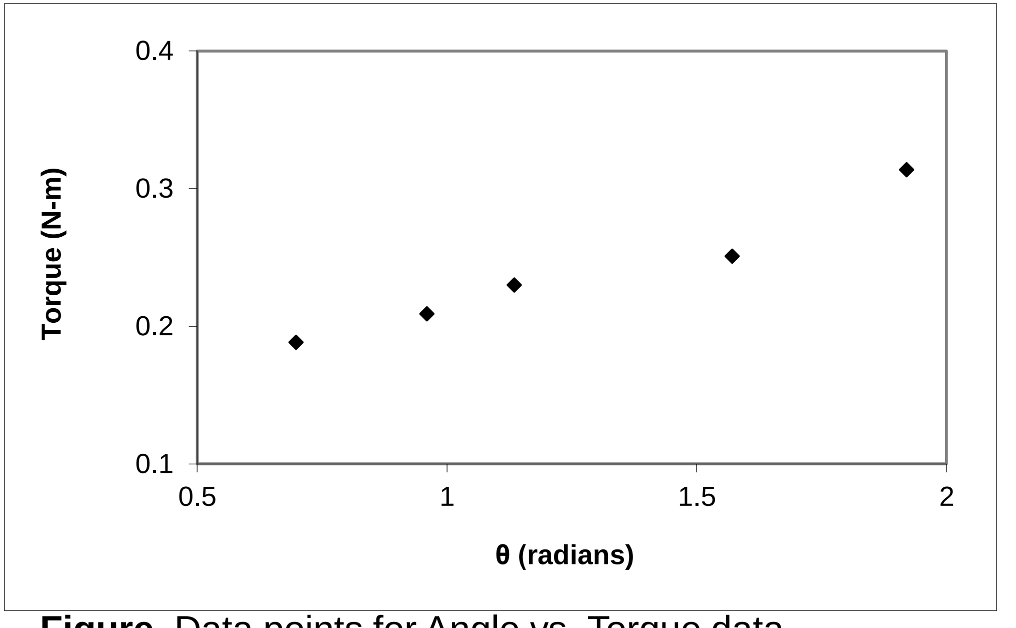


Figure. Data points for Angle vs. Torque data

Example 1 cont.

The following table shows the summations needed for the calculations of the constants in the regression model.

Table. Tabulation of data for calculation of important summations

	θ	T	θ^2	$T\theta$
	Radians	*N-m*	*Radians²*	*N-m-Radians*
	0.698132	0.188224	0.487388	0.131405
	0.959931	0.209138	0.921468	0.200758
	1.134464	0.230052	1.2870	0.260986
	1.570796	0.250965	2.4674	0.394215
	1.919862	0.313707	3.6859	0.602274
$\sum_{i=1}^{5} =$	6.2831	1.1921	8.8491	1.5896

Using equations described for a_0 and a_1 with $n = 5$

$$k_2 = \frac{n\sum_{i=1}^{5}\theta_i T_i - \sum_{i=1}^{5}\theta_i \sum_{i=1}^{5} T_i}{n\sum_{i=1}^{5}\theta_i^2 - \left(\sum_{i=1}^{5}\theta_i\right)^2}$$

$$= \frac{5(1.5896) - (6.2831)(1.1921)}{5(8.8491) - (6.2831)^2}$$

$$= 9.6091 \times 10^{-2}$$ *N-m/rad*

Example 1 cont.

Use the average torque and average angle to calculate k_1

$$\bar{T} = \frac{\sum_{i=1}^{5} T_i}{n}$$

$$= \frac{1.1921}{5}$$

$$= 2.3842 \times 10^{-1}$$

$$\bar{\theta} = \frac{\sum_{i=1}^{5} \theta_i}{n}$$

$$= \frac{6.2831}{5}$$

$$= 1.2566$$

Using,

$$k_1 = \bar{T} - k_2 \, \bar{\theta}$$

$$= 2.3842 \times 10^{-1} - (9.6091 \times 10^{-2})(1.2566)$$

$$= 1.1767 \times 10^{-1} \ N\text{-}m$$

Example 1 Results

Using linear regression, a trend line is found from the data

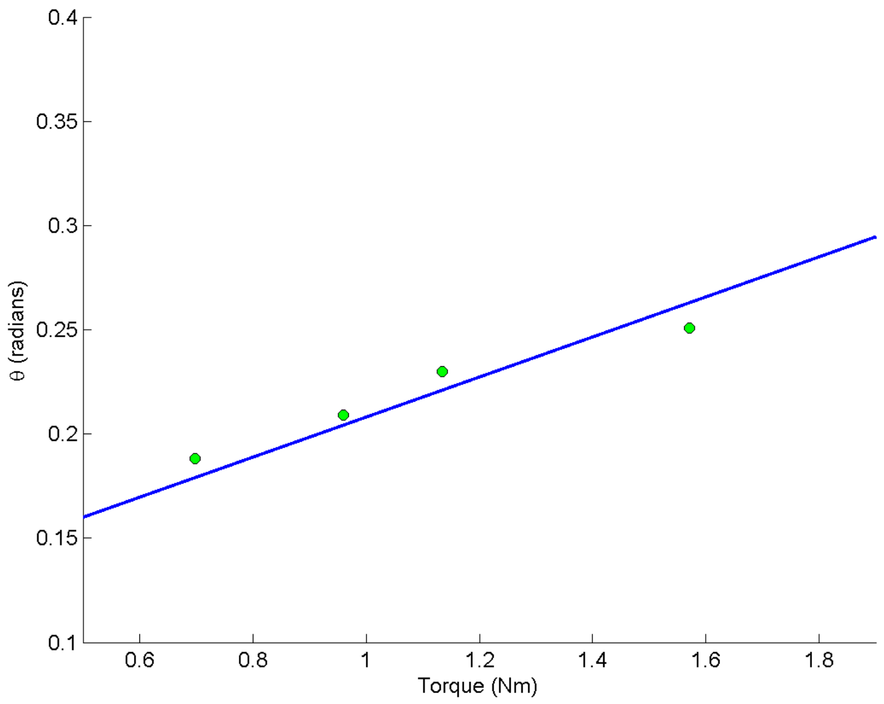


Figure. Linear regression of Torque versus Angle data

Example 2

To find the longitudinal modulus of composite, the following data is collected. Find the longitudinal modulus E using the regression model $\sigma = E\varepsilon$ and the sum of the square of the residuals.

Table. Stress vs. Strain data

Strain	Stress
(%)	*(MPa)*
0	0
0.183	306
0.36	612
0.5324	917
0.702	1223
0.867	1529
1.0244	1835
1.1774	2140
1.329	2446
1.479	2752
1.5	2767
1.56	2896

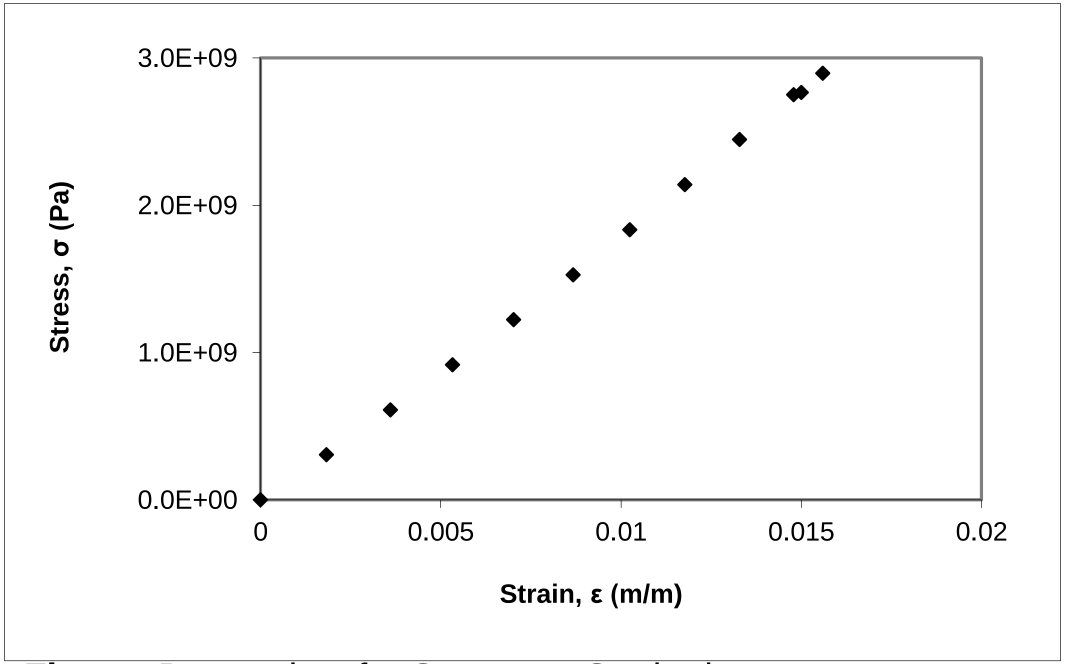


Figure. Data points for Stress vs. Strain data

Example 2 cont.

Residual at each point is given by

$$\gamma_i = \sigma_i - E\varepsilon_i$$

The sum of the square of the residuals then is

$$S_r = \sum_{i=1}^{n} \gamma_i^2$$

$$= \sum_{i=1}^{n} \left(\sigma_i - E\varepsilon_i\right)^2$$

Differentiate with respect to E

$$\frac{\partial S_r}{\partial E} = \sum_{i=1}^{n} 2\left(\sigma_i - E\varepsilon_i\right)(-\varepsilon_i) = 0$$

Therefore $E = \dfrac{\sum_{i=1}^{n} \sigma_i \varepsilon_i}{\sum_{i=1}^{n} \varepsilon_i{}^2}$

Example 2 cont.

Table. Summation data for regression model

i	ε	σ	ε 2	εσ
1	0.0000	0.0000	0.0000	0.0000
2	1.8300×10^{-3}	3.0600×10^{8}	3.3489×10^{-6}	5.5998×10^{5}
3	3.6000×10^{-3}	6.1200×10^{8}	1.2960×10^{-5}	2.2032×10^{6}
4	5.3240×10^{-3}	9.1700×10^{8}	2.8345×10^{-5}	4.8821×10^{6}
5	7.0200×10^{-3}	1.2230×10^{9}	4.9280×10^{-5}	8.5855×10^{6}
6	8.6700×10^{-3}	1.5290×10^{9}	7.5169×10^{-5}	1.3256×10^{7}
7	1.0244×10^{-2}	1.8350×10^{9}	1.0494×10^{-4}	1.8798×10^{7}
8	1.1774×10^{-2}	2.1400×10^{9}	1.3863×10^{-4}	2.5196×10^{7}
9	1.3290×10^{-2}	2.4460×10^{9}	1.7662×10^{-4}	3.2507×10^{7}
10	1.4790×10^{-2}	2.7520×10^{9}	2.1874×10^{-4}	4.0702×10^{7}
11	1.5000×10^{-2}	2.7670×10^{9}	2.2500×10^{-4}	4.1505×10^{7}
12	1.5600×10^{-2}	2.8960×10^{9}	2.4336×10^{-4}	4.5178×10^{7}
$\sum_{i=1}^{12}$			1.2764×10^{-3}	2.3337×10^{8}

With

$$\sum_{i=1}^{12}\varepsilon_i^2 = 1.2764\times10^{-3}$$

and

$$\sum_{i=1}^{12}\sigma_i\varepsilon_i = 2.3337\times10^{8}$$

Using

$$E = \frac{\sum_{i=1}^{12}\sigma_i\varepsilon_i}{\sum_{i=1}^{12}\varepsilon_i^2}$$

$$= \frac{2.3337\times10^{8}}{1.2764\times10^{-3}}$$

$$= 182.84\ GPa$$

Example 2 Results

The equation $\sigma = 182.84\varepsilon$ describes the data.

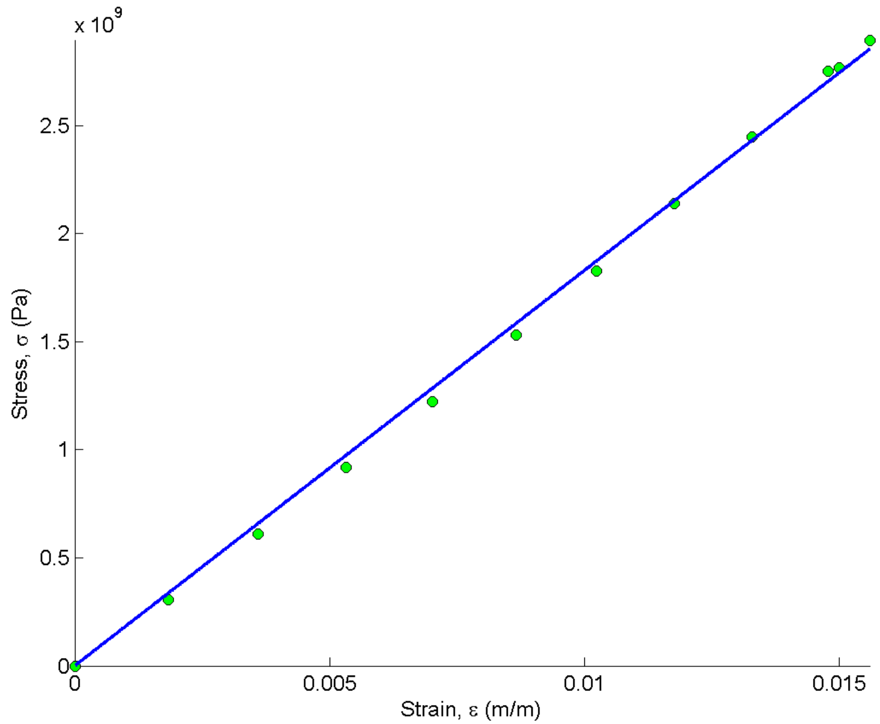


Figure. Linear regression for Stress vs. Strain data

Nonlinear Regression

Some popular nonlinear regression models:

1. Exponential model: $(y = ae^{bx})$
2. Power model: $(y = ax^{b})$
3. Saturation growth model: $\left(y = \dfrac{ax}{b+x}\right)$
4. Polynomial model: $(y = a_0 + a_1 x + ... + a_m x^m)$

Nonlinear Regression
Exponential Model

Given $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ best fit $y = ae^{bx}$ to the data.

