King Abdulaziz University

Mechanical Engineering Department

MEP 365 Thermal Measurements

Ch. 5 Error analysis (Uncertainty)

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Contents

1-Introduction to errors and uncertainty

In real measurements, the true value is not known, we estimate the parable error in the measurement using uncertainty

Uncertainty is very useful in reporting measured values

$$x' = \bar{x} \pm u_{\bar{x}} \qquad (P\%)$$

Uncertainty analysis : method used to quantify the $u_{\bar{x}}$ in the above equation

1-Introduction to errors and uncertainty

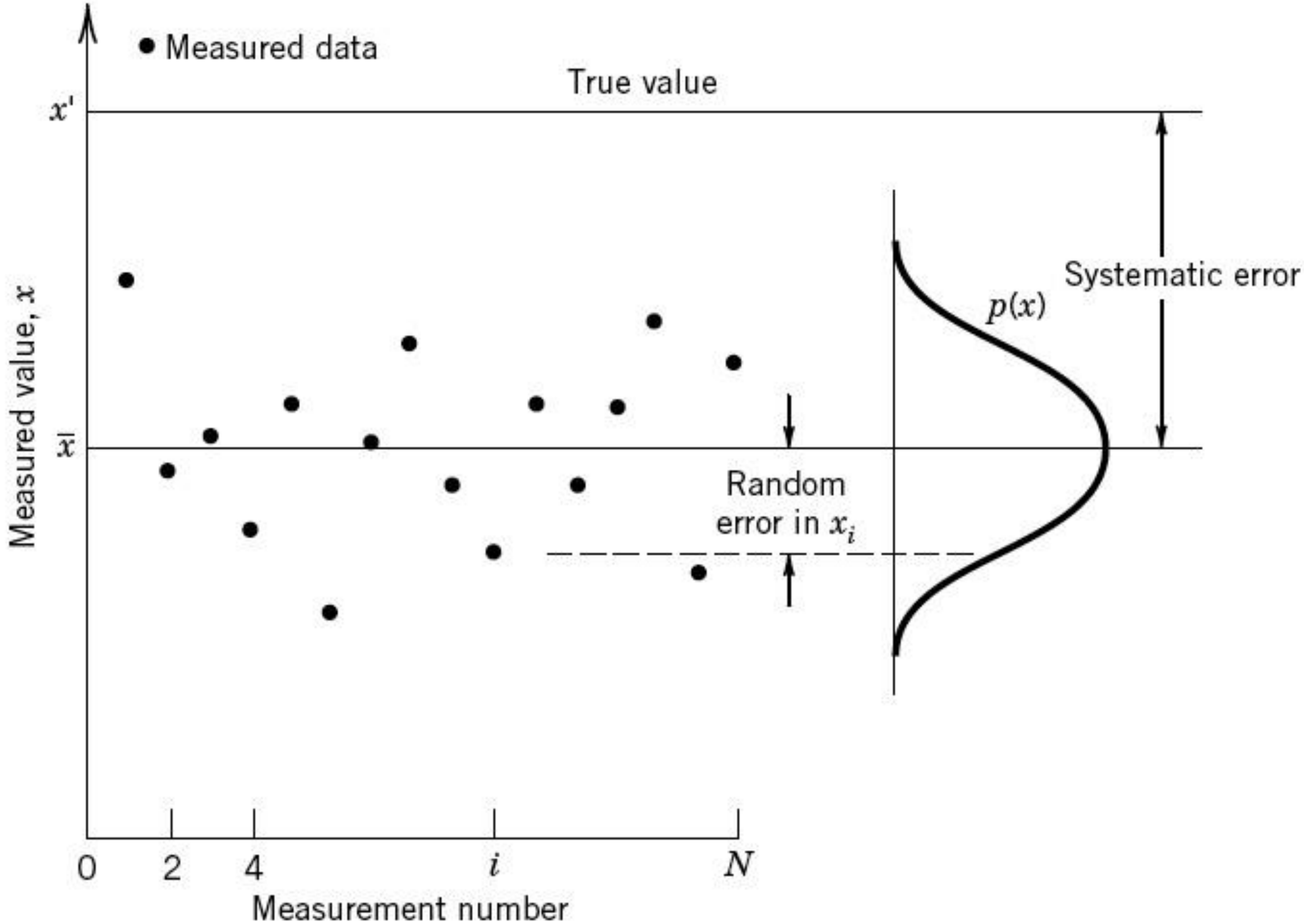


Figure 5.1. Distribution of errors upon repeated measurements.

