

**SIGNALS AND SYSTEMS USING MATLAB**  
**Chapter 4 — Frequency Analysis: The Fourier Series**

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# Eigenfunctions

$x(t) = e^{j\Omega_0 t}$ ,  $-\infty < t < \infty$ , input to a causal and stable LTI system

steady state output  $y(t) = e^{j\Omega_0 t} H(j\Omega_0)$

$$H(j\Omega_0) = \int_0^{\infty} h(\tau) e^{-j\Omega_0 \tau} d\tau = H(s)|_{s=j\Omega_0}$$

frequency response at  $\Omega_0$

$x(t) = e^{j\Omega_0 t}$  is eigenfunction of LTI system

**Example:** RC circuit, voltage source be  $v_s(t) = 4 \cos(t + \pi/4)$ ,  $R = 1 \Omega$ ,  
 $C = 1\text{F}$

transfer function  $H(s) = \frac{V_c(s)}{V_s(s)} = \frac{1}{s+1}$

$$H(j1) = \frac{\sqrt{2}}{2} \angle -\pi/4 \text{ frequency response at } \Omega_0 = 1$$

steady-state output  $v_c(t) = 4|H(j1)| \cos(t + \pi/4 + \angle H(j1)) = 2\sqrt{2} \cos(t)$

Example: Low-pass filter using RC circuit

Input  $v_s(t) = 1 + \cos(10,000t)$  to series RC circuit ( $R = C = 1$ )

$$v_s(t) = v_c(t) + \frac{dv_c(t)}{dt}$$

if input  $v_s(t) = e^{j\Omega t}$  output  $v_c(t) = e^{j\Omega t} H(j\Omega)$ , then in o.d.e.

$$e^{j\Omega t} = e^{j\Omega t} H(j\Omega)(1 + j\Omega) \Rightarrow H(j\Omega) = \frac{1}{1 + j\Omega} = \frac{1}{\sqrt{1 + \Omega^2}}$$

$$v_s(t) = \cos(0t) + \cos(10,000t)$$

$$v_c(t) \approx 1 + \frac{1}{10,000} \cos(10,000t - \pi/2) \approx 1$$

attenuates higher frequency component (i.e., low-pass filter)

# Complex exponential Fourier series

**Fourier Series** of periodic signal  $x(t)$ , of fundamental period  $T_0$ , is infinite sum of **ortho-normal** complex exponentials of frequencies multiples of **fundamental frequency**  $\Omega_0 = 2\pi/T_0$  (rad/sec) of  $x(t)$ :

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$
$$\text{FS coefficients } X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt$$

$\{e^{jk\Omega_0 t}\}$  are **ortho-normal** Fourier basis

$$\begin{aligned} \frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{jk\Omega_0 t} [e^{j\ell\Omega_0 t}]^* dt &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} e^{j(k-\ell)\Omega_0 t} dt \\ &= \begin{cases} 0 & k \neq \ell \text{ orthogonal} \\ 1 & k = \ell \text{ normal} \end{cases} \end{aligned}$$

# Line spectrum

- Parseval's power relation

$P_x$  : power of periodic signal  $x(t)$  of fundamental period  $T_0$

$$P_x = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X_k|^2, \quad \text{for any } t_0$$

- Periodic  $x(t)$  is represented in frequency by

- Magnitude line spectrum  $|X_k|$  vs  $k\Omega_0$
- Phase line spectrum  $\angle X_k$  vs  $k\Omega_0$
- Power line spectrum  $|X_k|^2$  vs  $k\Omega_0$
- Real-valued periodic signal  $x(t)$ , of fundamental period  $T_0$ ,

$X_k = X_{-k}^*$  or equivalently

(i)  $|X_k| = |X_{-k}|$ , i.e., magnitude  $|X_k|$  is even function of  $k\Omega_0$ .

