SIGNALS AND SYSTEMS USING MATLAB
Chapter 1 — Continuous–time Signals

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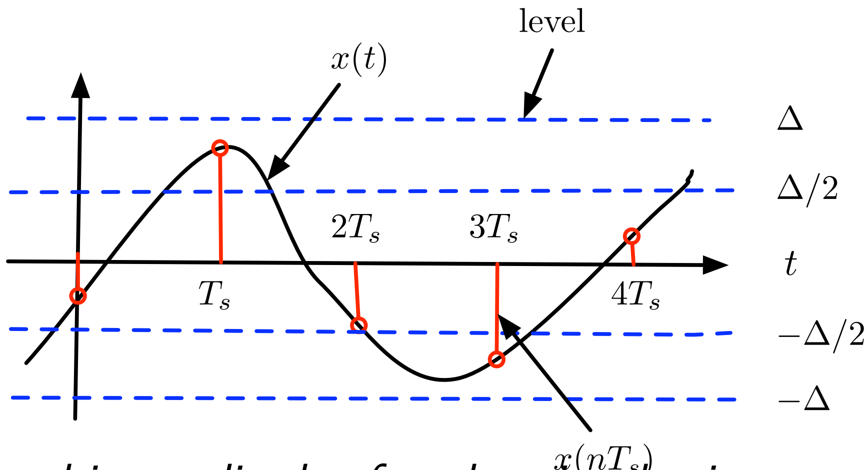
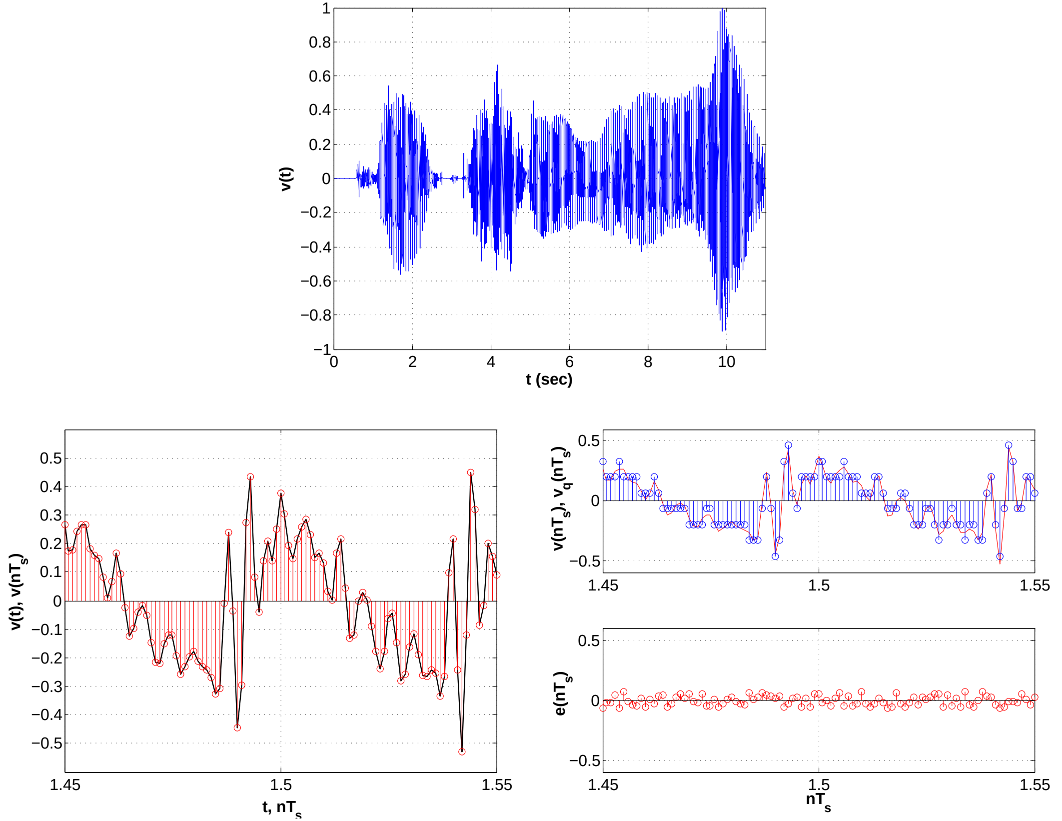
Classification of time–dependent signals