Levels of uncertainty

Design stage uncertainty

Advanced stage and/or single measurement uncertainty

Multiple measurement uncertainty

2-Design stage uncertainty

Uncertainty before the measurement takes place

1-Interpolation error $u_0 = \pm \frac{1}{2}\text{resolution}$

2-Instrument error u_c

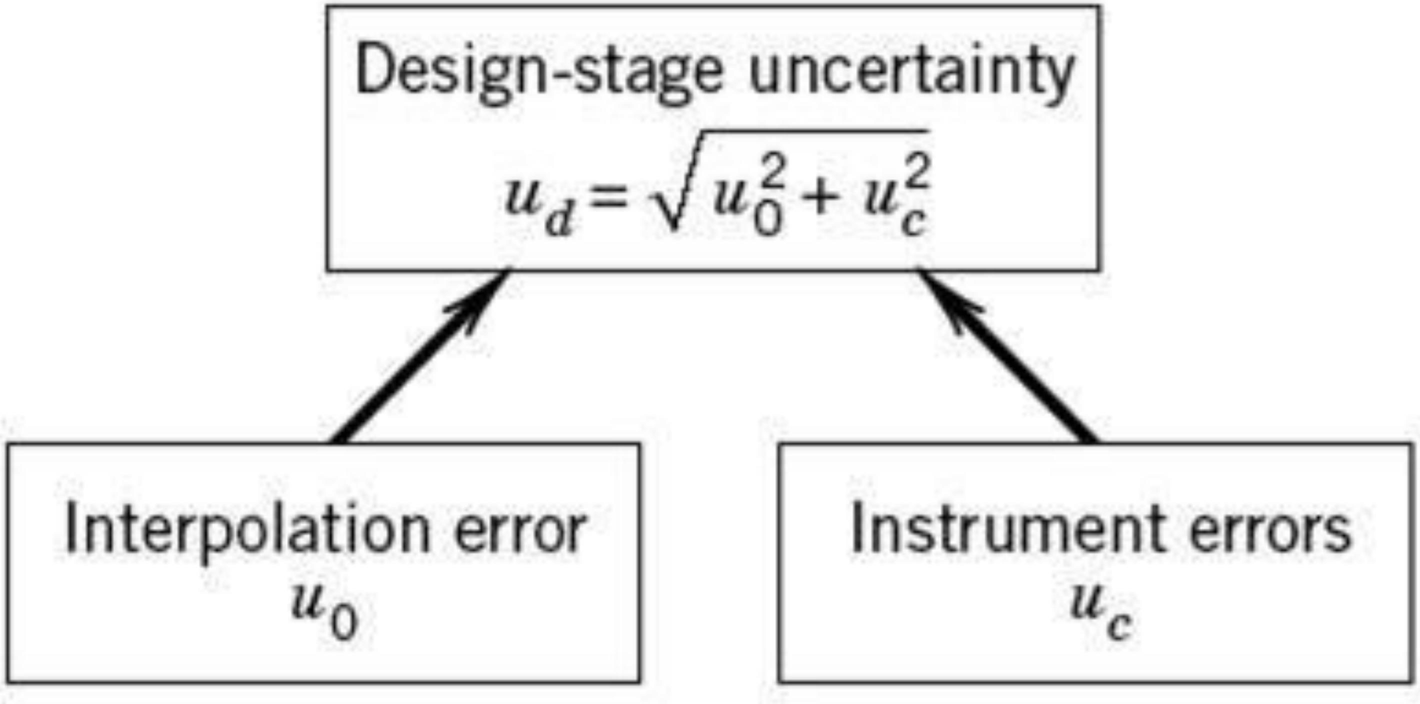
Using **RSS** (Root Sum Squares method)

Instrument Uncertainty $$u_c = \left[{e_1}^2 + {e_2}^2 + {e_2}^2 + \ldots . {e_m}^2 \right]^{1/2}$$

For example $$u_c = \left[{e_h}^2 + {e_L}^2 + {e_K}^2 + {e_R}^2 \right]^{1/2}$$

Design stage uncertainty $$u_d = \pm \left[{u_0}^2 + {u_c}^2 \right]^{1/2}$$

Design stage uncertainty



Must consider the design stage uncertainty for the Instrument (or **transducer)** and the **readout device**

2-Design stage uncertainty

Combination of transducer and readout design stage uncertainty

In case you have a transducer and a read out device you can find the design stage uncertainty for the transducer and the design stage uncertainty of the readout and then combining the two into one uncertainty