(ii)  $\angle X_k = -\angle X_{-k}$ , i.e., phase  $\angle X_k$  is odd function of  $k\Omega_0$

# Trigonometric Fourier series

Real-valued, periodic signal  $x(t)$ , of fundamental period  $T_0$ ,

$$\begin{aligned} x(t) &= \underbrace{X_0}_{\text{dc-component}} + 2 \sum_{k=1}^{\infty} \underbrace{|X_k| \cos(k\Omega_0 t + \theta_k)}_{k^{\text{th}} \text{ harmonic}} \\ &= c_0 + 2 \sum_{k=1}^{\infty} [c_k \cos(k\Omega_0 t) + d_k \sin(k\Omega_0 t)] \quad \Omega_0 = \frac{2\pi}{T_0} \end{aligned}$$

Fourier coefficients  $\{c_k, d_k\}$

$$\begin{aligned} c_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \cos(k\Omega_0 t) dt \quad k = 0, 1, \dots \\ d_k &= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) \sin(k\Omega_0 t) dt \quad k = 1, 2, \dots \end{aligned}$$

Sinusoidal basis functions  $\{\sqrt{2} \cos(k\Omega_0 t), \sqrt{2} \sin(k\Omega_0 t)\}$ ,  $k = 0, \pm 1, \dots$ , are orthonormal in  $[0, T_0]$

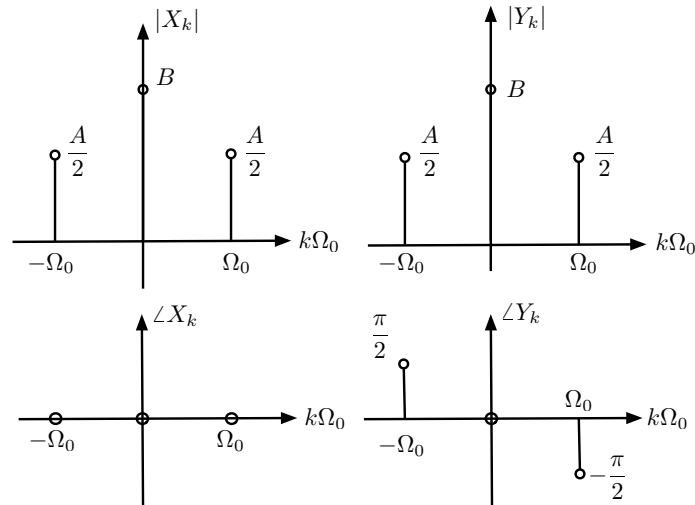
Example:  $x(t) = B + A \cos(\Omega_0 t + \theta)$  periodic of fundamental period  $T_0$

trigonometric Fourier series:  $X_0 = B$ ;  $|X_1| = A/2$ ,  $\angle X_1 = \theta$

exponential Fourier series:

$$x(t) = B + \frac{A}{2} \left[ e^{j(\Omega_0 t + \theta)} + e^{-j(\Omega_0 t + \theta)} \right]$$

$$X_0 = B, \quad X_1 = \frac{Ae^{j\theta}}{2}, \quad X_{-1} = X_1^* = \frac{Ae^{-j\theta}}{2}$$



*Line spectrum of  $x(t) = B + A \cos(\Omega_0 t)$  and of  $y(t) = B + A \sin(\Omega_0 t)$  (right).*

## Fourier coefficients from Laplace

$x(t)$ , periodic of fundamental period  $T_0$

period:  $x_1(t) = x(t)[u(t - t_0) - u(t - t_0 - T_0)]$ , any  $t_0$

$$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]_{s=jk\Omega_0} \quad \Omega_0 = \frac{2\pi}{T_0} \text{ (fundamental frequency), } k = 0, \pm 1, \dots$$

Example:  $x(t)$  periodic,  $T_0 = 2$ ,  $x_1(t) = u(t) - u(t - 1)$

$$x(t) = \sum_{m=-\infty}^{\infty} x_1(t - 2m) = \sum_{k=-\infty}^{\infty} X_k e^{jk\pi t}$$

$$X_k = \frac{1}{2} \mathcal{L}[x_1(t)]_{s=jk\pi} = \frac{1 - e^{-jk\pi}}{jk\pi} = e^{-jk\pi/2} \frac{\sin(k\pi/2)}{k\pi/2}$$