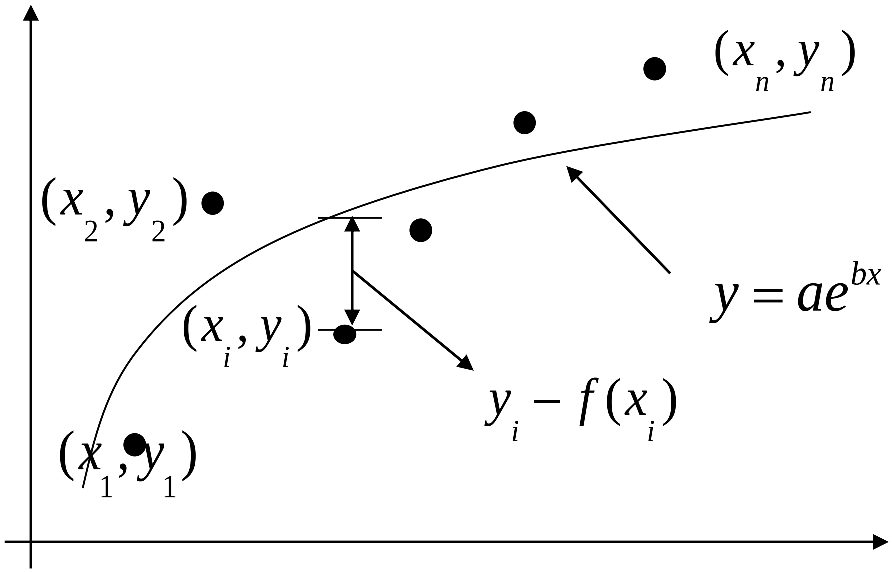


Figure. Exponential model of nonlinear regression for y vs. x data

Finding Constants of Exponential Model

The sum of the square of the residuals is defined as

$$S_r = \sum_{i=1}^{n} \left(y_i - a e^{b x_i} \right)^2$$

Differentiate with respect to *a* and *b*

$$\frac{\partial S_r}{\partial a} = \sum_{i=1}^{n} 2\left(y_i - a e^{b x_i} \right)\left(-e^{b x_i} \right) = 0$$

$$\frac{\partial S_r}{\partial b} = \sum_{i=1}^{n} 2\left(y_i - a e^{b x_i} \right)\left(-a x_i e^{b x_i} \right) = 0$$

Finding Constants of Exponential Model

Rewriting the equations, we obtain

$$-\sum_{i=1}^{n} y_i e^{bx_i} + a\sum_{i=1}^{n} e^{2bx_i} = 0$$

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - a\sum_{i=1}^{n} x_i e^{2bx_i} = 0$$

Finding Constants of Exponential Model

Solving the first equation for *a* yields

$$a = \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}}$$

Substituting a back into the previous equation

$$\sum_{i=1}^{n} y_i x_i e^{bx_i} - \frac{\sum_{i=1}^{n} y_i e^{bx_i}}{\sum_{i=1}^{n} e^{2bx_i}} \sum_{i=1}^{n} x_i e^{2bx_i} = 0$$

The constant *b* can be found through numerical methods such as bisection method.

Example 1-Exponential Model

Many patients get concerned when a test involves injection of a radioactive material. For example for scanning a gallbladder, a few drops of Technetium-99m isotope is used. Half of the techritium-99m would be gone in about *6* hours. It, however, takes about *24* hours for the radiation levels to reach what we are exposed to in day-to-day activities. Below is given the relative intensity of radiation as a function of time.

Table. Relative intensity of radiation as a function of time.

t(hrs)	0	1	3	5	7	9
γ	1.000	0.891	0.708	0.562	0.447	0.355

Example 1-Exponential Model cont.

The relative intensity is related to time by the equation

$$\gamma = Ae^{\lambda t}$$

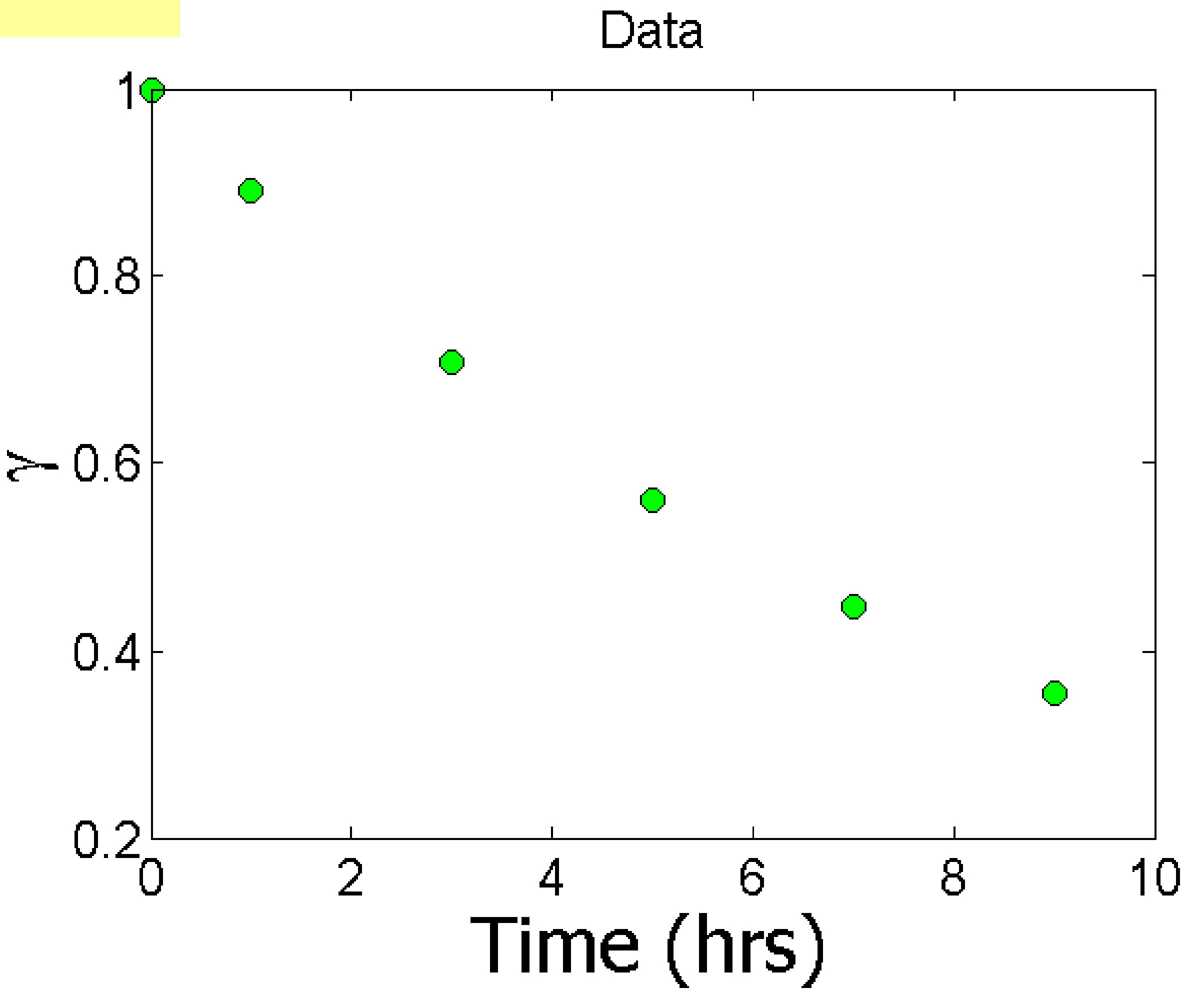
Find:

a) The value of the regression constants A and λ

b) The half-life of Technium-99m

c) Radiation intensity after 24 hours

Plot of data



Constants of the Model

$$\gamma = Ae^{\lambda t}$$

The value of λ is found by solving the nonlinear equation

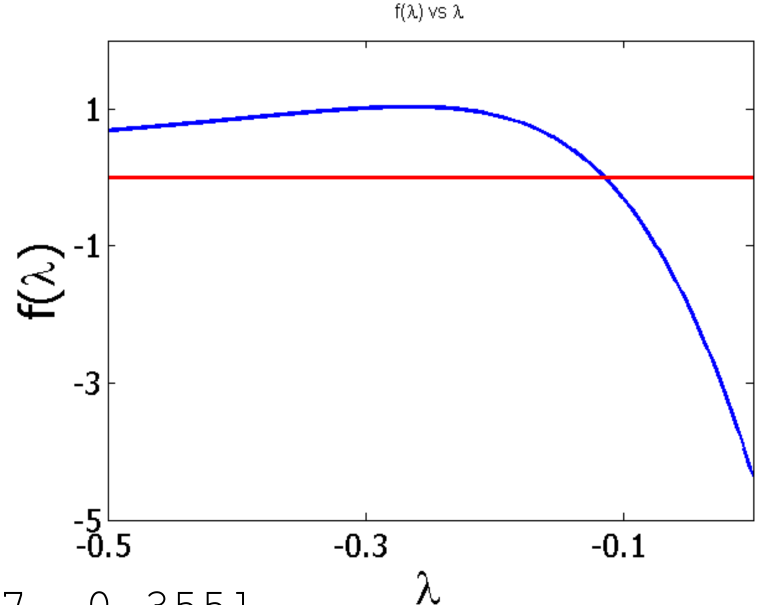
$$f(\lambda) = \sum_{i=1}^{n} \gamma_i t_i e^{\lambda t_i} - \frac{\sum_{i=1}^{n} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n} e^{2\lambda t_i}} \sum_{i=1}^{n} t_i e^{2\lambda t_i} = 0$$

$$A = \frac{\sum_{i=1}^{n} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n} e^{2\lambda t_i}}$$

Setting up the Equation in MATLAB

$$f(\lambda)=\sum_{i=1}^{n}\gamma_i t_i e^{\lambda t_i}-\frac{\sum_{i=1}^{n}\gamma_i e^{\lambda t_i}}{\sum_{i=1}^{n}e^{2\lambda t_i}}\sum_{i=1}^{n}t_i e^{2\lambda t_i}=0$$

$$\lambda=-0.1151$$




```
t=[0 1  3  5  7  9]
gamma=[1  0.891  0.708  0.562  0.447  0.355]
syms lamda
sum1=sum(gamma.*t.*exp(lamda*t));
sum2=sum(gamma.*exp(lamda*t));
sum3=sum(exp(2*lamda*t));
sum4=sum(t.*exp(2*lamda*t));
f=sum1-sum2/sum3*sum4;
```

Calculating the Other Constant

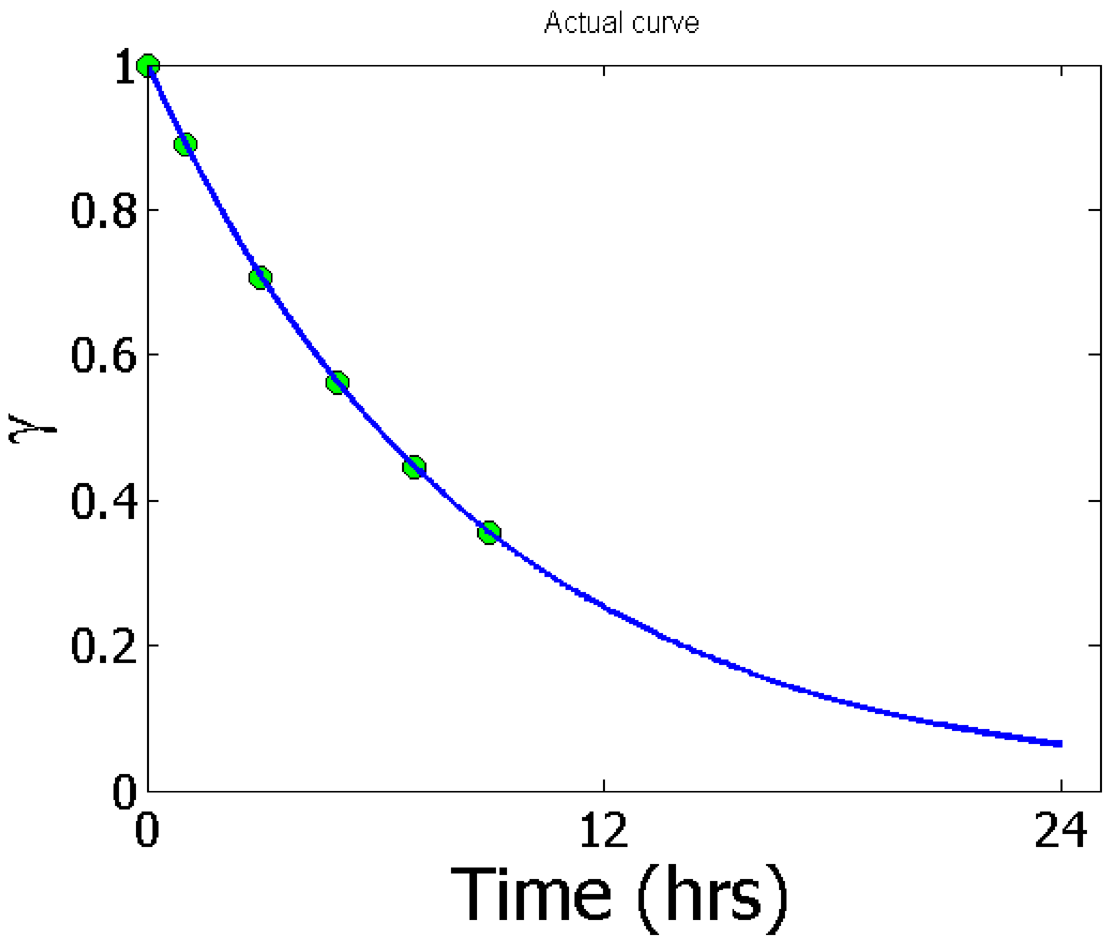
The value of *A* can now be calculated

$$A = \frac{\sum_{i=1}^{6} \gamma_i e^{\lambda t_i}}{\sum_{i=1}^{6} e^{2\lambda t_i}} = 0.9998$$

The exponential regression model then is

$$\gamma = 0.9998\, e^{-0.1151t}$$

Graph of data and regression curve



Relative Intensity After 24 hrs

The relative intensity of radiation after 24 hours

$$\gamma = 0.9998 \times e^{-0.1151(24)}$$

$$= 6.3160 \times 10^{-2}$$

This result implies that only

$$\frac{6.316 \times 10^{-2}}{0.9998} \times 100 = 6.317\%$$

radioactive intensity is left after 24 hours.

Nonlinear Regression
Polynomial Model

Given $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ best fit $y = a_0 + a_1 x + \ldots + a_m x^m$

$(m \leq n-2)$ to a given data set.

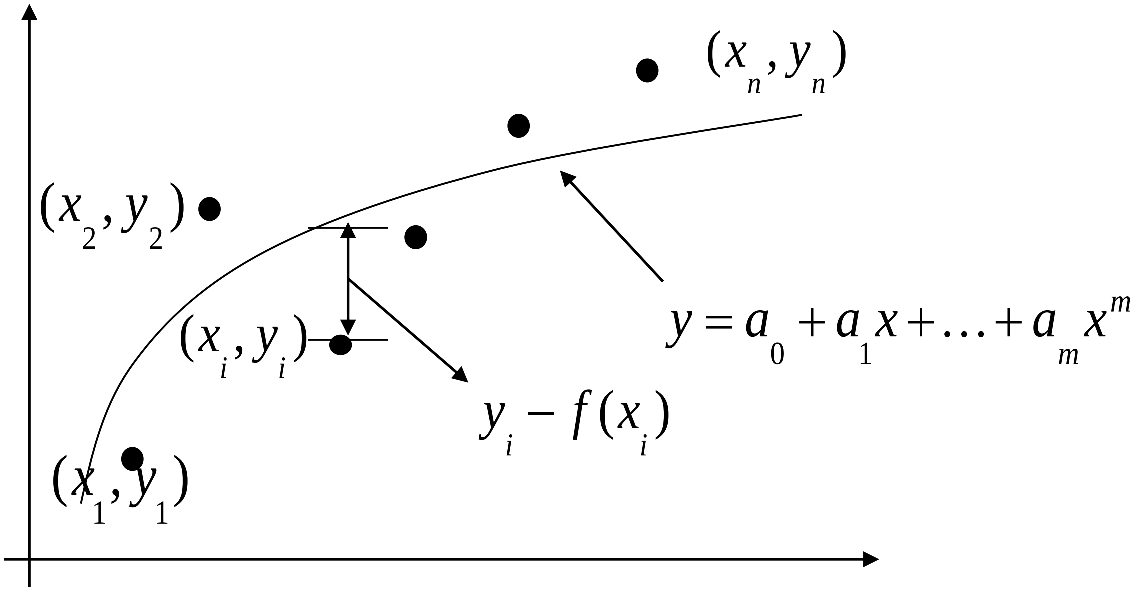


Figure. Polynomial model for nonlinear regression of y vs. x data

Polynomial Model cont.