Transducer

$$(u_d)_P = \pm\sqrt{(u_0)_P^2 + (u_c)_P^2}$$

Readout

$$(u_d)_E = \pm\sqrt{(u_0)_E{}^2 + (u_c)_E{}^2}$$

Combined

$$u_d = \sqrt{(u_d)_E{}^2 + (u_d)_P{}^2}$$

Example 5.2

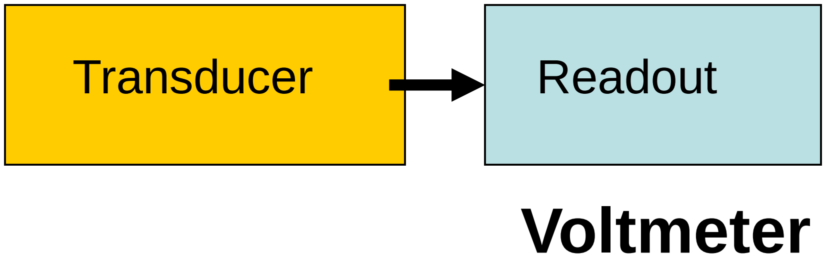
Voltmeter and pressure transducer. Expected reading 3 psi

Required: Design stage uncertainty, u_d

Voltmeter (output device)

Resolution 10 μV

Accuracy 0.001 % of reading



Pressure Transducer

Range $\pm$ 5psi

Sensitivity 1V/psi

Input power 10VDC

Linearity error 2.5 mv/psi (reading) over range

Sensitivity error 2mV/psi (reading) over range

Resolution : negligible

Example 5.2 continue

For the transducer

$$(u_d)_P = \pm\sqrt{(u_c)^2{}_P + (u_0)^2{}_P}$$

Instrument uncertainty $(u_c)_P$

$$(u_c)_P = \pm\sqrt{e_1{}^2 + e_2{}^2} = \pm\sqrt{(2.5*3)^2 + (2*3)^2} = \pm 9.61 mV$$

Resolution uncertainty for the transducer

$$(u_0)_P = 0$$

$$(u_d)_P = \pm 9.61\, mV$$

For the output (Voltmeter) device

$$(u_0)_E = \pm\frac{10}{2} = \pm 5\mu V$$

$$(u_c)_E = \pm 0.001\frac{1}{100} * 3psi(1V / psi) * 10^6 = \pm 30\ \mu V$$

Uncertainty for the output device

$$(u_d)_E = \pm\sqrt{(u_0)_E{}^2 + (u_c)_E{}^2} = \sqrt{5^2 + 30^2} = 30.4\ \mu\text{V} = \pm 0.0304\,\text{mV}$$

Uncertainty for the combination of the transducer and the readout

$$u_d = \sqrt{(u_d)_E{}^2 + (u_d)_P{}^2} = \sqrt{0.0304^2 + 9.61^2} = \pm 9.61\,\text{mV}$$

$$(u_d)_P = \pm 9.61\, mV$$

$$(u_d)_E = \pm 0.0304\,\text{mV}$$

$$u_d = \pm\sqrt{(u_d)_E{}^2 + (u_d)_P{}^2} = \pm\sqrt{0.03^2 + 9.61^2} = \pm 9.61\,\text{mV}$$

Notice that essentially the overall design uncertainty is due to the transducer i.e. having a better (more accurate) voltmeter (output device) will not improve the design stage uncertainty

3-Identifying error sources

Since the measurement involve three stages i,e. Calibration, data acquisition, and data reduction, then errors can be grouped into these three groups i.e.

- ❖Calibration errors
- ❖Data acquisition errors
- ❖Data reduction errors

3-Identifying error sources

All types of errors can be grouped into:

Systematic (bias, B) error

Random (or precision, P) error)

Grouping of errors is not important
The important thing is how to treat all these errors and produce a final uncertainty value

Systematic (bias) error

Each repeated measurement contain the same (fixed) amount of the systematic error. It can be either high or low ±B. It can be reduced by comparison method such as

1-Calibration

2-Concomitant methods

3-Interlaboraty comparison

4-Experience

Calibration errors can be reduced to very small value but it cannot be totally eliminated.

Systematic (bias) error

The systematic error is represented by a range such as $\pm b$

The systematic error at any level of confidence is $\pm t_{v,P} b$

For our calculation for probability 95%, the systematic uncertainty will be assumed to be given by

$$b = \pm B/2$$

Notice that as the degree of freedom goes to $\propto$

$$t_{v,P} = 1.96$$

$$t_{vP} \approx 2.0$$

b is called systematic standard uncertainty

Table 4.4 Student's *t* Distribution

ν	t_{50}	t_{90}	t_{95}	t_{99}
1	1.000	6.314	12.706	63.657
2	0.816	2.920	4.303	9.925
3	0.765	2.353	3.182	5.841
4	0.741	2.132	2.770	4.604
5	0.727	2.015	2.571	4.032
6	0.718	1.943	2.447	3.707
7	0.711	1.895	2.365	3.499
8	0.706	1.860	2.306	3.355
9	0.703	1.833	2.262	3.250
10	0.700	1.812	2.228	3.169
11	0.697	1.796	2.201	3.106
12	0.695	1.782	2.179	3.055
13	0.694	1.771	2.160	3.012
14	0.692	1.761	2.145	2.977
15	0.691	1.753	2.131	2.947
16	0.690	1.746	2.120	2.921
17	0.689	1.740	2.110	2.898
18	0.688	1.734	2.101	2.878
19	0.688	1.729	2.093	2.861
20	0.687	1.725	2.086	2.845
21	0.686	1.721	2.080	2.831
30	0.683	1.697	2.042	2.750
40	0.681	1.684	2.021	2.704
50	0.680	1.679	2.010	2.679
60	0.679	1.671	2.000	2.660
∞	0.674	1.645	1.960	2.576