## Reflection and even and odd periodic signals

$x(t)$  periodic of fundamental period  $T_0$ , Fourier coefficients  $\{X_k\}$

- Reflection: Fourier coefficients of  $x(-t)$  are  $\{X_{-k}\}$
- Even  $x(t)$ :  $\{X_k\}$  are real

$$x(t) = X_0 + 2 \sum_{k=1}^{\infty} X_k \cos(k\Omega_0 t)$$

Odd  $x(t)$ :  $\{X_k\}$  are imaginary

$$x(t) = 2 \sum_{k=1}^{\infty} jX_k \sin(k\Omega_0 t)$$

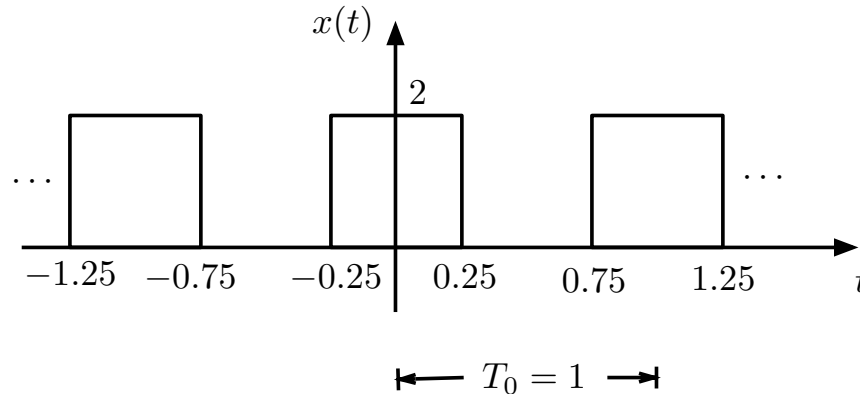
- Any periodic signal  $x(t)$  then  $x(t) = x_e(t) + x_o(t)$ ,  $x_e(t)$  and  $x_o(t)$  even and odd components

$$X_k = X_{ek} + X_{ok}$$

$$X_{ek} = 0.5[X_k + X_{-k}]$$

$$X_{ok} = 0.5[X_k - X_{-k}]$$

Example: periodic pulse train  $x(t)$ , of fundamental period  $T_0 = 1$



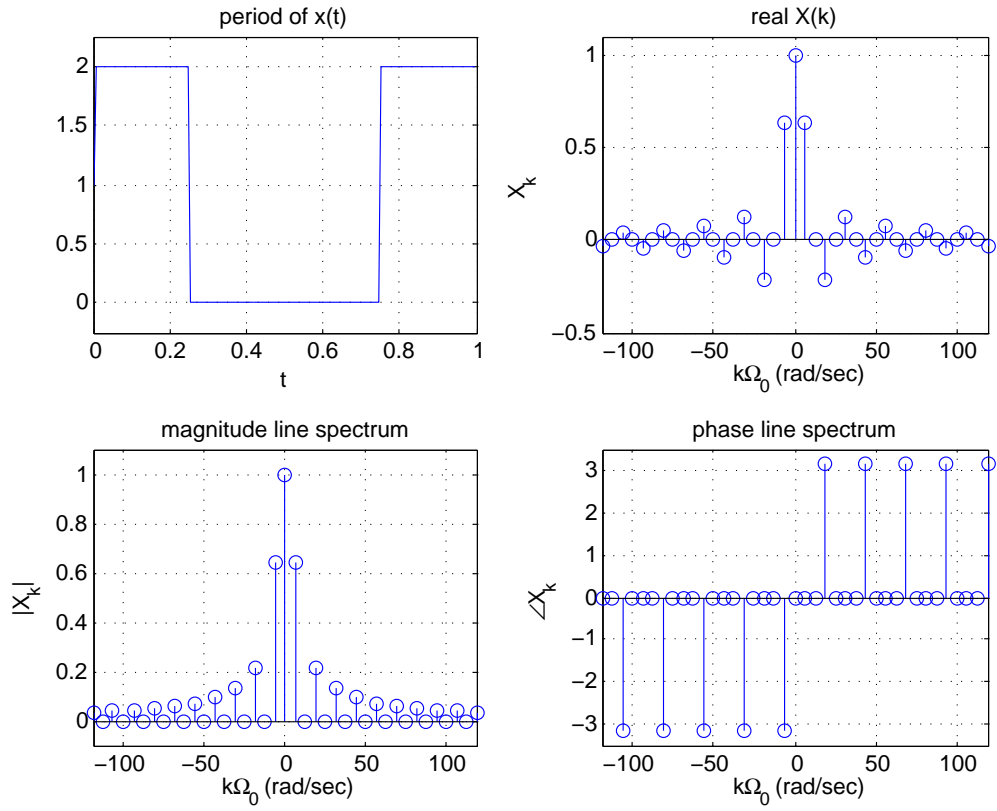
Integral formula: 
$$X_k = \frac{1}{T_0} \int_{-T_0/4}^{3T_0/4} x(t) e^{-j\Omega_0 kt} dt = \frac{\sin(\pi k/2)}{(\pi k/2)}, \quad k \neq 0$$