The residual at each data point is given by

$$E_i = y_i - a_0 - a_1 x_i - . \ . \ . - a_m x_i^m$$

The sum of the square of the residuals then is

$$S_r = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n} \left(y_i - a_0 - a_1 x_i - . \ . \ . - a_m x_i^m \right)^2$$

Polynomial Model cont.

To find the constants of the polynomial model, we set the derivatives with respect to a_i where $i = 1, \ldots m,$ equal to zero.

$$\frac{\partial S_r}{\partial a_0} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \ . \ . - a_m x_i^m\right)(-1) = 0$$

$$\frac{\partial S_r}{\partial a_1} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \ . \ . - a_m x_i^m\right)(-x_i) = 0$$

$$\vdots \qquad \vdots \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$\frac{\partial S_r}{\partial a_m} = \sum_{i=1}^{n} 2.\left(y_i - a_0 - a_1 x_i - . \ . \ . - a_m x_i^m\right)(-x_i^m) = 0$$

Polynomial Model cont.

These equations in matrix form are given by

$$\begin{bmatrix} n & \left(\sum_{i=1}^{n} x_i\right) & . \quad . \quad . & \left(\sum_{i=1}^{n} x_i^m\right) \\ \left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) & . \quad . \quad . & \left(\sum_{i=1}^{n} x_i^{m+1}\right) \\ . \quad . & . \quad . \quad . \quad . & . \quad . \quad . & . \quad . \\ \left(\sum_{i=1}^{n} x_i^m\right) & \left(\sum_{i=1}^{n} x_i^{m+1}\right) & . \quad . \quad . & \left(\sum_{i=1}^{n} x_i^{2m}\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ . \ . \ . \\ a_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_i y_i \\ . \quad . \quad . \\ \sum_{i=1}^{n} x_i^m y_i \end{bmatrix}$$

The above equations are then solved for $a_0, a_1, \ldots, a_m$

Example 2 Polynomial Model

Regress the thermal expansion coefficient vs. temperature data to a second order polynomial.

Table. Data points for temperature vs α

Temperature, T (ºF)	Coefficient of thermal expansion, α (in/in/ºF)
80	6.47×10^{-6}
40	6.24×10^{-6}
−40	5.72×10^{-6}
−120	5.09×10^{-6}
−200	4.30×10^{-6}
−280	3.33×10^{-6}
−340	2.45×10^{-6}

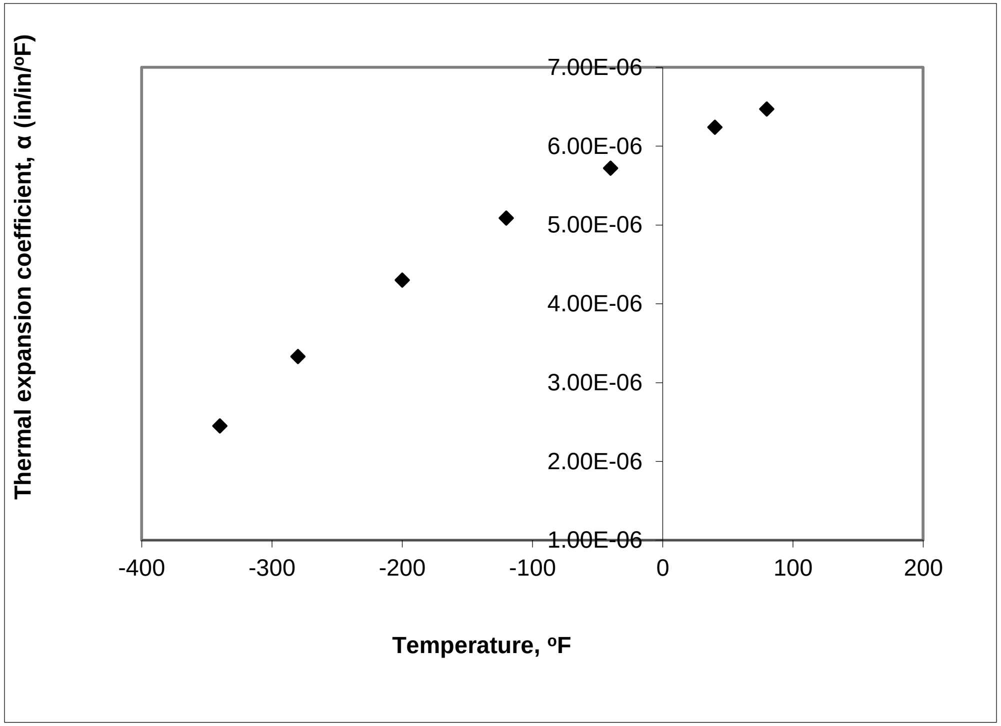


Figure. Data points for thermal expansion coefficient vs temperature.

Example 2 Polynomial Model cont.

We are to fit the data to the polynomial regression model

$$\alpha = a_0 + a_1 T + a_2 T^2$$

The coefficients a_0, a_1, a_2 are found by differentiating the sum of the square of the residuals with respect to each variable and setting the values equal to zero to obtain

$$\begin{bmatrix} n & \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) \\ \left(\sum_{i=1}^{n} T_i\right) & \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) \\ \left(\sum_{i=1}^{n} T_i^2\right) & \left(\sum_{i=1}^{n} T_i^3\right) & \left(\sum_{i=1}^{n} T_i^4\right) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} \alpha_i \\ \sum_{i=1}^{n} T_i \, \alpha_i \\ \sum_{i=1}^{n} T_i^2 \alpha_i \end{bmatrix}$$

Example 2 Polynomial Model cont.

The necessary summations are as follows

Table. Data points for temperature vs. α

Temperature, T (°F)	Coefficient of thermal expansion, α (in/in/°F)
80	6.47×10^{-6}
40	6.24×10^{-6}
−40	5.72×10^{-6}
−120	5.09×10^{-6}
−200	4.30×10^{-6}
−280	3.33×10^{-6}
−340	2.45×10^{-6}

$$\sum_{i=1}^{7} T_i^2 = 2.5580\times10^{5}$$

$$\sum_{i=1}^{7} T_i^3 = -7.0472\times10^{7}$$

$$\sum_{i=1}^{7} T_i^4 = 2.1363\times10^{10}$$

$$\sum_{i=1}^{7} \alpha_i = 3.3600\times10^{-5}$$

$$\sum_{i=1}^{7} T_i\alpha_i = -2.6978\times10^{-3}$$

$$\sum_{i=1}^{7} T_i^2\alpha_i = 8.5013\times10^{-1}$$

Example 2 Polynomial Model cont.

Using these summations, we can now calculate a_0, a_1, a_2

$$\begin{bmatrix} 7.0000 & -8.6000\times10^{2} & 2.5800\times10^{5} \\ -8.600\times10^{2} & 2.5800\times10^{5} & -7.0472\times10^{7} \\ 2.5800\times10^{5} & -7.0472\times10^{7} & 2.1363\times10^{10} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3.3600\times10^{-5} \\ -2.6978\times10^{-3} \\ 8.5013\times10^{-1} \end{bmatrix}$$

Solving the above system of simultaneous linear equations we have

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6.0217\times10^{-6} \\ 6.2782\times10^{-9} \\ -1.2218\times10^{-11} \end{bmatrix}$$

The polynomial regression model is then

$$\alpha = a_0 + a_1 T + a_2 T^2$$

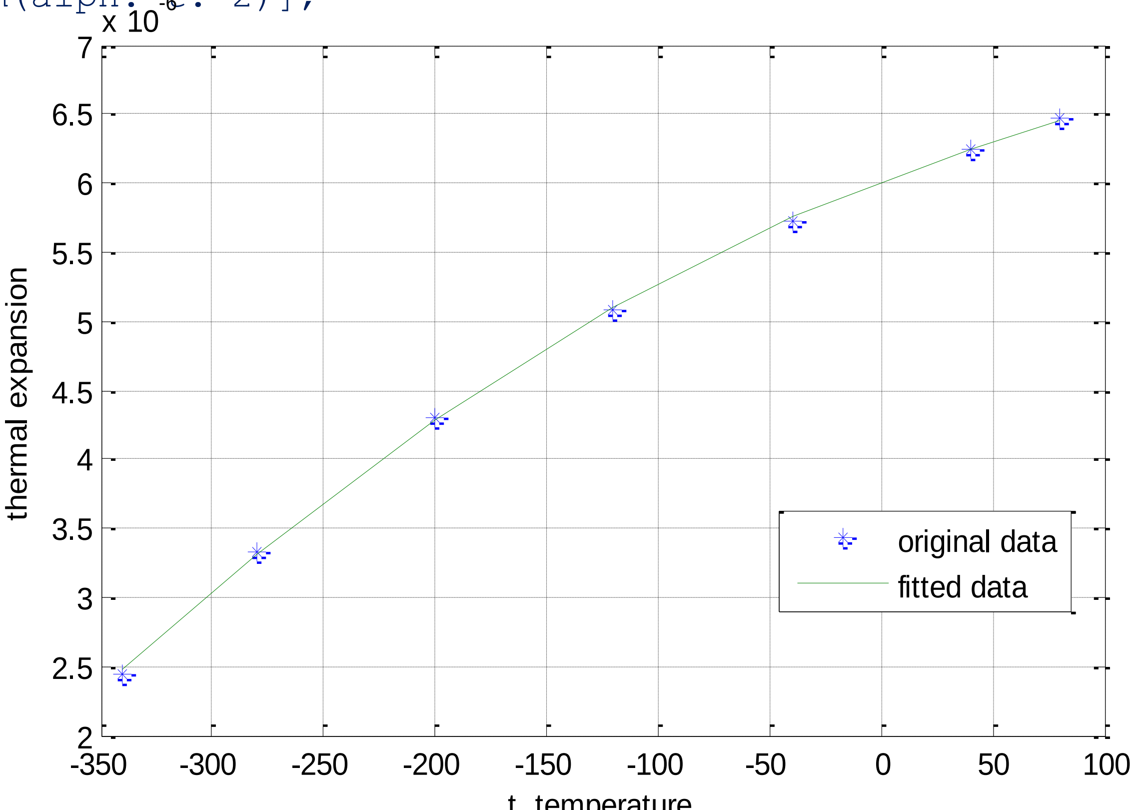
$$= 6.0217\times10^{-6} + 6.2782\times10^{-9}\,\mathrm{T} - 1.2218\times10^{-11}\,\mathrm{T}^2$$

Example 2 Polynomial Model cont.

Solution using Matlab

First approach

```
t=[80 40 -40 -120 -200 -280 -340];
alph=[6.47*10^(-6) 6.24*10^(-6) 5.72*10^(-6) 5.09*10^(-6) 4.3*10^(-6) 3.33*10^(-6)
2.45*10^(-6)];
A=[7 sum(t) sum(t.^2);sum(t) sum(t.^2) sum(t.^3);sum(t.^2) sum(t.^3) sum(t.^4)];
b=[sum(alph) sum(t.*alph) sum(alph.*t.^2)];
a=inv(A)*b';
alph_e=a(1)+a(2)*t+a(3)*t.^2;
figure(1)
plot(t,alph,'*',t,alph_e)
```



Example 2 Polynomial Model cont.

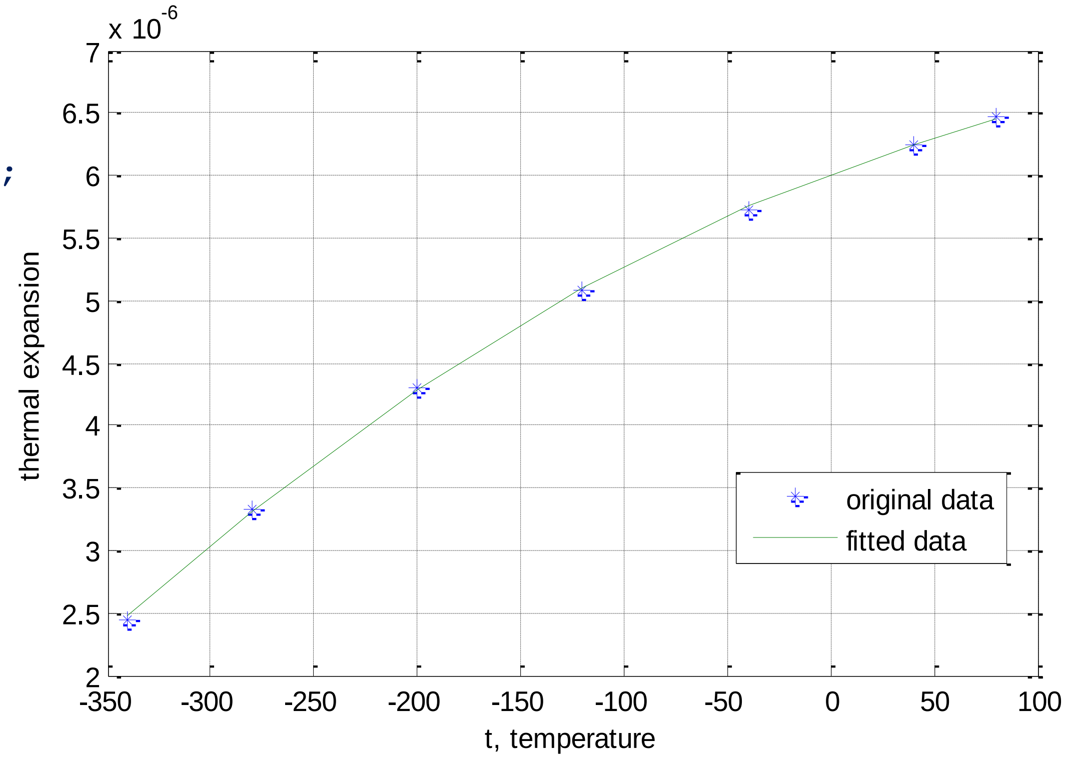
Solution using Matlab

Second approach using polyfit function

```
t=[80 40 -40 -120 -200 -280 -340];
alph=[6.47*10^(-6) 6.24*10^(-6) 5.72*10^(-6) 5.09*10^(-6) 4.3*10^(-6) 3.33*10^(-6)
2.45*10^(-6)];

P=polyfit(t,alpha,2);
Pe=fliplr(P);
alph_e2=Pe(1)+Pe(2)*t+Pe(3)*t.^2;

figure(2)
plot(t,alph,'*',t,alph_e2)
```



INTERPOLATION

Chapter 18

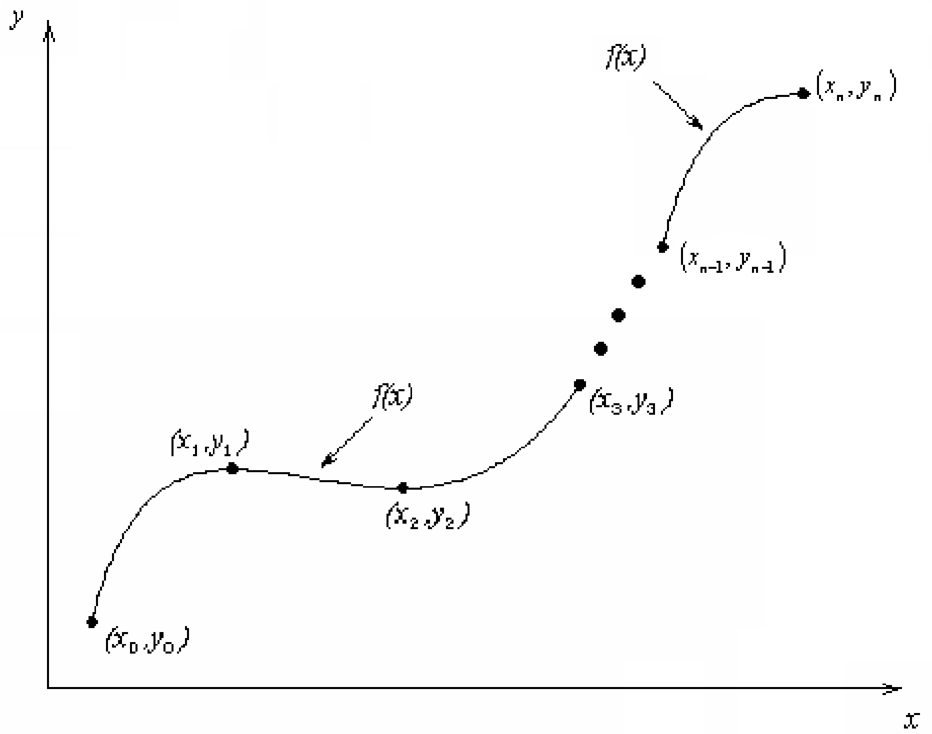
What is Interpolation?