$t_{\propto,P} = 1.96$

Or approximately =2

Random (or precision) error)

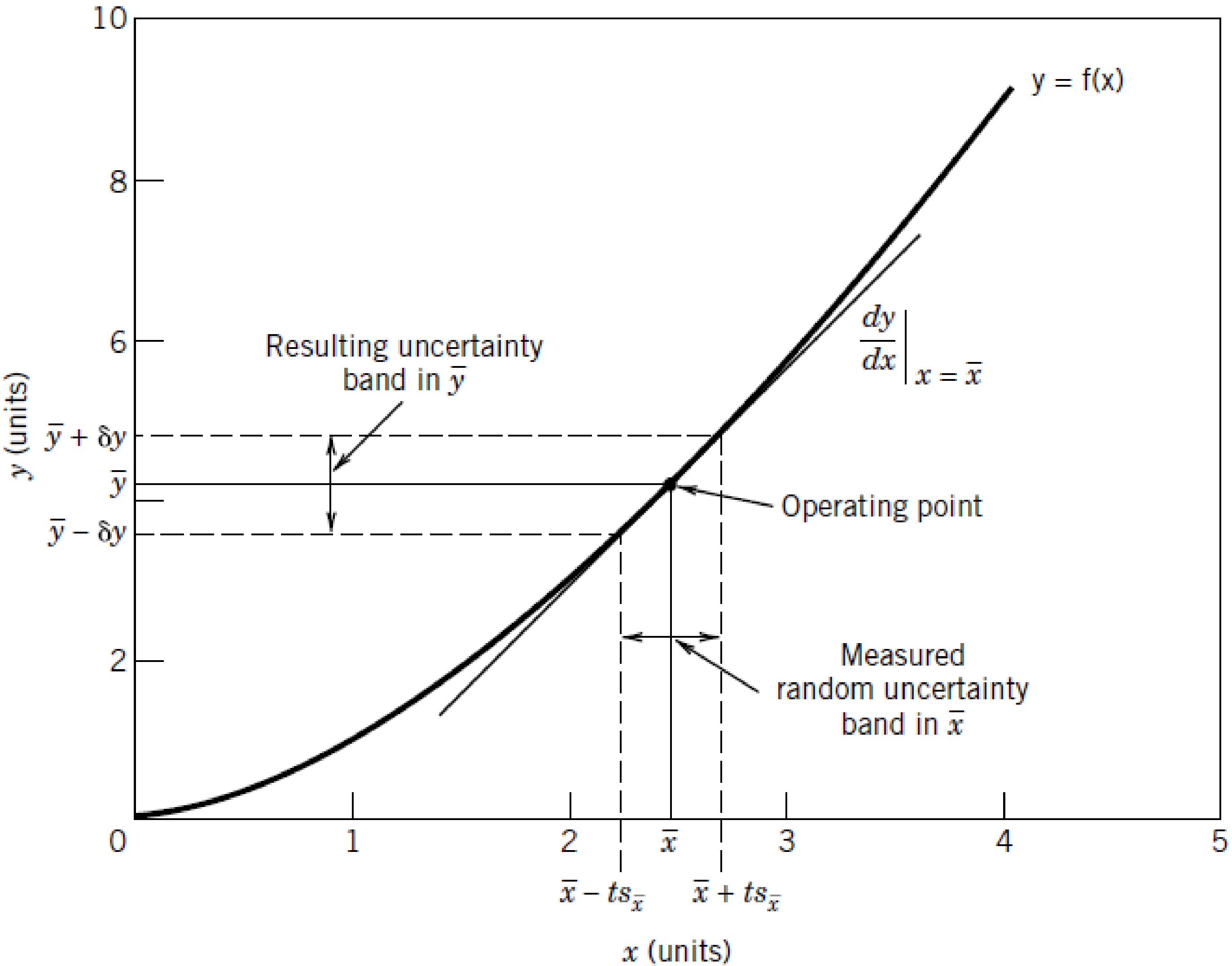
Repeated measurements under same conditions results in variation of the variable. By using statistical analysis one could find the mean value and the region around the mean where x varies. Error can be only estimate with certain probability. The confidence interval for variable x is given by

$$\pm t_{v,P} s_{\bar{x}}$$

Standard Random uncertainty $s_{\bar{x}}$

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}}$$

4- Error Propagation



If we have an uncertainty in x

How it will be reflected on y

If y=y(x)

Figure 5.3 Relationship between a measured variable and a resultant calculated using the value of that variable.