$$X_0 = \frac{1}{T_0} \int_{-T_0/4}^{3T_0/4} x(t) dt = \int_{-1/4}^{1/4} 2 dt = 1$$

Laplace transform:  $x_1(t - 0.25) = 2[u(t) - u(t - 0.5)], \quad X_1(s) = 2(e^{0.25s} - e^{-0.25s})$

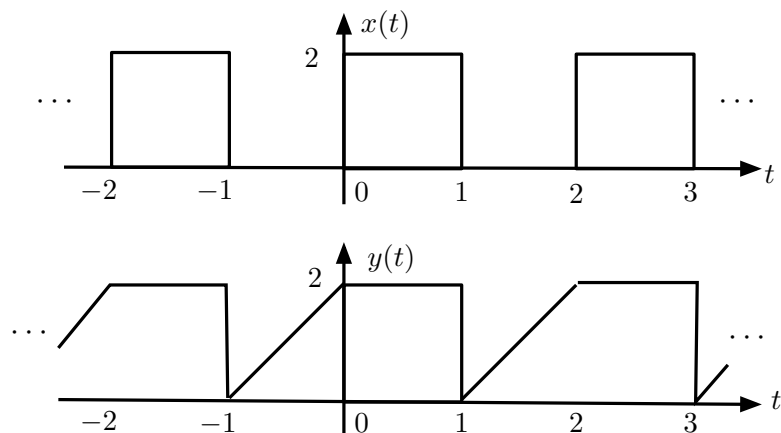
$$X_k = \frac{1}{T_0} \mathcal{L}[x_1(t)]|_{s=jk\Omega_0} = \frac{\sin(\pi k/2)}{\pi k/2} \quad k \neq 0$$

Fourier series: 
$$x(t) = \sum_{k=-\infty}^{\infty} \frac{\sin(\pi k/2)}{(\pi k/2)} e^{jk2\pi t}$$



Top: period of  $x(t)$  and real  $X_k$  vs  $k\Omega_0$ ; bottom magnitude and phase line spectra

## Example: Non-symmetric periodic signals



$$z(t) = x(t + 0.5), \text{ even, period: } z_1(t) = 2[u(t + 0.5) - u(t - 0.5)]$$

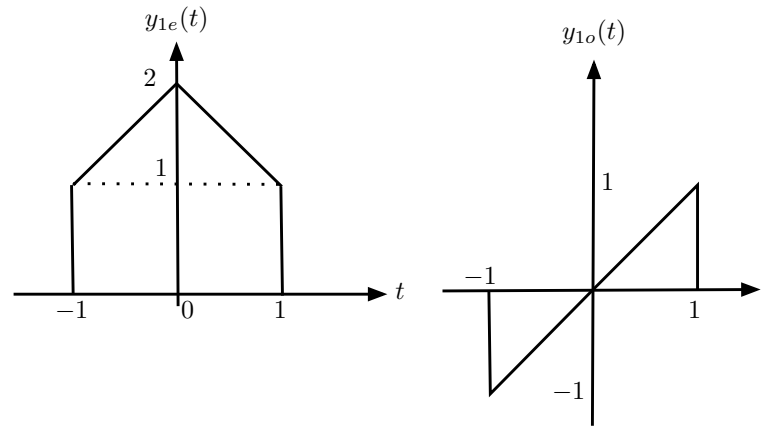
$$Z_1(s) = \frac{2}{s}[e^{0.5s} - e^{-0.5s}]$$