- *Predictability:* random or deterministic
- *Variation of time and amplitude:* continuous-time, discrete–time, or digital
- *Energy/power:* finite or infinite energy/power
- *Repetitive behavior:* periodic or aperiodic
- *Symmetry with respect to time origin:* even or odd
- *Support:* Finite or infinite support (outside support signal is always zero)

- Analog to digital converter (ADC or A/D converter): converts analog signals into digital signals
- Digital to analog converter (DAC or D/A converter): converts digital to analog signals



Discretization in time and in amplitude of analog signal using sampling period T_s and quantization level Δ. In time, samples are taken at uniform times $\{nT_s\}$, and in amplitude the range of amplitudes is divided into a finite number of levels so that each sample value is approximated by one of them



Segment of voice signal on top is sampled and quantized. Bottom left: voice segment (continuous line) and the sampled signal (vertical samples) using a sampling period $T_s = 10^{-3}$ *sec. Bottom-right: sampled and quantized signal at the top, and quantization error, difference between the sampled and the quantized signals, at the bottom.*

$$\boxed{\begin{array}{rl} x(.) : \mathcal{R} \rightarrow \mathcal{R} & (\mathcal{C}) \\ t \rightarrow & x(t) \end{array}}$$

Example: complex signal $y(t) = (1+j)e^{j\pi t/2}, \quad 0 \leq t \leq 10, \quad 0$ otherwise

$$\begin{aligned} y(t) &= \sqrt{2}e^{j(\pi t/2+\pi/4)} \\ &= \begin{cases} \sqrt{2}\left[\cos(\pi t/2+\pi/4) + j\sin(\pi t/2+\pi/4)\right], & 0 \leq t \leq 10, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

If $\quad x(t) = \sqrt{2}\cos(\pi t/2 + \pi/4), -\infty < t < \infty$

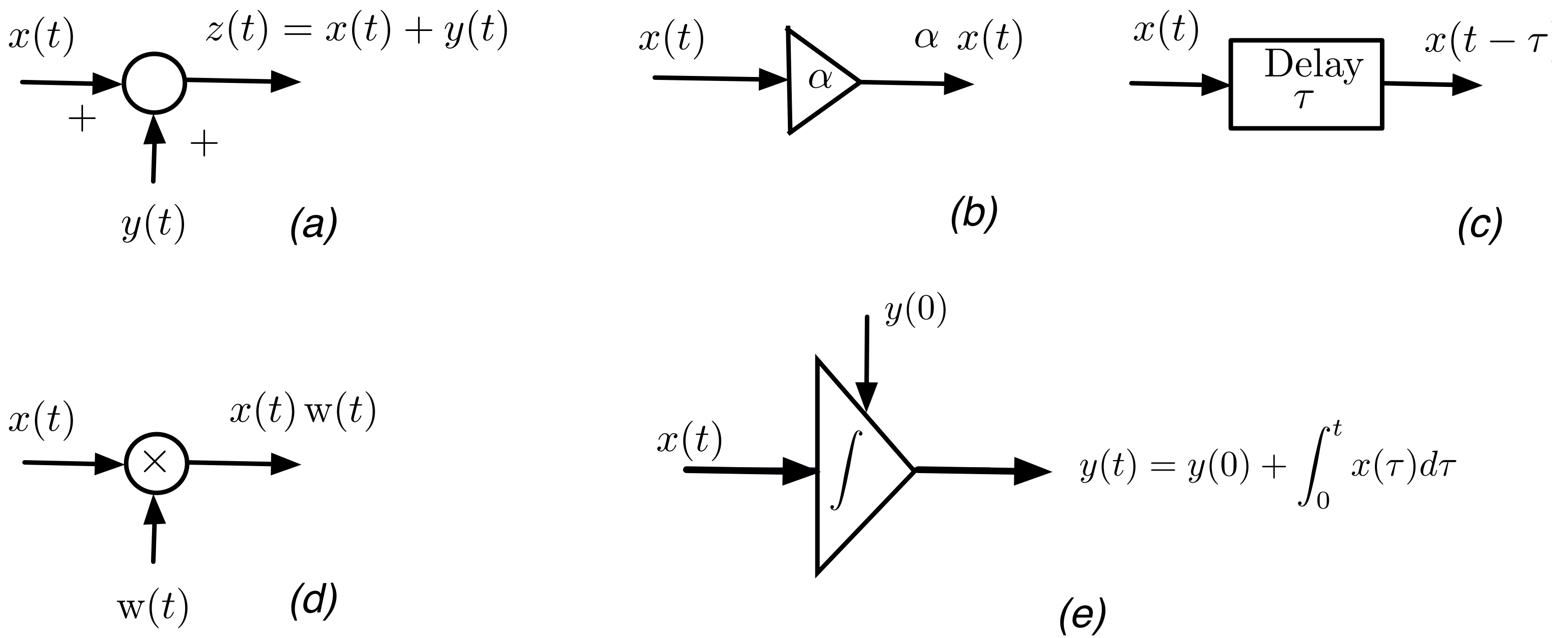
$p(t) = 1, \quad 0 \leq t \leq 10, \quad 0$ otherwise then