4- Error propagation

Consider y=f(x)

Statistical analysis was done for x $\quad \bar{x} \pm t_{v,P} s_{\bar{x}}$

Required: propagation of error in y

$$\bar{y} \pm \delta y = f(\bar{x} \pm t s_{\bar{x}})$$

Using Taylor series

$$\bar{y} \pm \delta y = f(\bar{x}) \pm \left[\left(\frac{dy}{dx} \right)_{x=\bar{x}} t s_{\bar{x}} + \left(\frac{d^2 y}{dx^2} \right)_{x=\bar{x}} (t s_{\bar{x}})^2 + \ldots\ldots \right]$$

For first approximation

$$\bar{y} = f(\bar{x}) \qquad \pm \delta y = \left(\frac{dy}{dx} \right)_{x=\bar{x}} t s_{\bar{x}} \qquad u_y = \left(\frac{dy}{dx} \right)_{x=\bar{x}} u_x$$

4- Error propagation
Extension to more than one variable function

Generally for result R which is function of x_1, x_2, x_3 ……x_L

$$R = R(x_1, x_2, x_3, \ldots\ldots x_L) \qquad R' = \bar{R} \pm u_R$$

$$\bar{R} = R(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$u_R = f_2(u_{x1}, u_{x2}, \ldots\ldots u_{xL})$$

Using **RSS** (Root Sum Squares method)

$$u_R = \pm\left[\sum_{i=1}^{L}\left(\frac{\partial R}{\partial x_i}u_{xi}\right)^2\right]^{1/2} = \pm\left[\sum_{i=1}^{L}(\theta_i u_{xi})^2\right]^{1/2}$$

Where $\theta_i = \left(\dfrac{\partial R}{\partial x_i}\right)_{x_i = \bar{x}_i}$

Example 5.3 on Error Propagation

y=KE. K=10.10 mm/V. E=5 V. u_K=±0.1 mm/V, u_E=±0.01V at 95 % confidence

Required u_y

$$y = KE$$

$$u_y = \pm\left[(\theta_K u_K)^2 + (\theta_E u_E)^2\right]^{1/2}$$

$$\theta_K = \frac{\partial y}{\partial K} = E \qquad \theta_E = \frac{\partial y}{\partial E} = K$$

$$u_y = \pm\left[(5*0.1)^2 + (10.1*0.01)^2\right]^{1/2} = \pm 0.51 mm \quad (95\%)$$

$$\bar{y} = KE = 10.1*5 = 50.5 \qquad \bar{y} \pm u_y = 50.5 \pm 0.51 \text{ mm} \quad (95\%)$$

Error Propagation using Numerical Approach

Sequential Perturbation

$$R = R(x_1, x_2, x_3, \ldots\ldots x_L)$$

at operating point $$R_o = R(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

Increase independent variable one by one

$$R_1^{\,+} = R(\bar{x}_1 + u_{x1}, \bar{x}_2, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$R_2^{\,+} = R(\bar{x}_1, \bar{x}_2 + u_{x2}, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$R_3^{\,+} = R(\bar{x}_1, \bar{x}_2, \bar{x}_3 + u_{x3}, \ldots\ldots \bar{x}_L)$$

$$\delta R_i^{\,+} = R_i^{\,+} - R_o$$

decrease independent variables

$$R_1^{\,-} = R(\bar{x}_1 - u_{x1}, \bar{x}_2, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$R_2^{\,-} = R(\bar{x}_1, \bar{x}_2 - u_{x2}, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$R_3^{\,-} = R(\bar{x}_1, \bar{x}_2, \bar{x}_3 - u_{x3}, \ldots\ldots \bar{x}_L)$$

$$\delta R_i^{\,-} = R_i^{\,-} - R_o$$

$$\delta R_i = \frac{\delta R_i^{\,+} - \delta R_i^{\,-}}{2} = \theta_i u_i \qquad u_R = \pm\left[\sum_{i=1}^{L} \left(\delta R_i\right)^2\right]^{1/2}$$

Example 5.3 y=KE. K=10.10 mm/V. E=5 V. $u_K=\pm0.1$ mm/V, $u_E=\pm0.01$V at 95 % confidence

Required u_y using Perturbation method

$y = KE$ Let R=y R_o=5*10.10=50.50 mm

i	x_i	R^+_i	R^-_i	δR^+_i	δR^-_i	δR_i
1	E	50.60	50.40	0.1	-0.1	0.1
2	K	51.00	50.00	0.5	-0.5	0.5

$$\delta R_i^+ = R_i^+ - R_o$$

$$\delta R_i^- = R_i^- - R_o$$

$$\delta R_i = \frac{\delta R_i^+ - \delta R_i^-}{2} = \theta_i u_i$$

$$u_R = \pm\left[\sum_{i=1}^{L}(\delta R_i)^2\right]^{1/2} = \pm\left[(0.1)^2 + (0.5)^2\right]^{1/2} = \pm 0.51$$

$$y' = 50.50 \pm 0.51 \text{ mm } (95\%)$$

5- Advanced stage uncertainty analysis

Not only design stage uncertainty but additional factors such as procedural and test control errors

Orders of advanced stage uncertainty

Zero order uncertainty

First order uncertainty

…….