$$Z_k = \frac{1}{2} \frac{2}{jk\pi} [e^{jk\pi/2} - e^{-jk\pi/2}] = \frac{\sin(0.5\pi k)}{0.5\pi k} \text{ real-valued}$$

$$x(t) = z(t - 0.5) = \sum_k Z_k e^{jk\Omega_0(t-0.5)} = \sum_k \underbrace{[Z_k e^{-jk\pi/2}]}_{X_k} e^{jk\pi t}$$

$X_k$  complex since  $x(t)$  neither even nor odd

Even and odd components of the period of  $y(t)$ ,  $-1 \leq t \leq 1$



$$y_{1e}(t) = \underbrace{[u(t+1) - u(t-1)]}_{\text{rectangular pulse}} + \underbrace{[r(t+1) - 2r(t) + r(t-1)]}_{\text{triangular pulse}}$$

$$y_{1o}(t) = \underbrace{[r(t+1) - r(t-1) - 2u(t-1)]}_{\text{triangular pulse}} - \underbrace{[u(t+1) - u(t-1)]}_{\text{rectangular pulse}}$$

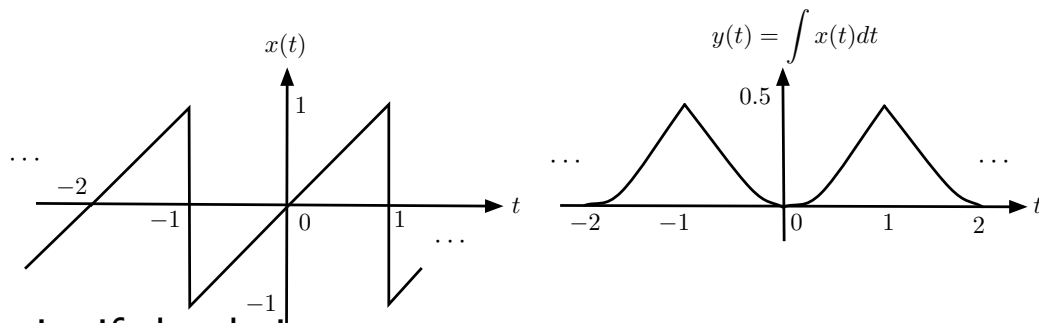
$$Y_{ek} = \frac{1}{T_0} Y_{1e}(s) \Big|_{s=jk\Omega_0} = \frac{1 - (-1)^k}{(k\pi)^2} \quad k \neq 0, \quad Y_{e0} = 1.5$$

$$Y_{ok} = \frac{1}{T_0} Y_{1o}(s) \Big|_{s=jk\Omega_0} = j \frac{(-1)^k}{k\pi} \quad k \neq 0, \quad Y_{o0} = 0$$