$$y(t) = [x(t) + jx(t-1)]p(t)$$

Given signals $x(t)$, $y(t)$, constants α and τ, and function $\mathrm{w}(t)$:

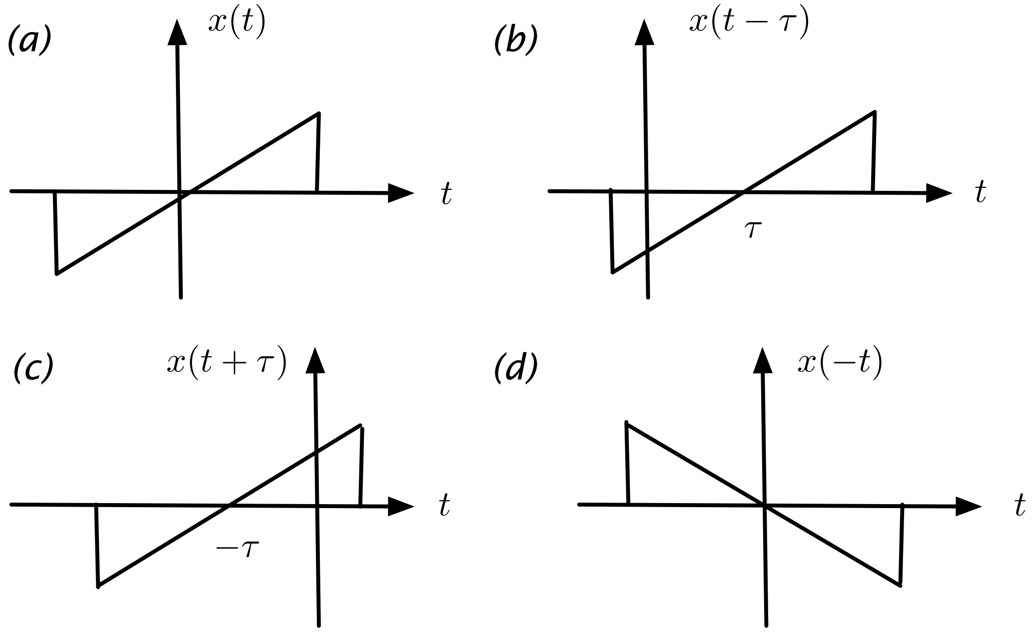
- *Signal addition/subtraction:* $x(t) + y(t)$, $x(t) - y(t)$
- *Constant multiplication:* $\alpha x(t)$
- *Time shifting*
  - $x(t - \tau)$ is $x(t)$ *delayed* by τ
  - $x(t + \tau)$ is $x(t)$ *advanced* by τ
- *Time scaling* $x(\alpha t)$
  - $\alpha = -1$, $x(-t)$ reversed in time or *reflected*
  - $\alpha > 1$, $x(\alpha t)$ is $x(t)$ *compressed*
  - $\alpha < 1$, $x(\alpha t)$ is $x(t)$ *expanded*
- Time windowing $x(t)\mathrm{w}(t)$, $\mathrm{w}(t)$ *window*
- Integration

$$y(t) = \int_{t_0}^{t} x(\tau) d\tau + y(t_0)$$



Basic signal operations: (a) adder, (b) constant multiplier, (c) delay, (d) time-windowing, (e) integrator

Delayed, advanced and reflected signals



Continuous-time signal (a), and its delayed (b), advanced (c), and reflected (d) versions.