Methods:

- Polynomial
- Lagrange Method of Interpolation
- Spline Interpolation Method
- Newton’s Divided Difference Method

What is Interpolation ?

For a given data (x0,y0), (x1,y1), …… (xn,yn), the interpolation is to find the value of 'y' at a value of 'x' that is not given.



Interpolation of discrete data.

Interpolation with Polynomials

Polynomials are the most common choice because they are easy to:

- Evaluate
- Differentiate, and
- Integrate

Direct Method

Given 'n+1' data points (x_0,y_0), (x_1,y_1),.............. (x_n,y_n), pass a polynomial of order 'n' through the data as given below:

$$y = a_0 + a_1 x + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n.$$

where a_0, a_1,................. a_n are real constants.

- Set up 'n+1' equations to find 'n+1' constants.
- To find the value 'y' at a given value of 'x', simply substitute the value of 'x' in the above polynomial.

Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

1- Find the velocity at t=16 seconds using the direct method for:

- **linear interpolation**
- **Quadratic interpolation**
- **Cubic interpolation**

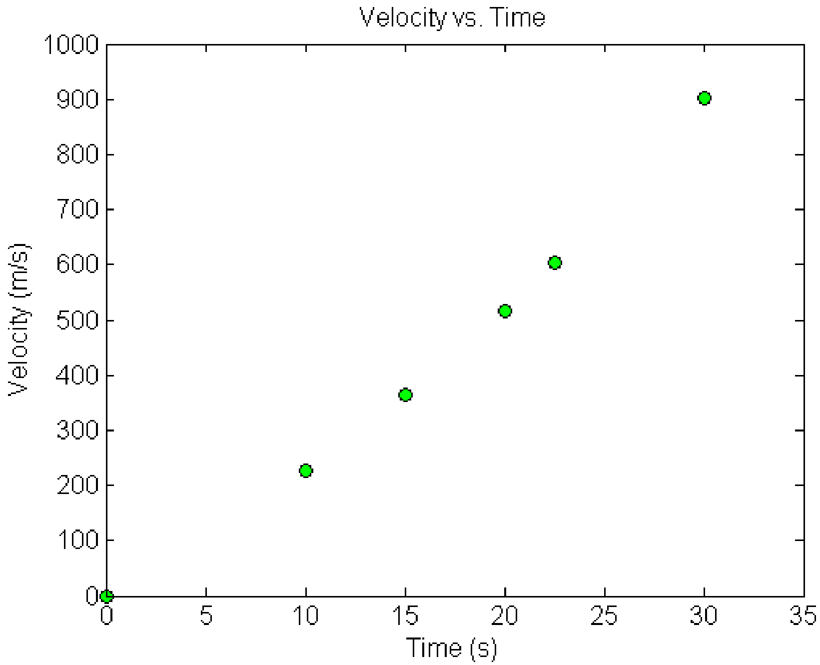
2- Find from cubic method:

a – The distance covered by the rocket from t=11s to t=16s.

b -The acceleration of the rocket at t=16s

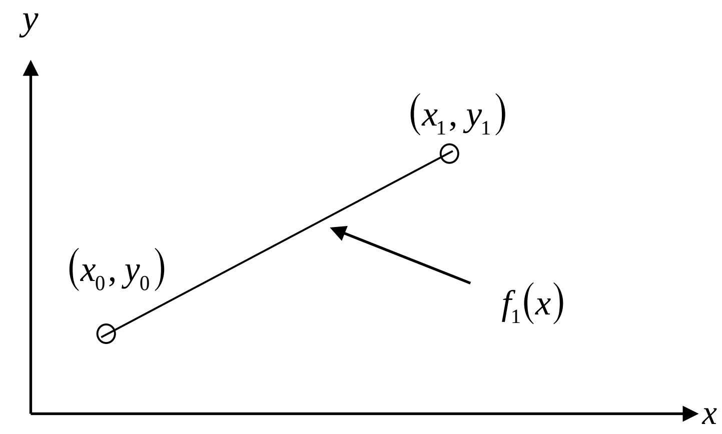
Table 1 Velocity as a function of time.

$t, (\text{s})$	$v(t), (\text{m/s})$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Velocity vs. time data for the rocket example

Linear Interpolation



Linear interpolation.

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

Solving the above two equations gives,

$$a_0 = -100.93 \qquad a_1 = 30.914$$

Hence

$$v(t) = -100.93 + 30.914t, \ 15 \le t \le 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$

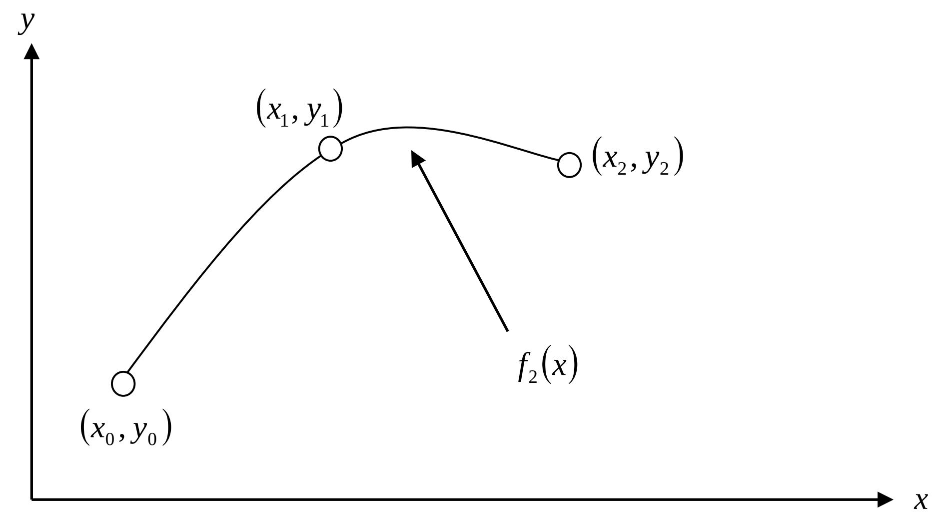
Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$



Quadratic interpolation.

Solving the above three equations gives

$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

Quadratic Interpolation (cont.)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \ 10 \le t \le 20$$

$$v(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$

$$= 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

$$= 0.38410\%$$

Cubic Interpolation

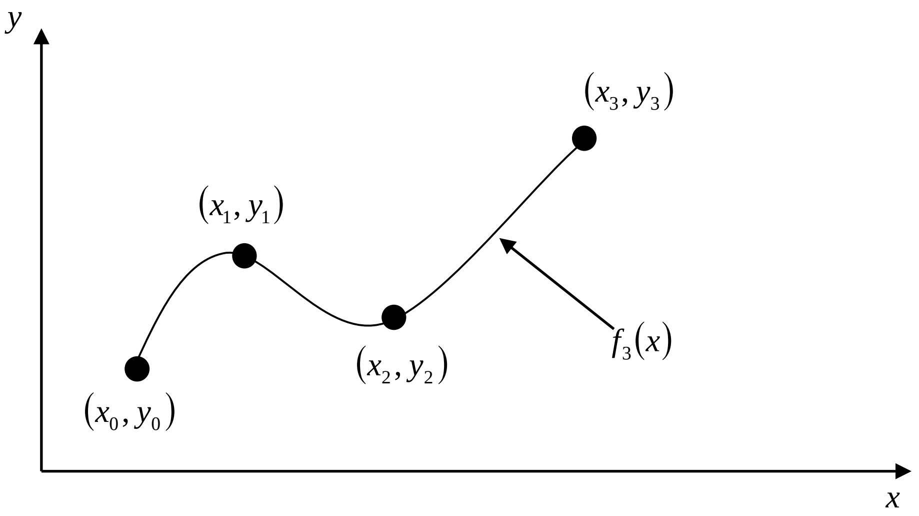
$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$



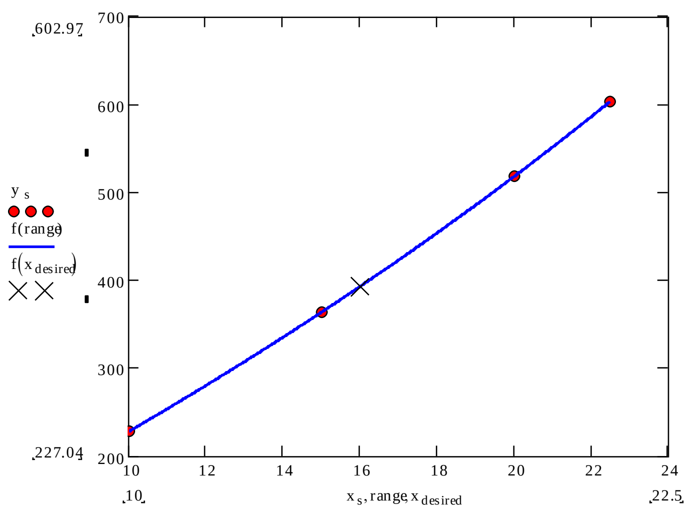
Cubic interpolation.

$a_0 = -4.2540 \quad a_1 = 21.266 \quad a_2 = 0.13204 \quad a_3 = 0.0054347$

Cubic Interpolation cont.

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$

$$v(16) = -4.2540 + 21.266(16) + 0.13204(16)^2 + 0.0054347(16)^3$$
$$= 392.06 \text{ m/s}$$



The absolute percentage relative approximate error $\left|\in_a\right|$ between second and third order polynomial is

$$\left|\in_a\right| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$
$$= 0.033269\%$$

Comparison

Table 4 Comparison of different orders of the polynomial.

Order of Polynomial	1	2	3
$v(t=16)$ m/s	393.7	392.19	392.06
Absolute Relative Approximate Error	----------	0.38410 %	0.033269 %

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$

$$= \int_{11}^{16} \left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)dt$$

$$= \left[-4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605 \text{ m}$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at t=16s given that

$$v(t) = -4.2540 + 21.266t + 0.13204^2 + 0.0054347t^3, 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt} v(t)$$

$$= \frac{d}{dt}\left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right)$$

$$= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \le t \le 22.5$$

$$a(16) = 21.266 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.665 \text{ m/s}^2$$

Lagrange Method of Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where 'n' in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ given at $(n+1)$ data points as $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ is a weighting function that includes a product of $(n-1)$ terms with terms of $j = i$ omitted.

Example

The upward velocity of a rocket is given as a function of time in the table .

1- Find the velocity at t=16 seconds using the Lagrang method for:

- linear interpolation
- **Quadratic interpolation**
- **Cubic interpolation**

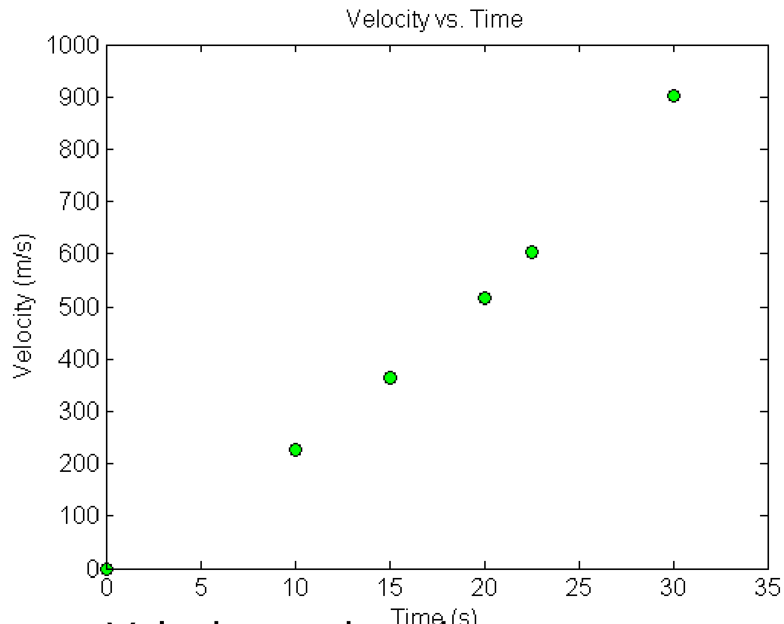
2- Find from cubic method:

a – The distance covered by the rocket from t=11s to t=16s.

b -The acceleration of the rocket at t=16s

Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Velocity vs. time data for the rocket example

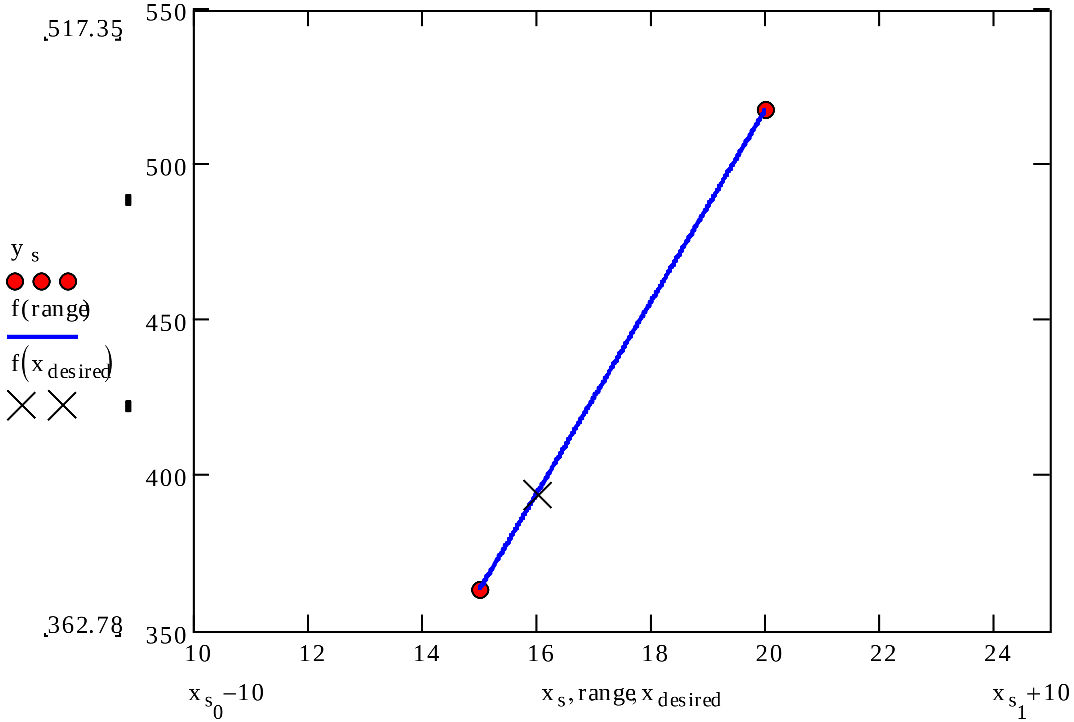
Linear Interpolation

$$v(t) = \sum_{i=0}^{1} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1)$$

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$



Linear Interpolation cont.