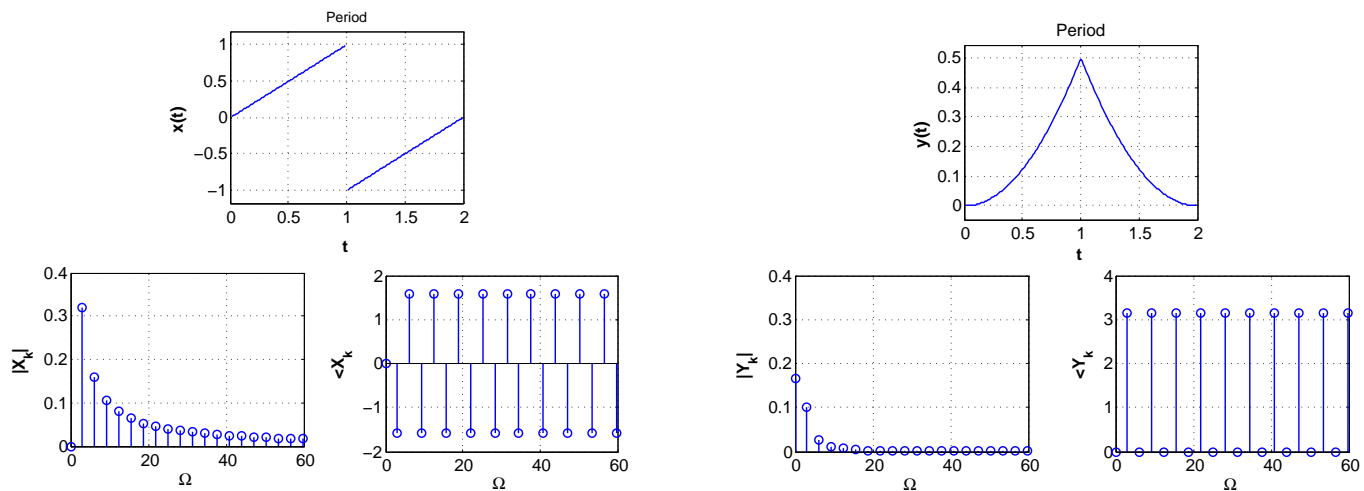
$$Y_k = \begin{cases} Y_{e0} + Y_{o0} = 1.5 + 0 = 1.5 & k = 0 \\ Y_{ek} + Y_{ok} = (1 - (-1)^k)/(k\pi)^2 + j(-1)^k/(k\pi) & k \neq 0 \end{cases}$$

# Example: Integration

$$y(t) = \int_{-\infty}^t x(t) dt$$



Integral does not exist if the dc is not zero



# Convergence of Fourier series

For Fourier series of  $x(t)$  to converge, it should:

- be absolutely integrable,
- have a finite number of maxima, minima and discontinuities.

FS equals  $x(t)$  at every continuity point and  $0.5[x(t + 0_+) + x(t + 0_-)]$  at every discontinuity point

Example: Approximate train of pulses with  $x_2(t) = \alpha + \beta \cos(\Omega_0 t)$  by

$$\begin{aligned} \text{Minimize } E_2 &= \frac{1}{T_0} \int_{T_0} |x(t) - x_2(t)|^2 dt, \text{ w.r.t. } \alpha, \beta \\ \frac{dE_2}{d\alpha} &= -\frac{1}{T_0} \int_{T_0} 2[x(t) - \alpha] dt = 0 \\ \frac{dE_2}{d\beta} &= -\frac{1}{T_0} \int_{T_0} 2[x(t) \cos(\Omega_0 t) - \beta \cos^2(\Omega_0 t)] dt = 0 \\ \alpha &= \frac{1}{T_0} \int_{T_0} x(t) dt, \\ \beta &= \frac{2}{T_0} \int_{T_0} x(t) \cos(\Omega_0 t) dt \end{aligned}$$

Periodic signal  $x(t)$

- Time-shifting:  $x(\pm t_0)$  remains periodic of the same fundamental period

$$x(t) \leftrightarrow \{X_k\} \Rightarrow x(t \mp t_0) \leftrightarrow X_k e^{\mp jk\Omega_0 t_0} = |X_k| e^{j(\angle X_k \mp k\Omega_0 t_0)}$$

only change in phase

- Frequency-shifting:

- $x(t)e^{j\Omega_1 t}$  is periodic of fundamental period  $T_0$  if  $\Omega_1 = M\Omega_0$ , for an integer  $M \geq 1$ ,
- for  $\Omega_1 = M\Omega_0$ ,  $M \geq 1$ , the Fourier coefficients  $X_k$  are shifted to frequencies  $k\Omega_0 + \Omega_1 = (k + M)\Omega_0$
- the modulated signal is real-valued by multiplying  $x(t)$  by  $\cos(\Omega_1 t)$ .