Example

$$x(t) = \begin{cases} t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

delayed by 1: $$x(t-1) = \begin{cases} t-1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

advanced by 1: $$x(t+1) = \begin{cases} t+1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

reflected: $$x(-t) = \begin{cases} -t & -1 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

reflected and delayed by 1: $$x(-t+1) = \begin{cases} -t+1 & 0 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

reflected and advanced by 1: $$x(-t-1) = \begin{cases} -t-1 & -2 \le t \le 0 \\ 0 & \text{otherwise} \end{cases}$$

compressed by 2: $$x(2t) = \begin{cases} 2t & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

expanded by 2: $$x(t/2) = \begin{cases} t/2 & -2 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{\begin{array}{lll} x(t) & \text{even}: & x(t) = x(-t) \\ x(t) & \text{odd}: & x(t) = -x(-t) \end{array}}$$

- *Even and odd decomposition:* For any signal $y(t)$

$$y(t) = y_e(t) + y_o(t)$$

$$\begin{aligned} y_e(t) &= 0.5\,[y(t) + y(-t)] \quad \text{even component} \\ y_o(t) &= 0.5\,[y(t) - y(-t)] \quad \text{odd component} \end{aligned}$$

Example $x(t) = \cos(2\pi t + \theta), \quad -\infty < t < \infty$

$$\text{even} \quad x(t) = x(-t) \to \cos(2\pi t + \theta) = \cos(-2\pi t + \theta) = \cos(2\pi t - \theta)$$

$$\theta = -\theta, \text{or } \theta = 0, \pi$$

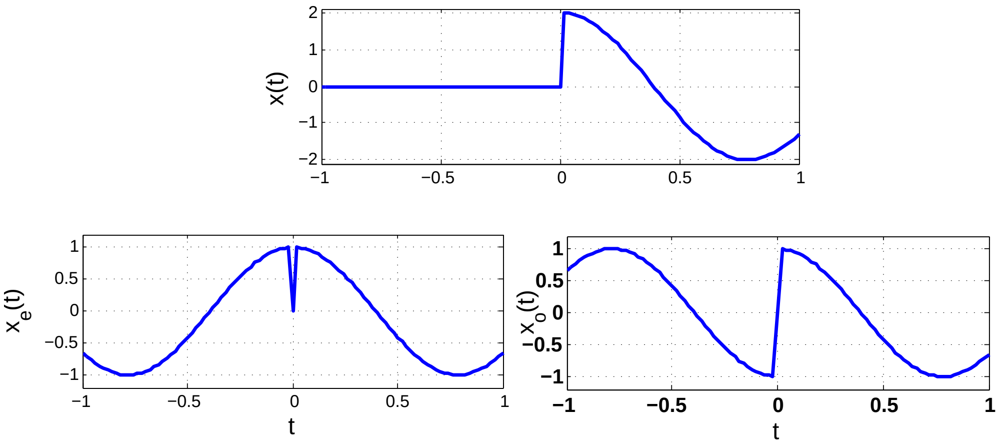
$$\begin{aligned} \text{odd} \quad x(t) = -x(-t) \to \cos(2\pi t + \theta) &= -\cos(-2\pi t + \theta) = \cos(-2\pi t + \theta \pm \pi) \\ &= \cos(2\pi t - \theta \mp \pi) \end{aligned}$$

$$\theta = -\theta \mp \pi, \text{or } \theta = \mp\pi/2$$

Example Given

$$x(t) = \begin{cases} 2\cos(4t) & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

not even or odd, its even and odd components are



If signal is 2 at $t = 0$

$$x_1(t) = \begin{cases} 2\cos(4t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

the odd component is same as before, and the even component is 2 at $t = 0$ and same as before otherwise