……..

Nth order uncertainty

5- Advanced stage uncertainty analysis

Zero order uncertainty

All variables are fixed except the physical act of observation. Only resolution (interpolation) error is considered. i.e. u_o

Zero order uncertainty is not adequate for reporting of test results

5- Advanced stage uncertainty analysis

First order uncertainty

The effect of time as an extraneous variable is considered. i.e. taking N measurements of the variable with time

$$u_1 = \pm t_{v,95} s_{\bar{x}} \qquad (95\%)$$

The uncertainty u_1 is including resolution effect into consideration

First order uncertainty is not adequate for reporting of test results

At each successive order of uncertainty other factors are considered

5- Advanced stage uncertainty analysis

N^{th} order uncertainty

Instruments uncertainty u_c entered into the scheme

$$u_N = \left[u_c{}^2 + \left(\sum_{i}^{N-1} u_i{}^2 \right) \right]^{1/2}$$

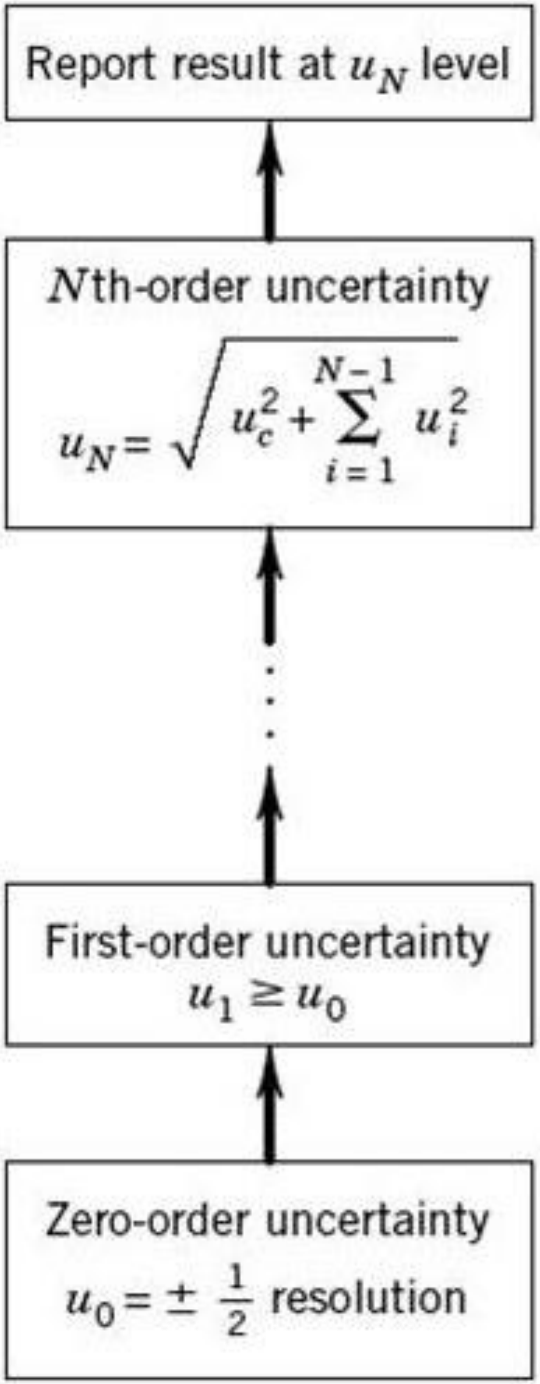
Uncertainty estimate at the Nth order allow for the direct comparison between results of similar tests obtained either using different instrument or different test facilities.

Uncertainty orders

Advanced stage

or single measurement uncertainty

$$u_N = \left[u_c{}^2 + u_1^2 + u_2^2 + ...\right]^{1/2}$$



Multiple measurement uncertainty analysis

Sufficient repetitions must be present in measured data

For set of measurements

Procedure to estimate uncertainty

1-Identify elemental errors

2-Estimate the magnitude of the systematic and random error

3-Calculate the uncertainty (expanded) for the result

Professional way to calculate uncertainty (similar to NIST)

NIST=National Institute of Standard and Technology

*https://www.**nist**.gov*

Multiple measurement uncertainty analysis

For each elementary error you have the systematic and random uncertainty i.e. $b_{\bar{x}}$ and $s_{\bar{x}}$.