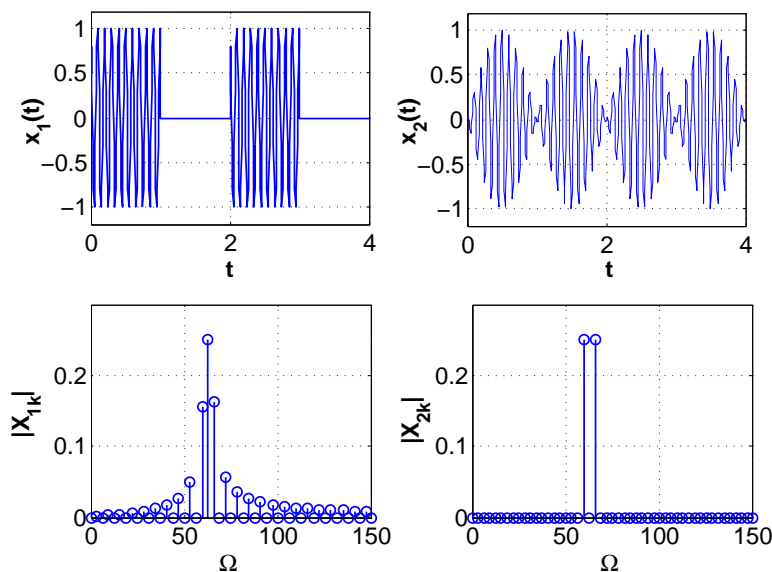
Example: Modulating  $\cos(20\pi t)$  with

- a periodic train of square pulses

$$x_1(t) = 0.5[1 + \text{sign}(\sin(\pi t))] = \begin{cases} 1 & \sin(\pi t) \geq 0 \\ 0 & \sin(\pi t) < 0 \end{cases}$$

- with a sinusoid

$$x_2(t) = \sin(\pi t).$$



*Modulated square-wave  $x_1(t) \cos(20\pi t)$  (left) and modulated cosine  $x_2(t) \cos(20\pi t)$*

# Response of LTI systems to periodic signals

Periodic input  $x(t)$  of causal and stable LTI system, with impulse response  $h(t)$ , by eigenfunction property of LTI systems

$$\text{Fourier series } x(t) = X_0 + 2 \sum_{k=1}^{\infty} |X_k| \cos(k\Omega_0 t + \angle X_k) \quad \Omega_0 = \frac{2\pi}{T_0}$$

$$y_{ss}(t) = X_0 |H(j0)| + 2 \sum_{k=1}^{\infty} |X_k| |H(jk\Omega_0)| \cos(k\Omega_0 t + \angle X_k + \angle H(jk\Omega_0))$$

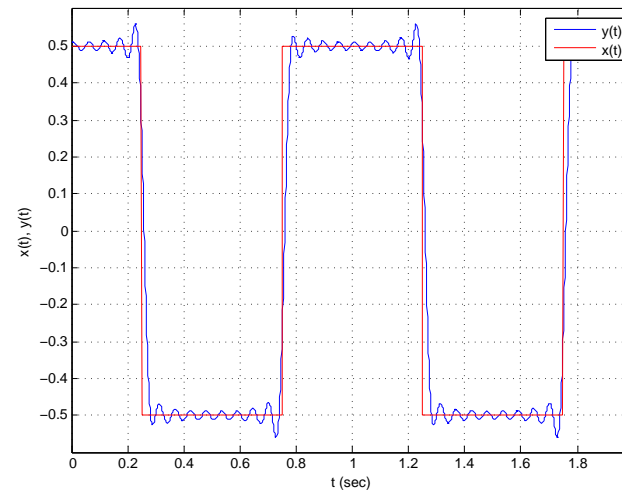
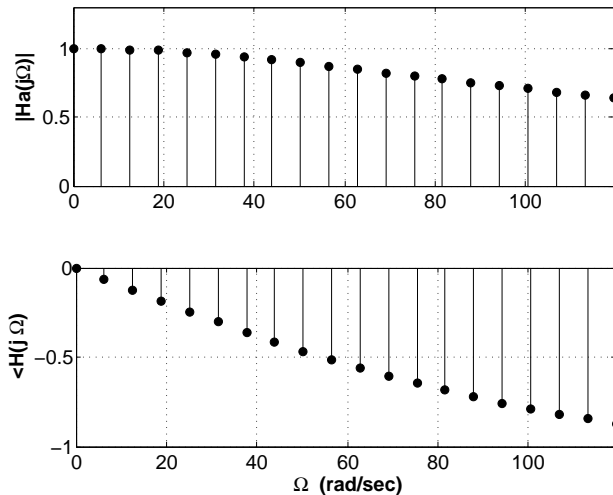
$$\text{where } H(jk\Omega_0) = |H(jk\Omega_0)| e^{j\angle H(jk\Omega_0)} \quad H(s)|_{s=jk\Omega_0}$$