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20}(362.78) + \frac{16 - 15}{20 - 15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

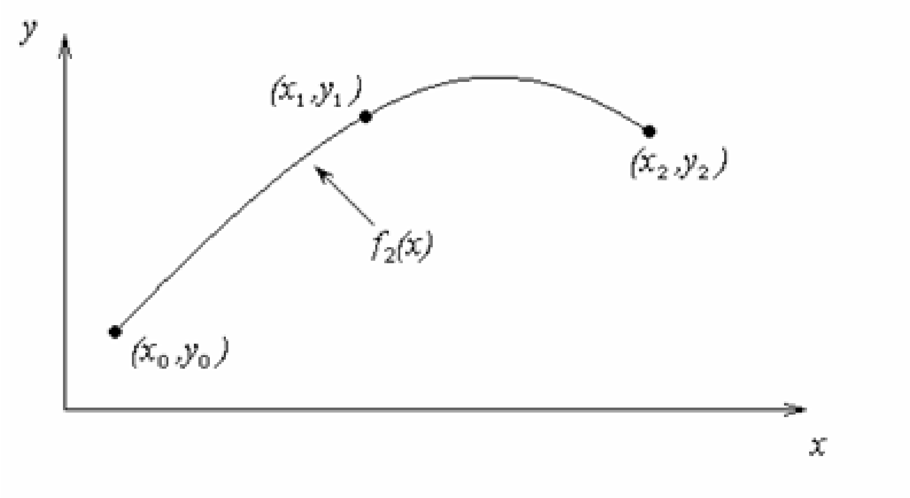
$$= 393.7 \text{ m/s.}$$

Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{2} L_i(t) v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Quadratic Interpolation

$$t_0 = 10, \; v(t_0) = 227.04$$

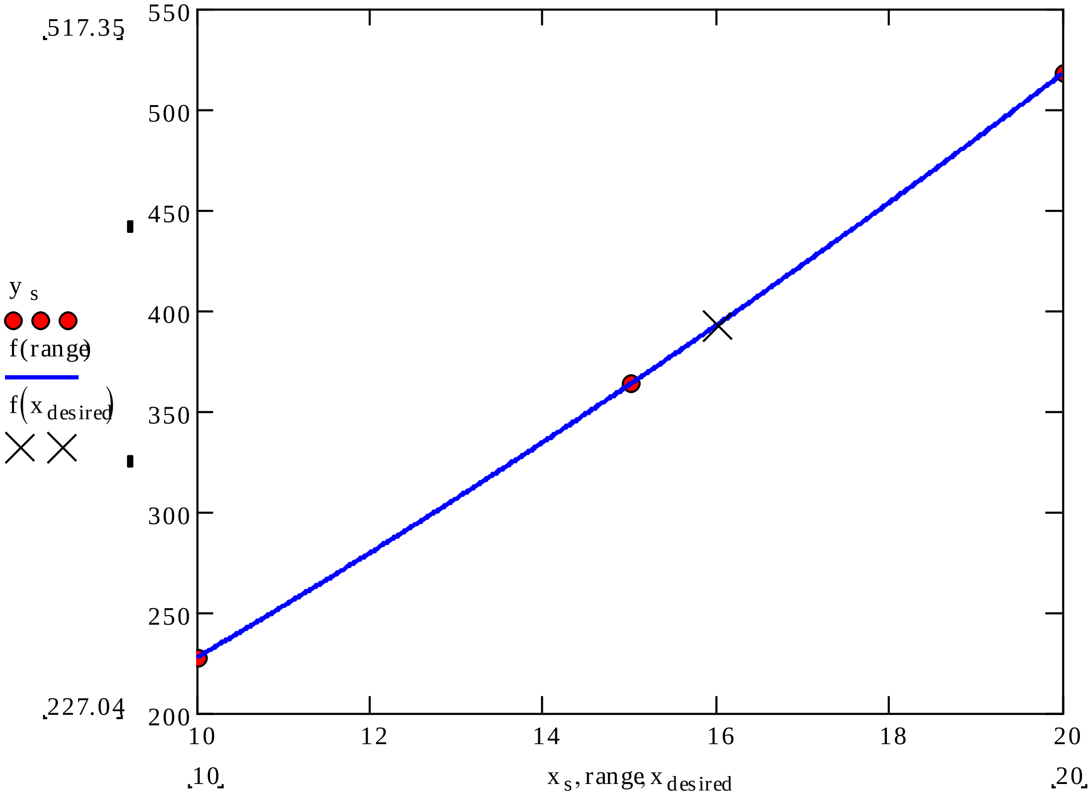
$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{2} \frac{t - t_j}{t_0 - t_j} = \left(\frac{t - t_1}{t_0 - t_1} \right) \left(\frac{t - t_2}{t_0 - t_2} \right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \left(\frac{t - t_0}{t_1 - t_0} \right) \left(\frac{t - t_2}{t_1 - t_2} \right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \left(\frac{t - t_0}{t_2 - t_0} \right) \left(\frac{t - t_1}{t_2 - t_1} \right)$$



Quadratic Interpolation (cont.)

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1)+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2)$$

$$\begin{aligned}v(16)&=\left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)(227.04)+\left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)(362.78)+\left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)(517.35)\\&=(-0.08)(227.04)+(0.96)(362.78)+(0.12)(527.35)\\&=392.19\text{ m/s}\end{aligned}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

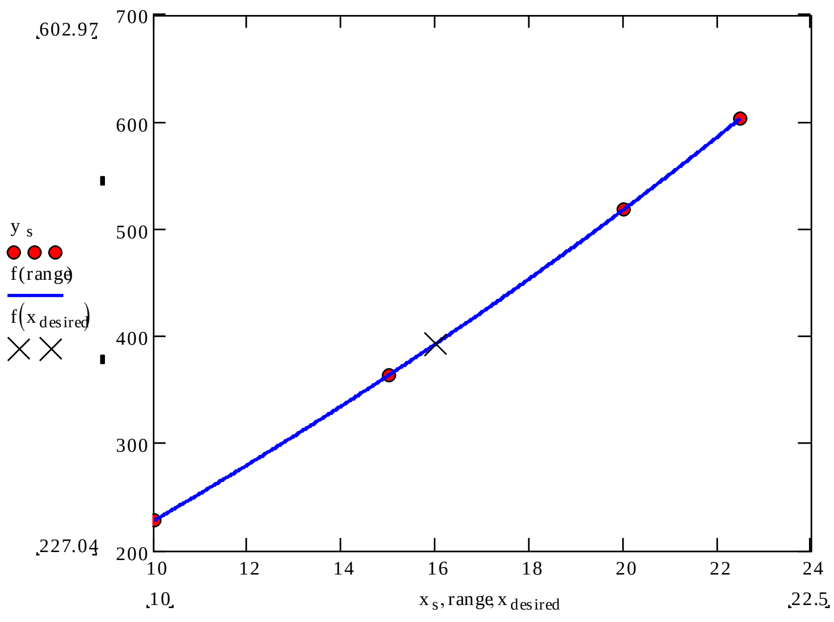
$$\begin{aligned}|\in_a|&=\left|\frac{392.19-393.70}{392.19}\right|\times 100\\&=0.38410\%\end{aligned}$$

Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^{3} L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$



Cubic Interpolation (cont.)

$$t_o = 10,\ v(t_o) = 227.04 \qquad t_1 = 15,\ v(t_1) = 362.78$$

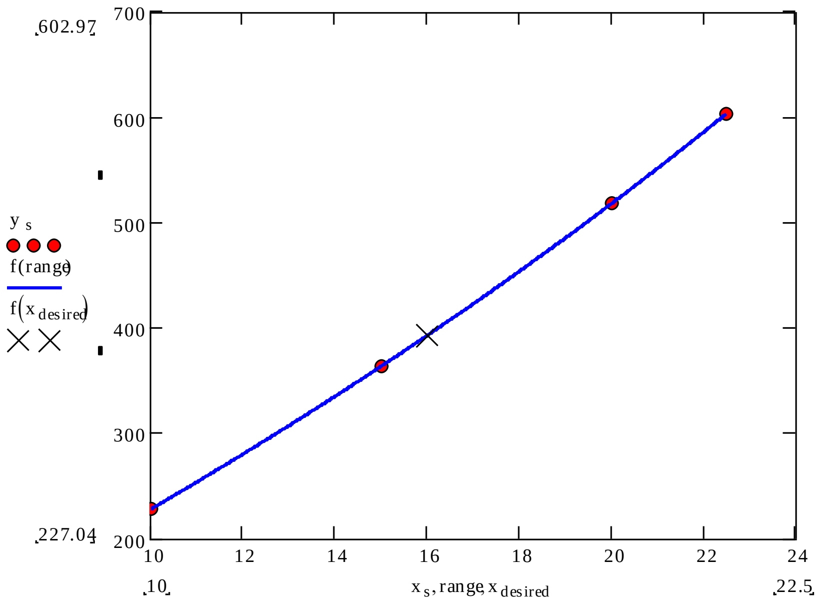
$$t_2 = 20,\ v(t_2) = 517.35 \qquad t_3 = 22.5,\ v(t_3) = 602.97$$

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{3} \frac{t - t_j}{t_0 - t_j} = \left(\frac{t - t_1}{t_0 - t_1}\right)\left(\frac{t - t_2}{t_0 - t_2}\right)\left(\frac{t - t_3}{t_0 - t_3}\right);$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{3} \frac{t - t_j}{t_1 - t_j} = \left(\frac{t - t_0}{t_1 - t_0}\right)\left(\frac{t - t_2}{t_1 - t_2}\right)\left(\frac{t - t_3}{t_1 - t_3}\right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{3} \frac{t - t_j}{t_2 - t_j} = \left(\frac{t - t_0}{t_2 - t_0}\right)\left(\frac{t - t_1}{t_2 - t_1}\right)\left(\frac{t - t_3}{t_2 - t_3}\right);$$

$$L_3(t) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{t - t_j}{t_3 - t_j} = \left(\frac{t - t_0}{t_3 - t_0}\right)\left(\frac{t - t_1}{t_3 - t_1}\right)\left(\frac{t - t_2}{t_3 - t_2}\right)$$



Cubic Interpolation (cont.)

$$v(t)=\left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_1)+\left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_2)$$

$$+\left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2)+\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3)$$

$$v(16)=\left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)\left(\frac{16-22.5}{10-22.5}\right)(227.04)+\left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)\left(\frac{16-22.5}{15-22.5}\right)(362.78)$$

$$+\left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)\left(\frac{16-22.5}{20-22.5}\right)(517.35)+\left(\frac{16-10}{22.5-10}\right)\left(\frac{16-15}{22.5-15}\right)\left(\frac{16-20}{22.5-20}\right)(602.97)$$

$$=(-0.0416)(227.04)+(0.832)(362.78)+(0.312)(517.35)+(-0.1024)(602.97)$$

$$=392.06\,\text{m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first and second order polynomial is

$$|\in_a|=\left|\frac{392.06-392.19}{392.06}\right|\times 100$$

$$=0.033269\%$$

Comparison

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	--------	0.38410%	0.033269%

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348)$$
$$+ (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$
$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \quad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$

$$\approx \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)dt$$

$$= [-4.245t + 21.265\frac{t^2}{2} + 0.13195\frac{t^3}{3} + 0.00544\frac{t^4}{4}]_{11}^{16}$$

$$= 1605 \text{ m}$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at t=16s given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \qquad 10 \le t \le 22.5$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3\right)$$

$$= 21.265 + 0.26390t + 0.01632t^2$$

$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$

$$= 29.665 \quad m/s^2$$

Spline Interpolation Method

Why Splines ?

$$f(x) = \frac{1}{1+25x^2}$$

Table : Six equidistantly spaced points in [-1, 1]

x	$y = \frac{1}{1+25x^2}$
-1.0	0.038461
-0.6	0.1
-0.2	0.5
0.2	0.5
0.6	0.1
1.0	0.038461

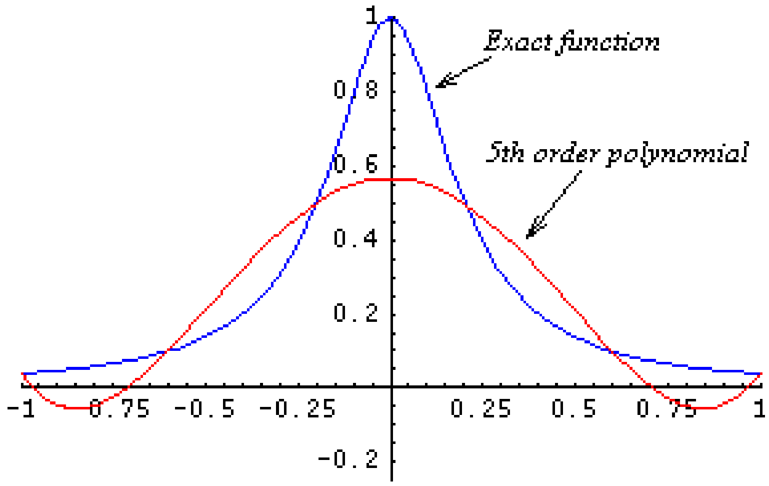
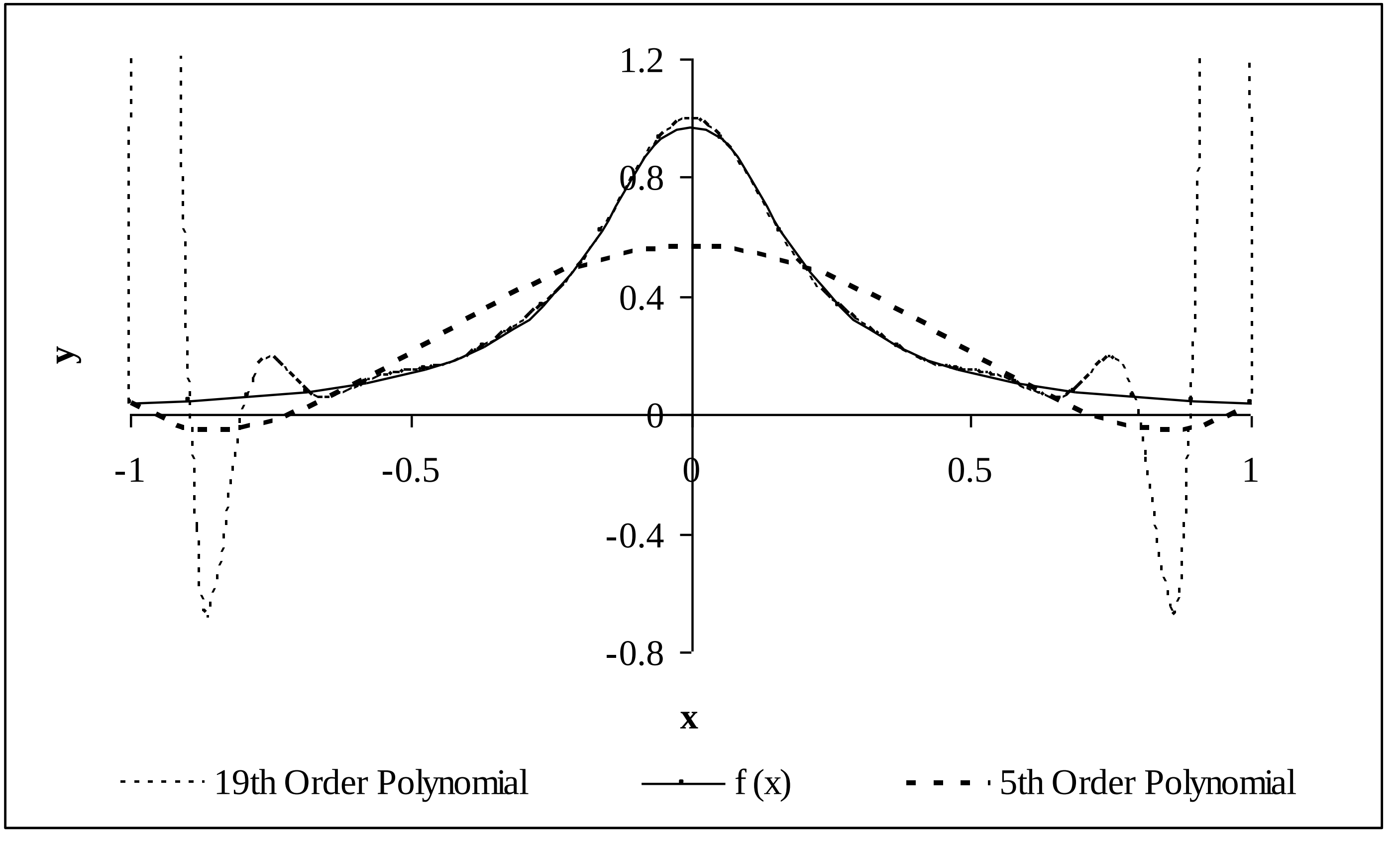