Combine systematic uncertainty $b_{\bar{x}} = \left((b_{\bar{x}})_1^2 + (b_{\bar{x}})_2^2 + (b_{\bar{x}})_3^2 + \ldots. + (b_{\bar{x}})_K^2 \right)^{1/2}$

Combine random uncertainty

$$s_{\bar{x}} = \left[(s_{\bar{x}_1})^2 + (s_{\bar{x}_2})^2 + (s_{\bar{x}_3})^2 + \cdots + (s_{\bar{x}_K})^2 \right]^{\frac{1}{2}} \qquad s_{\bar{x}} = \frac{s_x}{\sqrt{N}}$$

Expanded uncertainty

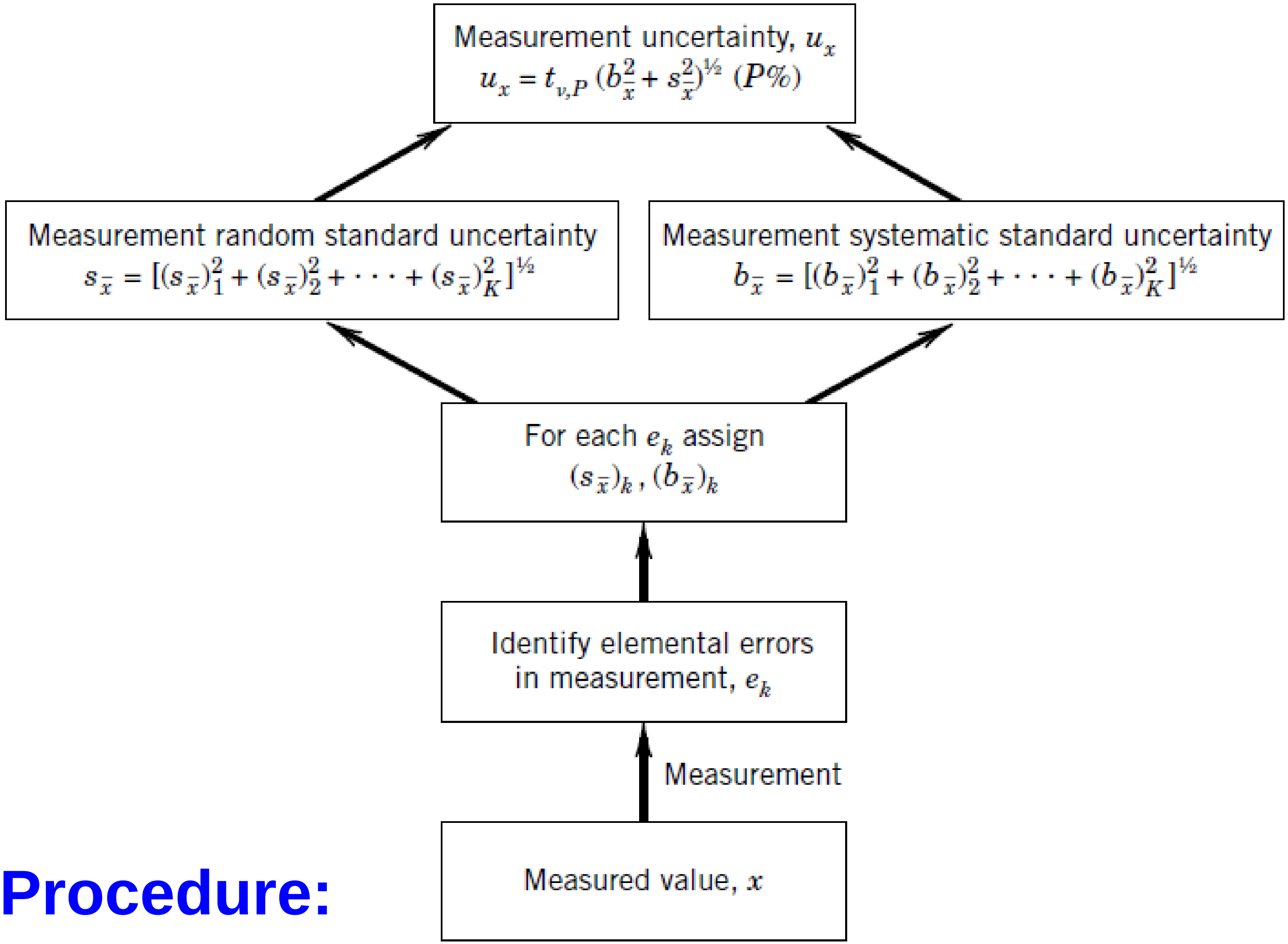
$$u = \pm t_{v,P} \left[b_{\bar{x}}{}^2 + s_{\bar{x}}{}^2 \right]^{1/2}$$

95% probability

Degree of freedom

$$v = \frac{\left(\sum_{k=1}^{K} (s_{\bar{x}}^2)_k + (b_{\bar{x}}^2)_k \right)^2}{\left(\sum_{k=1}^{K} (s_{\bar{x}}^4)_k / v_k + \sum_{k=1}^{K} (b_{\bar{x}}^4)_k / v_k \right)}$$

Multiple measurement uncertainty analysis



Procedure:

Figure 5.6 Multiple-measurement uncertainty procedure for combining uncertainties.