> $x(t)$ is periodic if
> (i) $x(t)$ defined in $-\infty < t < \infty$, and
> (ii) there is $T_0 > 0$, the fundamental period of $x(t)$,
> such that $x(t + kT_0) = x(t)$, integer$k$

Example $x(t) = e^{j2t}$ and $y(t) = e^{j\pi t}$

- $x(t) = \cos(2t) + j\sin(2t)$ periodic with $T_0 = 2\pi/2 = \pi$
- $y(t) = \cos(\pi t) + j\sin(\pi t)$ periodic with $T_1 = 2\pi/\pi = 2$
- $z(t) = x(t) + y(t)$ is not periodic as $T_0/T_1 \neq M/N$ where M, N integers
- $w(t) = x(t)y(t) = e^{j(2+\pi)t} = \cos(\Omega_2 t) + j\sin(\Omega_2 t)$, $\Omega_2 = 2 + \pi \to w(t)$ periodic with $T_2 = 2\pi/(2+\pi)$
- $p(t) = (1 + x(t))(1 + y(t)) = 1 + x(t) + y(t) + x(t)y(t)$ not periodic

$$\text{Energy of } x(t): \quad E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt,$$

$$\text{Power of } x(t): \quad P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

- $x(t)$ is *finite–energy*, or *square integrable*, if $E_x < \infty$
- $x(t)$ is *finite–power* if $P_x < \infty$

Example

- $x(t) = e^{-at}$, $a > 0, t \geq 0$ and 0 otherwise is finite energy and zero power
- $y(t) = (1 + j)e^{j\pi t/2}$, $0 \leq t \leq 10$, and 0 otherwise is finite energy and zero power

$$E_y = \int_0^{10} |(1 + j)e^{j\pi t/2}|^2 dt = 2 \int_0^{10} dt = 20$$

$x(t)$ period of fundamental period T_0 is

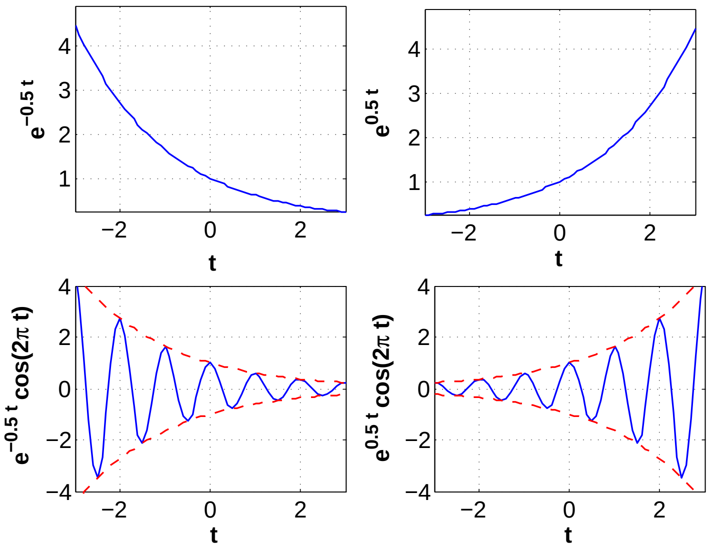
$$P_x = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)dt$$

for any t_0, i.e., the average energy in a period of the signal Let $T = NT_0$, integer $N > 0$:

$$\begin{aligned} P_x &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x^2(t)dt = \lim_{N\to\infty}\frac{1}{2NT_0}\int_{-NT_0}^{NT_0} x^2(t)dt \\ &= \lim_{N\to\infty}\frac{1}{2NT_0}\left[N\int_{-T_0}^{T_0} x^2(t)dt\right] = \frac{1}{2T_0}\int_{-T_0}^{T_0} x^2(t)dt \\ &= \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x^2(t)dt \end{aligned}$$

- Complex exponential

$$\begin{aligned} x(t) &= Ae^{at} = |A|e^{j\theta}e^{(r+j\Omega_0)t} \\ &= |A|e^{rt}\left[\cos(\Omega_0 t+\theta)+j\sin(\Omega_0 t+\theta)\right] \quad -\infty < t < \infty \end{aligned}$$



Analog exponentials: decaying exponential (top left), growing exponential (top right), modulated exponential decaying and growing (bottom left and right).