Figure : 5th order polynomial vs. exact function

Why Splines ?

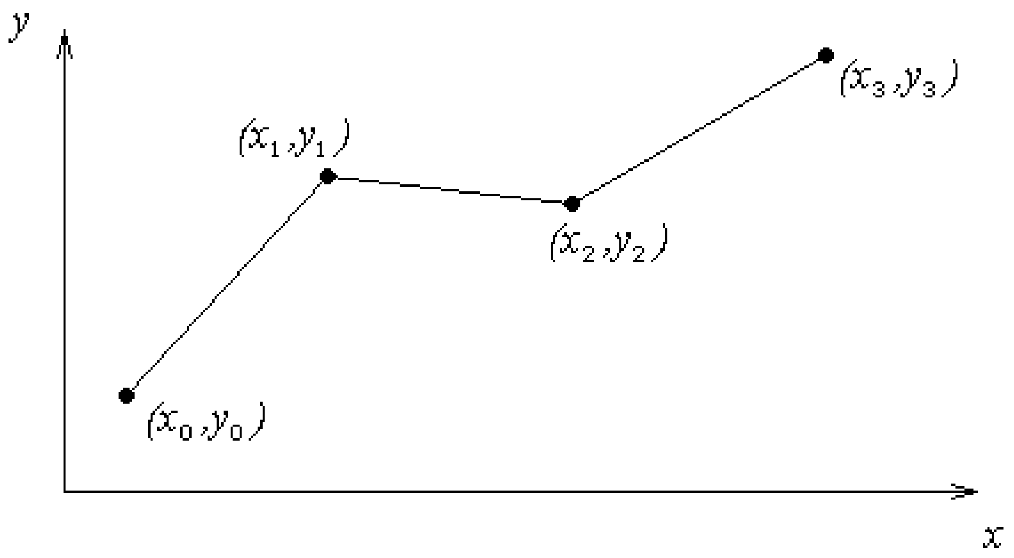


Higher order polynomial interpolation is a bad idea

Linear Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1})(x_n, y_n)$, fit linear splines to the data. This simply involves forming the consecutive data through straight lines. So if the above data is given in an ascending order, the linear splines are given by $\left(y_i = f(x_i)\right)$

Figure : Linear splines



Linear Interpolation (cont.)

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0), \qquad x_0 \le x \le x_1$$

$$= f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1), \qquad x_1 \le x \le x_2$$

$$.$$

$$.$$

$$.$$

$$= f(x_{n-1}) + \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}(x - x_{n-1}), \quad x_{n-1} \le x \le x_n$$

Note the terms of

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

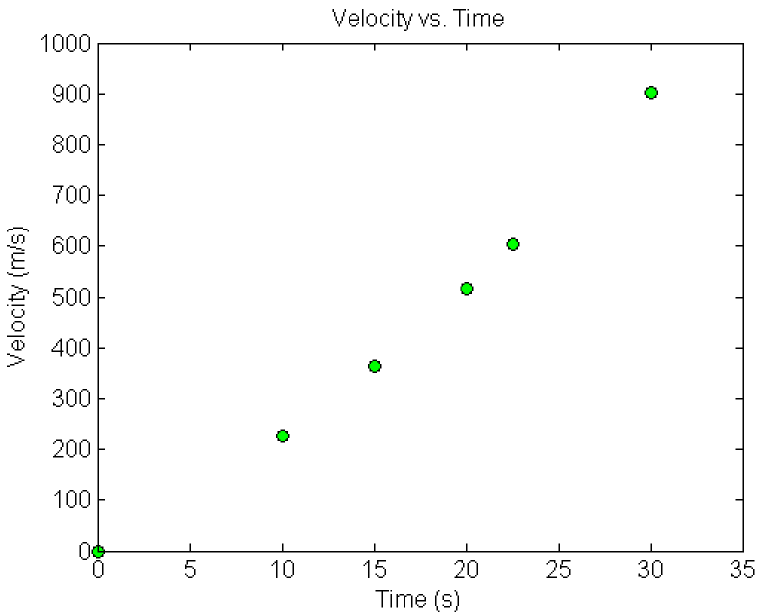
in the above function are simply slopes between x_{i-1} and x_i.

Example 1

The upward velocity of a rocket is given as a function of time in Table. Find the velocity at t=16 seconds using linear splines.

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Velocity vs. time data for the rocket example

Linear Interpolation

$t_0 = 15, \qquad v(t_0) = 362.78$

$t_1 = 20, \qquad v(t_1) = 517.35$

$$v(t) = v(t_0) + \frac{v(t_1) - v(t_0)}{t_1 - t_0}(t - t_0)$$

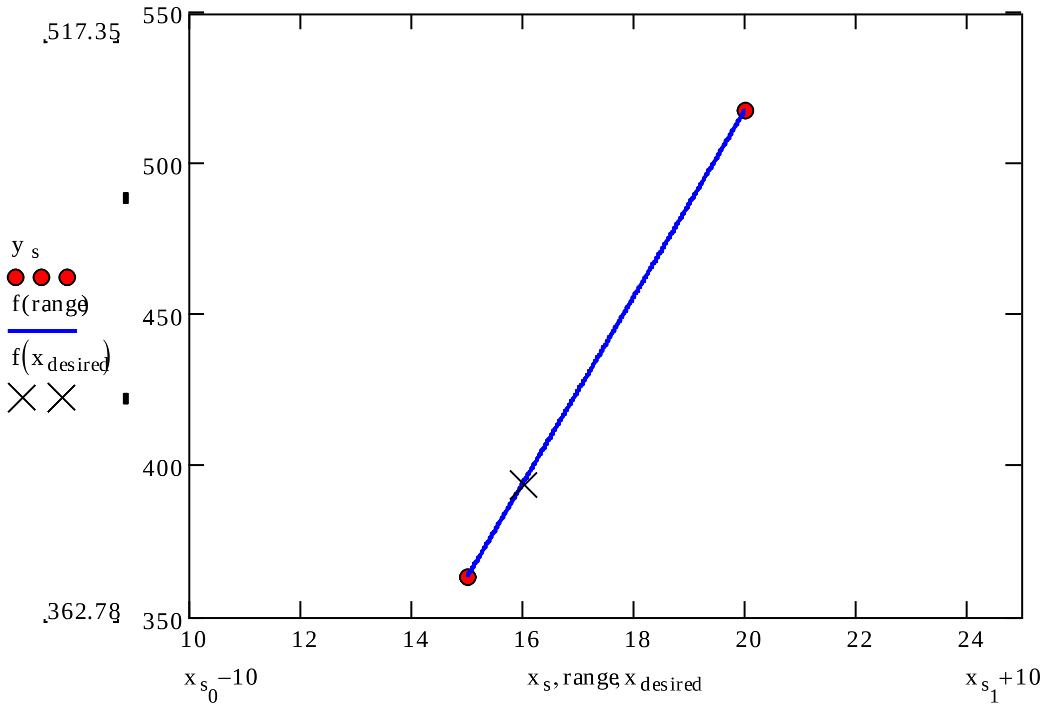
$$= 362.78 + \frac{517.35 - 362.78}{20 - 15}(t - 15)$$

$$v(t) = 362.78 + 30.913(t - 15)$$

At $t = 16$,

$$v(16) = 362.78 + 30.913(16 - 15)$$

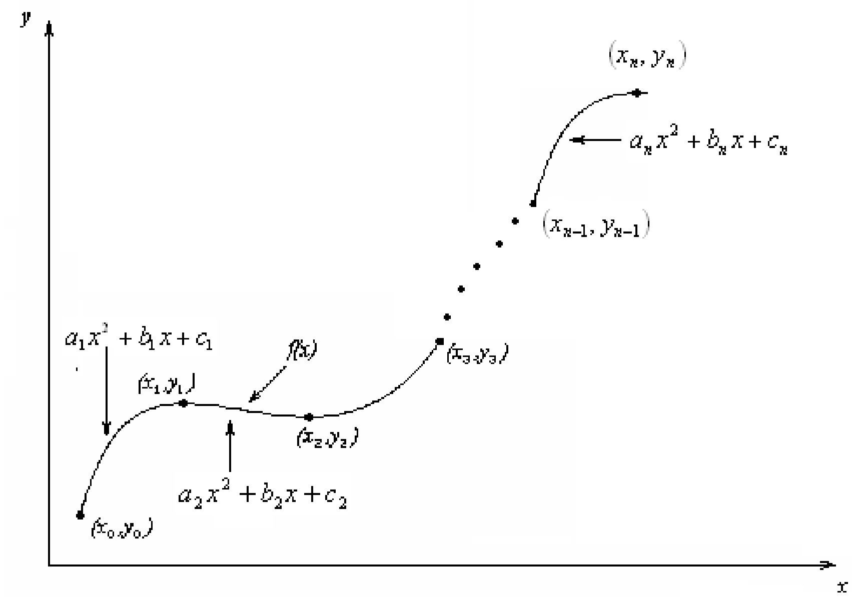
$$= 393.7 \text{ m/s}$$



Quadratic Interpolation

Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



Find a_i, b_i, c_i, $i = 1, 2, \ldots$, n

Quadratic Interpolation (cont.)

Each quadratic spline goes through two consecutive data points

$$a_1 x_0^{\ 2} + b_1 x_0 + c_1 = f(x_0)$$

$$a_1 x_1^{\ 2} + b_1 x_1 + c_1 = f(x_1)$$

.

.

.

$$a_i x_{i-1}^{\ 2} + b_i x_{i-1} + c_i = f(x_{i-1})$$

$$a_i x_i^{\ 2} + b_i x_i + c_i = f(x_i)$$

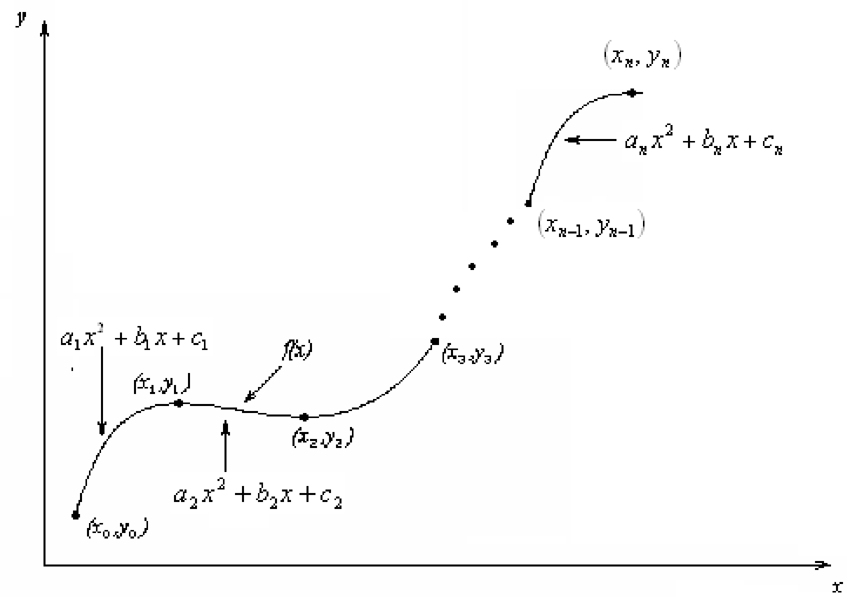
.

.

.

$$a_n x_{n-1}^{\ 2} + b_n x_{n-1} + c_n = f(x_{n-1})$$

$$a_n x_n^{\ 2} + b_n x_n + c_n = f(x_n)$$



This condition gives 2n equations

Quadratic Interpolation (cont.)

The first derivatives of two quadratic splines are continuous at the interior points.

For example, the derivative of the first spline

$$a_1 x^2 + b_1 x + c_1 \quad \text{is} \qquad 2a_1 x + b_1$$

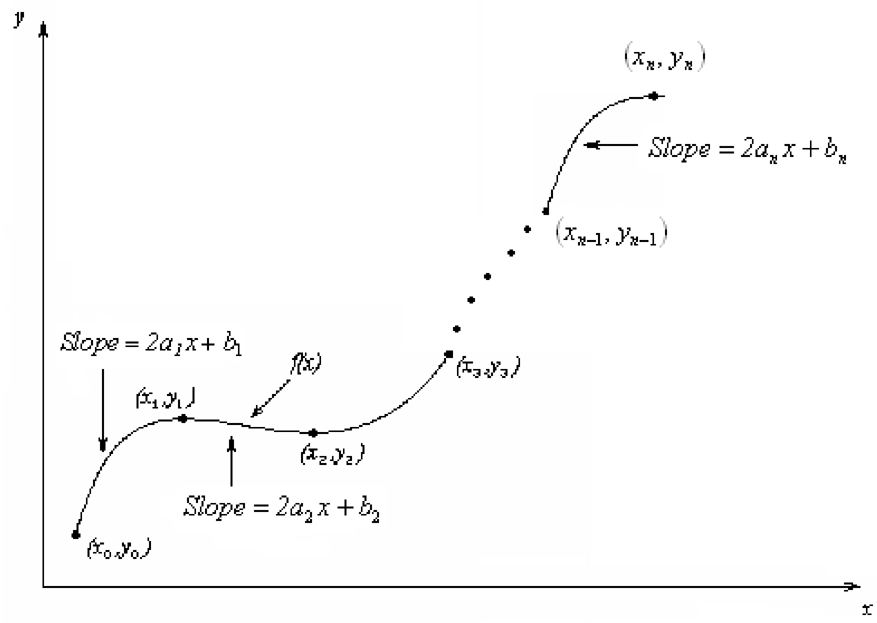
The derivative of the second spline

$$a_2 x^2 + b_2 x + c_2 \quad \text{is} \qquad 2a_2 x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1 x_1 + b_1 = 2a_2 x_1 + b_2$$

$$2a_1 x_1 + b_1 - 2a_2 x_1 - b_2 = 0$$



Quadratic Interpolation (cont.)