frequency response of the system at  $k\Omega_0$

Example: Low-pass filtering using RC circuit with

$$\text{transfer function } H(s) = \frac{1}{1 + s/100}$$

$$\text{input } x(t) = \sum_{k=-\infty, \neq 0}^{\infty} \frac{\sin(k\pi/2)}{k\pi/2} e^{j2k\pi t}$$



Left: magnitude and phase response of the low-pass RC filter at harmonic frequencies. Right: response due to the train of pulses  $x(t)$ . Actual signal values are given by the dashed line, and the filtered signal is indicated by the continuous line

# Derivatives and integrals of Periodic Signals

- Derivative: Derivative  $dx(t)/dt$  of periodic signal  $x(t)$  is periodic of the same fundamental period. If  $\{X_k\}$  are the coefficients of the Fourier series of  $x(t)$ , the Fourier coefficients of  $dx(t)/dt$  are

$$jk\Omega_0 X_k, \quad \Omega_0 \text{ fundamental frequency of } x(t)$$

- Integral: Zero-mean, periodic signal  $y(t)$  with Fourier coefficients  $\{Y_k\}$ ,

$$\text{integral } z(t) = \int_{-\infty}^t y(\tau) d\tau$$

$$Z_k = \frac{Y_k}{jk\Omega_0} \quad k \text{ integer } \neq 0$$

$$Z_0 = - \sum_{m \neq 0} Y_m \frac{1}{jm\Omega_0} \quad \Omega_0 = \frac{2\pi}{T_0}$$

## Example: Derivative

period of  $x(t)$  :  $x_1(t) = 2r(t) - 4r(t - 0.5) + 2r(t - 1)$ ,  $0 \leq t \leq 1$ ,  $T_0 = 1$

$$g(t) = \frac{dx(t)}{dt} \Rightarrow X_k = \frac{G_k}{jk\Omega_0} \quad k \neq 0$$

period of  $g(t)$  :  $g_1(t) = dx_1(t)/dt = 2u(t) - 4u(t - 0.5) + 2u(t - 1)$

$$X_k = \frac{G_k}{jk\Omega_0} = \frac{(-1)^{(k+1)} (\cos(\pi k) - 1)}{\pi^2 k^2} \quad k \neq 0$$

$X_0 = 0.5$  from plot of  $x(t)$

## Integral

$$x(t) = \int_{-\infty}^t g(\tau) d\tau, \quad (G_0 = 0)$$

$$X_k = \frac{G_k}{j\Omega_0 k} = \frac{(-1)^{(k+1)} (\cos(\pi k) - 1)}{\pi^2 k^2} \quad k \neq 0$$

$$X_0 = - \sum_{m=-\infty, m \neq 0}^{\infty} \frac{G_m}{j2m\pi} = 0.5 \sum_{m=-\infty, m \neq 0}^{\infty} (-1)^{m+1} \left[ \frac{\sin(\pi m/2)}{(\pi m/2)} \right]^2$$

## Basic Properties of Fourier Series

Signals and constants	$x(t), y(t)$ periodic with period $T_0, \alpha, \beta$	$X_k, Y_k$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
Parseval's power relation	$P_x = \frac{1}{T_0} \int_{T_0}  x(t) ^2 dt$	$P_x = \sum_k  X_k ^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\Omega_0 X_k$
Integration	$\int_{-\infty}^t x(t') dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\Omega_0} k \neq 0$
Time shifting	$x(t - \alpha)$	$e^{-j\alpha\Omega_0} X_k$
Frequency shifting	$e^{jM\Omega_0 t} x(t)$	$X_{k-M}$
Symmetry	$x(t)$ real	$ X_k  =  X_{-k} $ even function of $k$ $\angle X_k = -\angle X_{-k}$ odd function of $k$
Multiplication	$z(t) = x(t)y(t)$	$Z_k = \sum_m X_m Y_{k-m}$