- Sinusoid

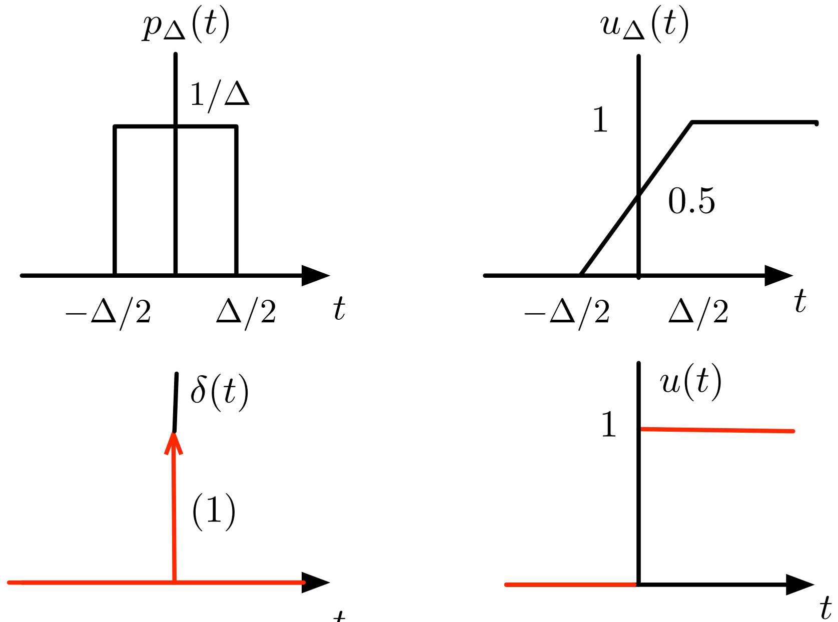
$$\boxed{A\cos(\Omega_0 t+\theta)=A\sin(\Omega_0 t+\theta+\pi/2)\qquad -\infty<t<\infty}$$

Modulation systems

$$A(t)\cos(\Omega(t)t+\theta(t))$$

- *Amplitude modulation or AM:* $A(t)$ changes according to the message, frequency and phase constant,
- *Frequency modulation or FM:* $\Omega(t)$ changes according to the message, amplitude and phase constant,
- *Phase modulation or PM:* $\theta(t)$ changes according to the message, amplitude and frequency constant

- Unit-impulse signal



Unit-impulse $\delta(t)$ and unit–step $u(t)$ as $\Delta \to 0$ in pulse $p_\Delta(t)$ and its integral $u_\Delta(t)$.

Unit-impulse

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \text{undefined} & t = 0 \end{cases}$$

$$\int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- Unit–step signal

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

- Ramp signal

$$r(t) = tu(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- Relations

$$\frac{dr(t)}{dt} = u(t), \quad \frac{d^2 r(t)}{dt^2} = \delta(t)$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$\int_{-\infty}^{t} \delta(\tau) d\tau = u(t), \quad \int_{-\infty}^{t} u(\tau) d\tau = r(t)$$

Example Triangular pulse

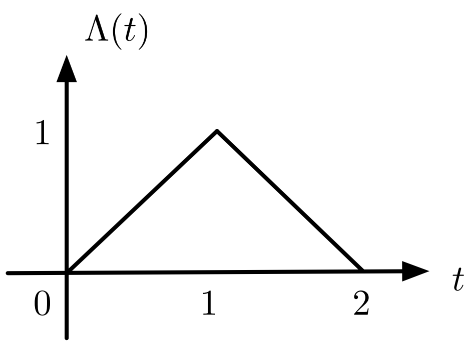
$$\Lambda(t) = \begin{cases} t & 0 \le t \le 1 \\ -t+2 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