Similarly at the other interior points,

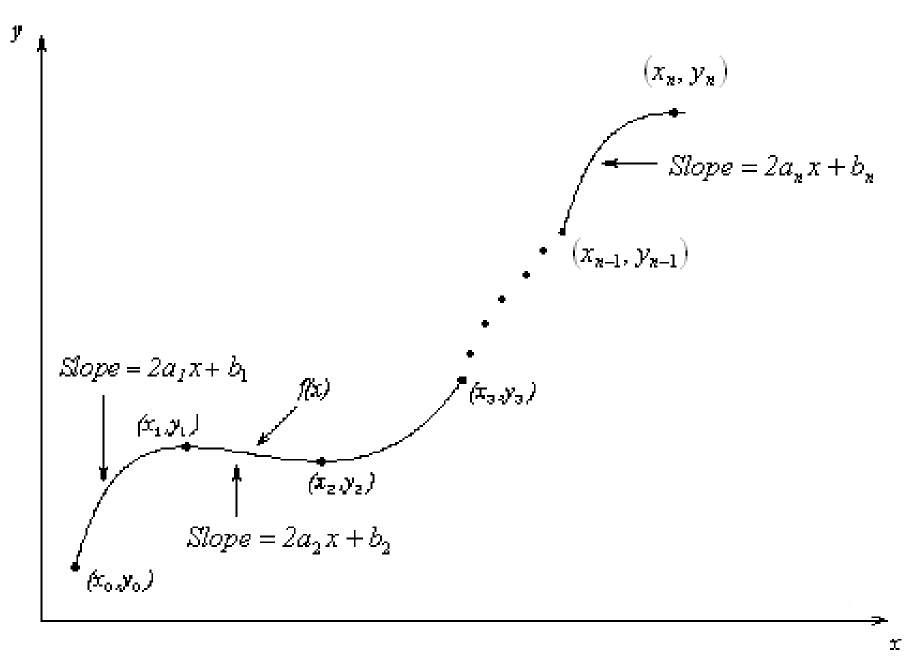
$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$



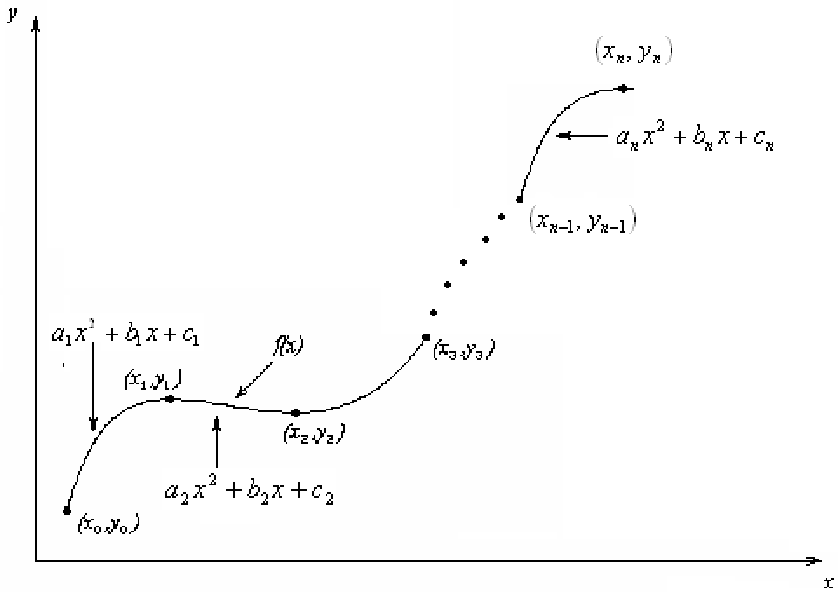
We have (n-1) such equations. The total number of equations is $(2n) + (n-1) = (3n-1)$.

We can assume that the first spline is linear, that is $a_1 = 0$

Quadratic Interpolation (cont.)

This gives us '3n' equations and '3n' unknowns. Once we find the '3n' constants, we can find the function at any value of 'x' using the splines,

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$



Example 2 Quadratic Spline

The upward velocity of a rocket is given as a function of time. Using quadratic splines

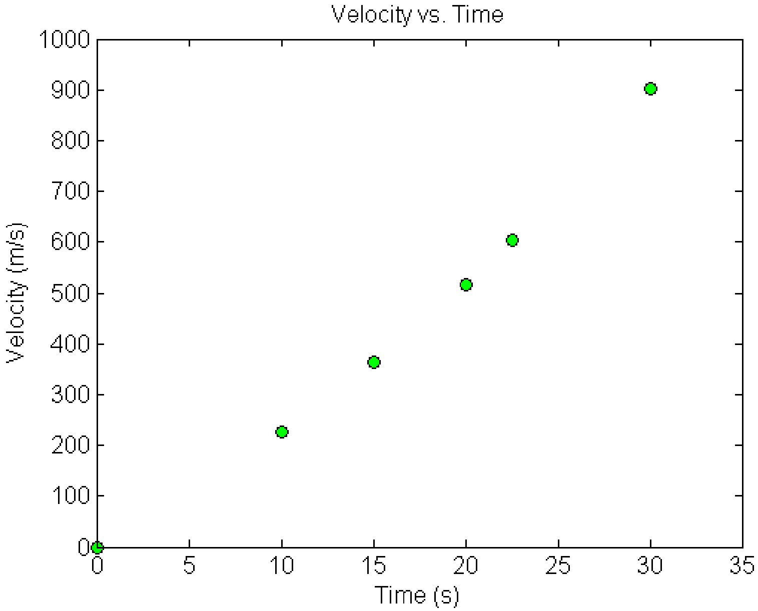
a) Find the velocity at t=16 seconds

b) Find the acceleration at t=16 seconds

c) Find the distance covered between t=11 and t=16 seconds

Table Velocity as a function of time

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Velocity vs. time data for the rocket example

Solution

$$
\begin{aligned}
v(t) &= a_1 t^2 + b_1 t + c_1, & 0 \le t \le 10 \\
&= a_2 t^2 + b_2 t + c_2, & 10 \le t \le 15 \\
&= a_3 t^2 + b_3 t + c_3, & 15 \le t \le 20 \\
&= a_4 t^2 + b_4 t + c_4, & 20 \le t \le 22.5 \\
&= a_5 t^2 + b_5 t + c_5, & 22.5 \le t \le 30
\end{aligned}
$$

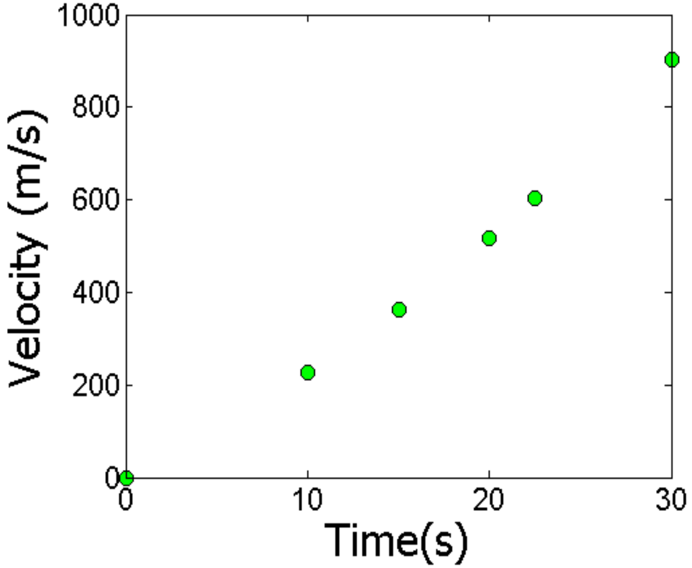
Let us set up the equations

Each Spline Goes Through Two Consecutive Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10$$

$$a_1(0)^2 + b_1(0) + c_1 = 0$$

$$a_1(10)^2 + b_1(10) + c_1 = 227.04$$



Each Spline Goes Through Two Consecutive Data Points

t	v(t)
s	**m/s**
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

$$a_2(10)^2 + b_2(10) + c_2 = 227.04$$

$$a_2(15)^2 + b_2(15) + c_2 = 362.78$$

$$a_3(15)^2 + b_3(15) + c_3 = 362.78$$

$$a_3(20)^2 + b_3(20) + c_3 = 517.35$$

$$a_4(20)^2 + b_4(20) + c_4 = 517.35$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97$$

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67$$

Derivatives are Continuous at Interior Data Points

$$v(t) = a_1 t^2 + b_1 t + c_1,\ 0 \le t \le 10$$
$$= a_2 t^2 + b_2 t + c_2, 10 \le t \le 15$$

$$\frac{d}{dt}\left(a_1 t^2 + b_1 t + c_1\right)\Big|_{t=10} = \frac{d}{dt}\left(a_2 t^2 + b_2 t + c_2\right)\Big|_{t=10}$$

$$\left(2a_1 t + b_1\right)\Big|_{t=10} = \left(2a_2 t + b_2\right)\Big|_{t=10}$$

$$2a_1(10) + b_1 = 2a_2(10) + b_2$$

$$20a_1 + b_1 - 20a_2 - b_2 = 0$$

Derivatives are continuous at Interior Data Points

At t=10

$$2a_1(10)+b_1-2a_2(10)-b_2=0$$

At t=15

$$2a_2(15)+b_2-2a_3(15)-b_3=0$$

At t=20

$$2a_3(20)+b_3-2a_4(20)-b_4=0$$

At t=22.5

$$2a_4(22.5)+b_4-2a_5(22.5)-b_5=0$$

$$a_1=0$$

Final Set of Equations

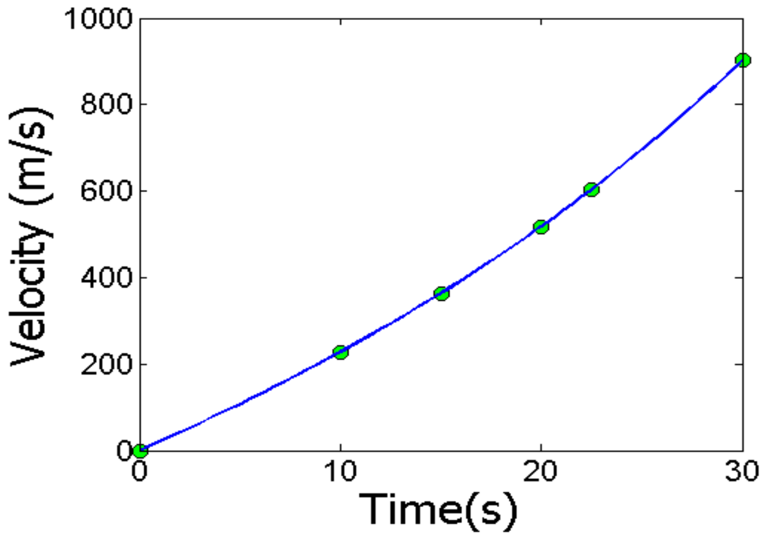
$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\ 20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Coefficients of Spline

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	−0.1356	35.66	−141.61
4	1.6048	−33.956	554.55
5	0.20889	28.86	−152.13

Final Solution

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$



Velocity at a Particular Point

a) Velocity at t=16

$$
\begin{aligned}
v(t) &= 22.704t, & 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, & 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, & 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, & 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, & 22.5 \le t \le 30
\end{aligned}
$$

$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24 \text{ m/s}
\end{aligned}
$$

Acceleration from Velocity Profile

b) The quadratic spline valid at t=16 is given by

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

$$v(t) = -0.1356t^2 + 35.66t - 141.61, \quad 15 \le t \le 20$$

$$a(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66, \quad 15 \le t \le 20$$

$$a(16) = -0.2712(16) + 35.66 = 31.321 \text{m/s}^2$$

Distance from Velocity Profile

c) Find the distance covered by the rocket from t=11s to t=16s.

$$S(16)-S(11)=\int_{11}^{16} v(t)dt$$

$$v(t)=0.8888t^2+4.928t+88.88,\ 10\le t\le 15$$

$$=-0.1356t^2+35.66t-141.61,\ 15\le t\le 20$$

$$S(16)-S(11)=\int_{11}^{16} v(t)dt=\int_{11}^{15} v(t)dt+\int_{15}^{16} v(t)dt$$

$$=\int_{11}^{15}\left(0.8888t^2+4.928t+88.88\right)dt+\int_{15}^{16}\left(-0.1356t^2+35.66t-141.61\right)dt$$

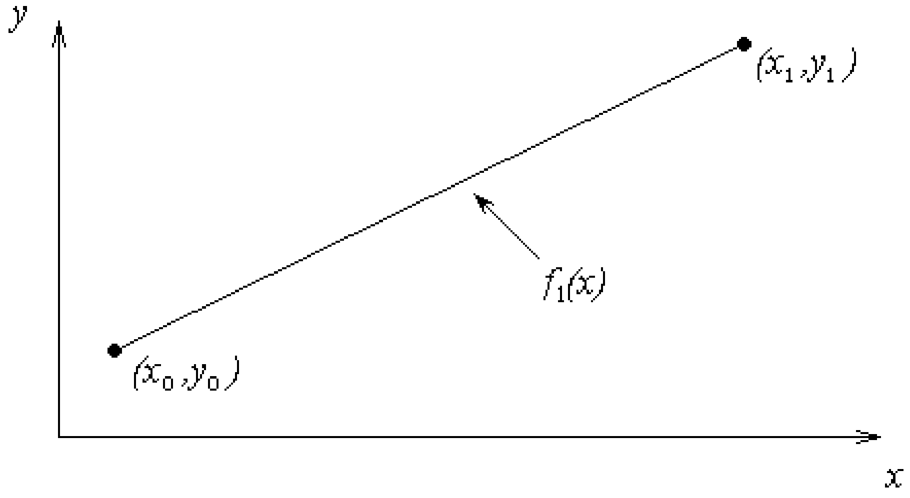
$$=1595.9\text{ m}$$

Newton’s Divided Difference Polynomial Method of Interpolation

Newton's Divided Difference Method

<u>Linear interpolation</u>: Given $(x_0, y_0), \ (x_1, y_1),$ pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$



where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

General Form

Given $(n+1)$ data points, $(x_0, y_0),(x_1, y_1),......,(x_{n-1}, y_{n-1}),(x_n, y_n)$ as

$$f_n(x) = b_0 + b_1(x - x_0) + ...+ b_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

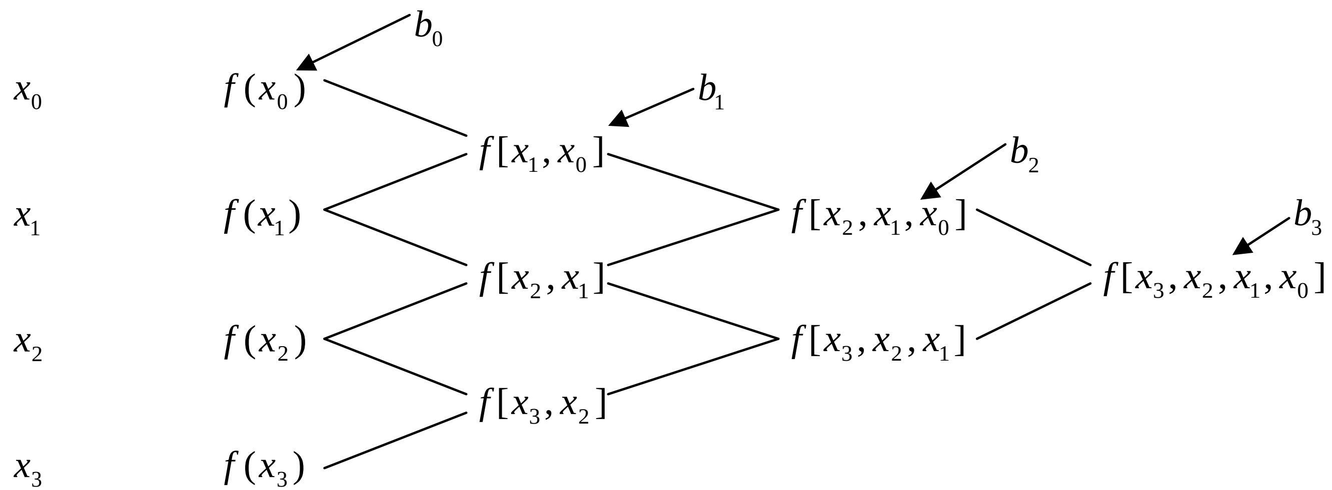
$$b_{n-1} = f[x_{n-1}, x_{n-2}, ..., x_0]$$

$$b_n = f[x_n, x_{n-1}, ..., x_0]$$

General form

The third order polynomial, given (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$
$$+ f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



Example

The upward velocity of a rocket is given as a function of time in the table .