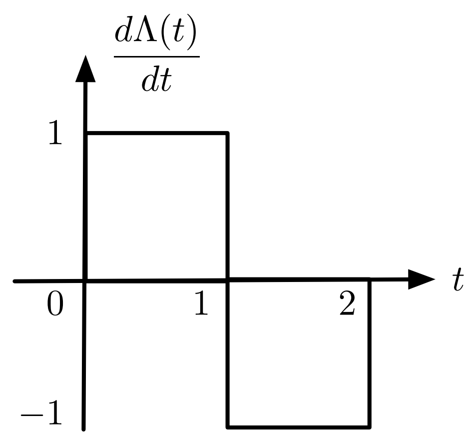
$$= r(t) - 2r(t-1) + r(t-2)$$

Derivative

$$\frac{d\Lambda(t)}{dt} = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$$= u(t) - 2u(t-1) + u(t-2)$$



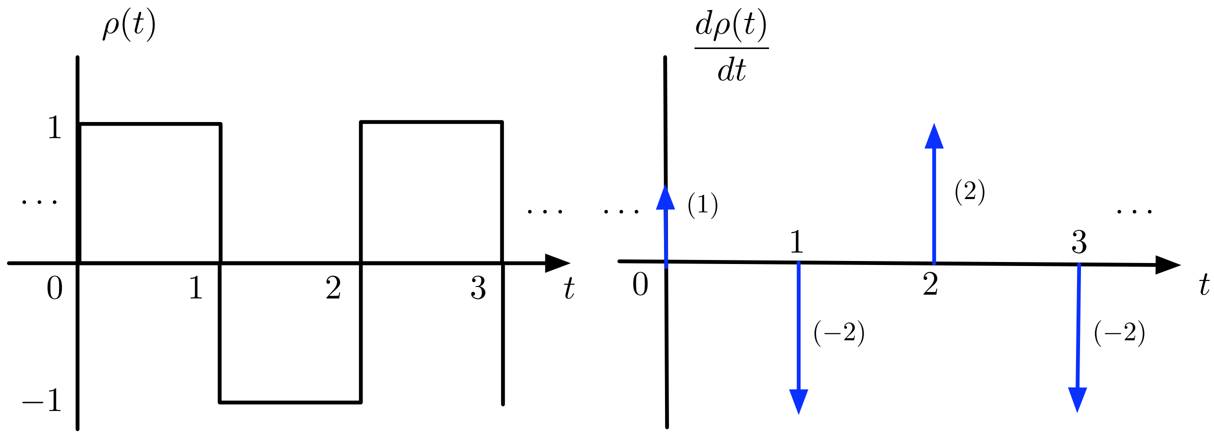


Example Causal train of pulses

$$\rho(t) = \sum_{k=0}^{\infty} s(t-2k), \qquad s(t) = u(t) - 2u(t-1) + u(t-2)$$

Derivative

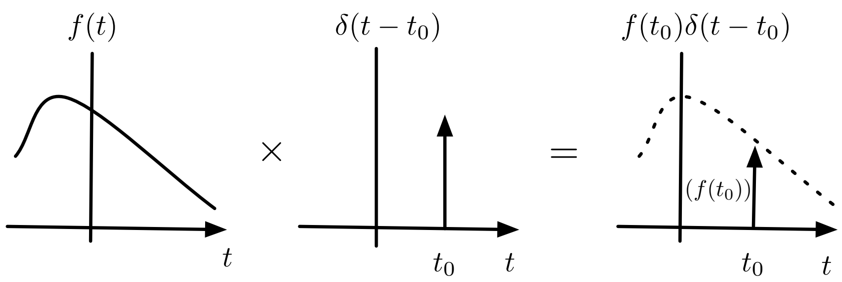
$$\frac{d\rho(t)}{dt} = \delta(t) + 2\sum_{k=1}^{\infty}\delta(t-2k) - 2\sum_{k=1}^{\infty}\delta(t-2k+1)$$



The number in () is area of the corresponding delta signal and it indicates the jump at the particular discontinuity, positive when increasing and negative when decreasing

- Sifting property of $\delta(t)$

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)dt = f(\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt = f(\tau), \quad \text{for any } \tau$$



- Generic representation

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$