1- **Find** the velocity at t=16 seconds using Newton's Divided Method for:

- **linear interpolation**
- **Quadratic interpolation**
- **Cubic interpolation**

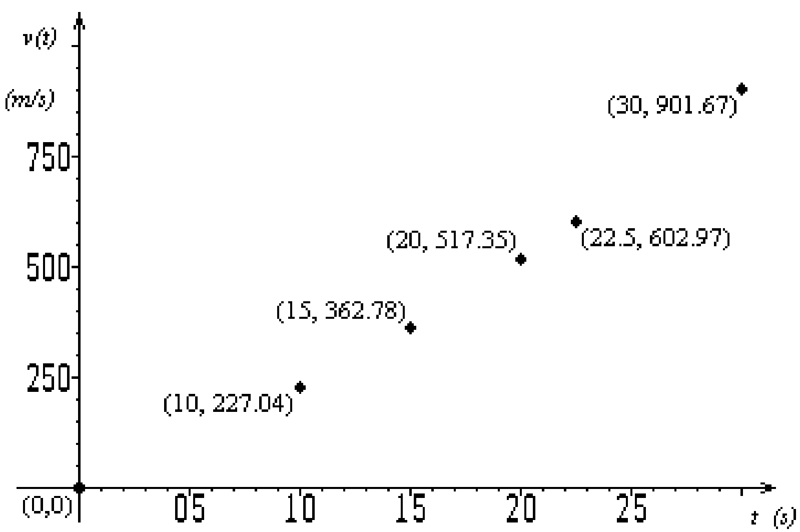
2- **Find** from cubic interpolation:

a –The distance covered by the rocket from t=11s to t=16s.

b - The acceleration of the rocket at t=16s

Velocity as a function of time

(s)	(m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Velocity vs. time data for the rocket example

Linear Interpolation

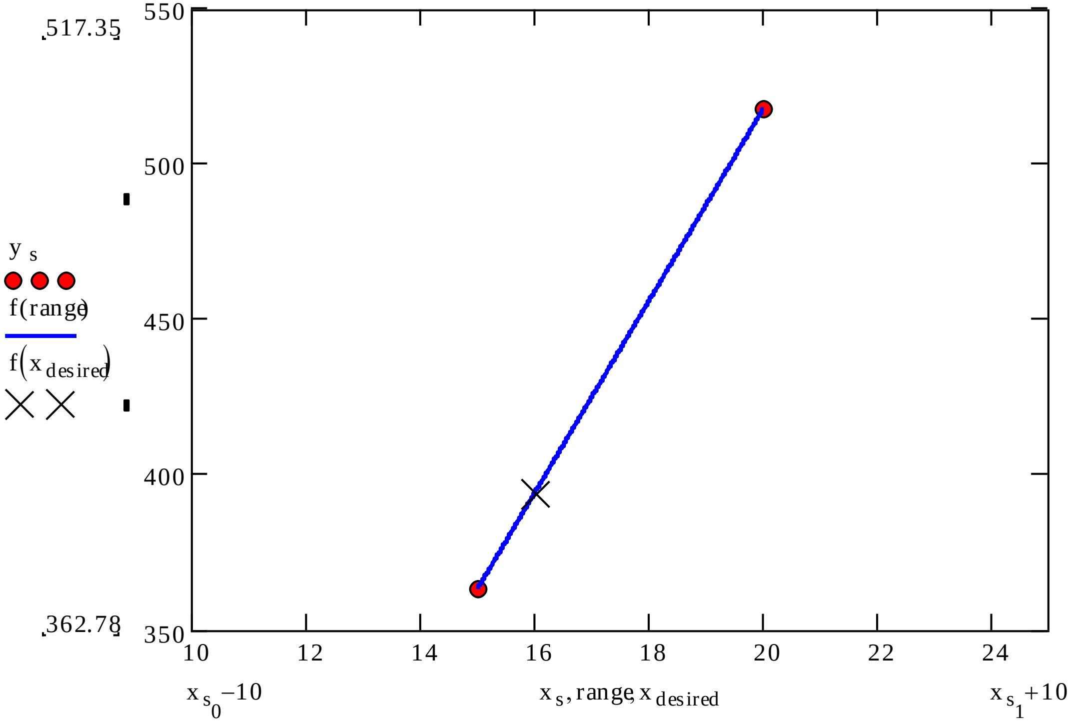
$$v(t) = b_0 + b_1(t - t_0)$$

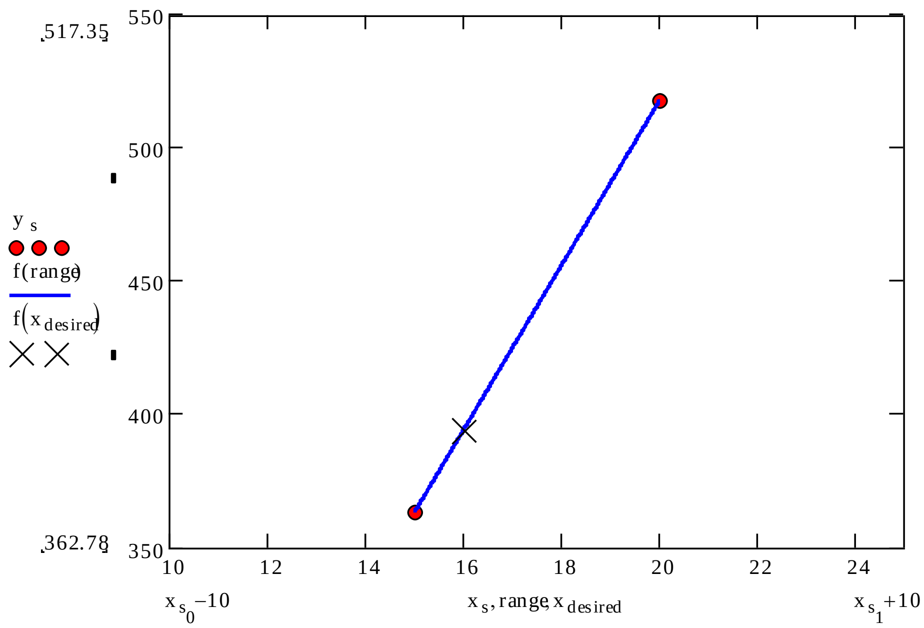
$t_0 = 15,\ v(t_0) = 362.78$

$t_1 = 20,\ v(t_1) = 517.35$

$b_0 = v(t_0) = 362.78$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$





$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15),\ 15 \le t \le 20$$

At $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

Quadratic Interpolation

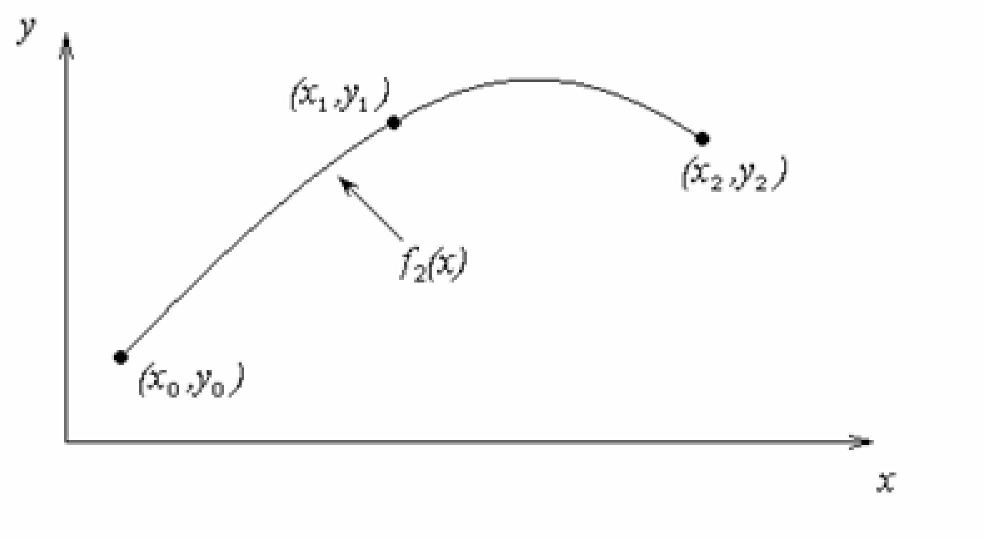
Given (x_0, y_0), (x_1, y_1), and (x_2, y_2), fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

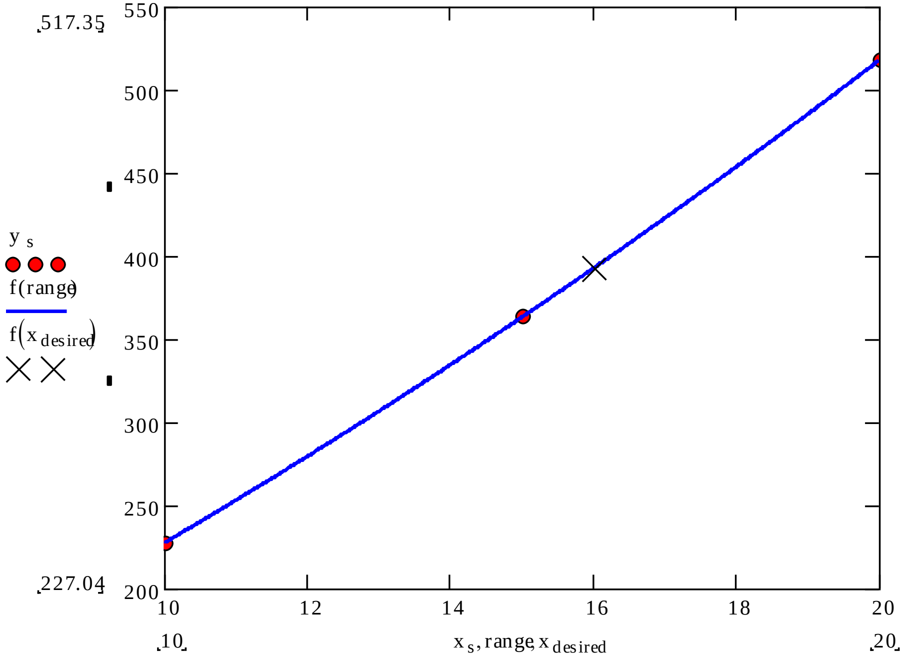
$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



Quadratic Interpolation (cont.)



$$t_0 = 10, \; v(t_0) = 227.04$$

$$t_1 = 15, \; v(t_1) = 362.78$$

$$t_2 = 20, \; v(t_2) = 517.35$$

Quadratic Interpolation (cont.)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\dfrac{v(t_2) - v(t_1)}{t_2 - t_1} - \dfrac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\dfrac{517.35 - 362.78}{20 - 15} - \dfrac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

Quadratic Interpolation (cont.)

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \qquad 10 \le t \le 20$$

At $t = 16$,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained between the results from the first order and second order polynomial is

$$|\in_a| = \left|\frac{392.19 - 393.69}{392.19}\right| \text{x}100$$

$$= 0.38502\ \%$$

Cubic Interpolation

The velocity profile is chosen as

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

we need to choose four data points that are closest to *t=16*

$$t_0 = 10, \quad v(t_0) = 227.04$$
$$t_1 = 15, \quad v(t_1) = 362.78$$
$$t_2 = 20, \quad v(t_2) = 517.35$$
$$t_3 = 22.5, \quad v(t_3) = 602.97$$

The values of the constants are found as:

$b_0 = 227.04$; $b_1 = 27.148$; $b_2 = 0.37660$; $b_3 = 5.4347*10^{-3}$

Cubic Interpolation (cont.)

$$
\begin{array}{lcccc}
 & b_0 & & & \\
t_0 = 10 & 227.04 & & & \\
 & & 27.148 \; (b_1) & & \\
t_1 = 15, & 362.78 & & 0.37660 \; (b_2) & \\
 & & 30.914 & & 5.4347x10^{-3} \; (b_3) \\
t_2 = 20, & 517.35 & & 0.44453 & \\
 & & 34.248 & & \\
t_3 = 22.5, & 602.97 & & &
\end{array}
$$

$b_0 = 227.04$; $b_1 = 27.148$; $b_2 = 0.37660$; $b_3 = 5.4347*10^{-3}$

Cubic Interpolation (cont.)

Hence

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$
$$= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15)$$
$$+ 5.4347 * 10^{-3}(t - 10)(t - 15)(t - 20)$$

At $t = 16,$

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15)$$
$$+ 5.4347 * 10^{-3}(16 - 10)(16 - 15)(16 - 20)$$
$$= 392.06 \text{ m/s}$$

The absolute relative approximate error $|\in_a|$ obtained is

$$|\in_a| = \left|\frac{392.06 - 392.19}{392.06}\right| x100$$

$$= 0.033427\ \%$$

Comparison Table

Order of Polynomial	1	2	3
v(t=16) m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	----------	0.38502 %	0.033427 %

Distance from Velocity Profile

Find the distance covered by the rocket from t=11s to t=16s ?

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) + 5.4347*10^{-3}(t-10)(t-15)(t-20) \qquad 10 \le t \le 22.5$$

So

$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3 \qquad 10 \le t \le 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)\,dt$$

$$= \left[-4.2541t + 21.265\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4} \right]_{11}^{16}$$

$$= 1605\, m$$

Acceleration from Velocity Profile

Find the acceleration of the rocket at *t=16*s given that

$$v(t) = -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}\left(-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3\right)$$

$$= 21.265 + 0.26408t + 0.016304t^2$$

$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.664\, m / s^2$$