Example 5.12

Stress σ uncertainty due random and systemic errors

$(b_{\bar{\sigma}})_1$=0.5 N/cm² $(b_{\bar{\sigma}})_2$=1.05 N/cm² (($b_{\bar{\sigma}})_3$=0 N/cm²

$(s_{\bar{\sigma}})_1$ =4.6 N/cm² $(s_{\bar{\sigma}})_2$=10.3 N/cm² $(s_{\bar{\sigma}})_3$=1.2 N/cm²

v_1=14 v_2=37 v_3=8

$$\bar{\sigma} = 223.4\ N/cm^2$$

$$s_{\bar{\sigma}} = \left((s_{\bar{\sigma}})_1^2 + (s_{\bar{\sigma}})_2^2 + (s_{\bar{\sigma}})_3^2\right)^{1/2} = 11.3\ N/cm^2$$

$$b_{\bar{\sigma}} = \left((b_{\bar{\sigma}})_1^2 + (b_{\bar{\sigma}})_2^2 + (b_{\bar{\sigma}})_2^2\right)^{1/2} = 1.16\ N/cm^2$$

$$v = \frac{\left(\sum_{k=1}^{K}(s_{\bar{x}}^2)_k + (b_{\bar{x}}^2)_k\right)^2}{\left(\sum_{k=1}^{K}(s_{\bar{x}}^4)_k / v_k + \sum_{k=1}^{K}(b_{\bar{x}}^4)_k / v_k\right)} = 49$$

$$t_{49,95} \approx 2$$

From Table 4.4

$$u_\sigma = \pm 2\left[b_{\bar{\sigma}}^2 + s_{\bar{\sigma}}^2\right]^{1/2} = \pm 2\left[2.3^2 + 11.3^2\right]^{1/2} = \pm 22.7\ N/cm^2$$

Best estimate of stress

$$\sigma' = 223.4 \pm 22.7\ \ N/cm^2$$

Propagation of uncertainty to the results using the concept of grouping the errors into systematic and random errors

$$R' = \bar{R} \pm u_R \qquad (P\%)$$

$$\bar{R} = f_1(\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\ldots \bar{x}_L)$$

$$u_R = f_2(b_{\bar{x}_1}, b_{\bar{x}_2}, b_{\bar{x}_3}, \ldots. b_{\bar{x}_L}, \ldots. s_{\bar{x}_1}, s_{\bar{x}_2}, s_{\bar{x}_3}, \ldots s_{\bar{x}_L})$$

$$s_R = \left(\sum_{i=1}^{L} \left[\theta_i s_{\bar{x}_i} \right]^2 \right)^{1/2} \qquad b_R = \left(\sum_{i=1}^{L} \left[\theta_i b_{\bar{x}_i} \right]^2 \right)^{1/2} \qquad \theta_i = \left. \frac{\partial R}{\partial x_i} \right|_{x=\bar{x}_i}$$

$$u_R = \pm t_{v,P} \left[b_R^2 + s_R^2 \right]^{1/2} \qquad (95\%)$$

OR

for large Data set (i.e. N→∞) $t_{v,95}=2$

$$v = \frac{\left\{ \sum_{i=1}^{L} \left(\theta_i s_{\bar{x}_i} \right)^2 \right\}^2}{\sum_{i-1}^{L} \left\{ \left(\theta_i s_{\bar{x}_i} \right)^4 / v_{\bar{x}_i} \right\}}$$

Example 5.13

Assume Ideal gas. Given pressure and temperature measurement. R=54.7 ft-lbf/lbm.R. Instrument pressure uncertainty is 1% of reading. Instrument temperature uncertainty is 0.6 R

$$N_P = 20 \qquad \bar{p} = 2253.91 psfa \qquad s_p = 167.21 psfa$$

$$N_T = 10 \qquad \bar{T} = 560.4 R \qquad s_T = 3.0 R$$

$$\rho = \frac{P}{RT}$$

Find density uncertainty? $\rho' = \rho \pm u_\rho$

Pressure

$$(b_{\bar{p}})_1 = (B_p)/2 = 0.01 * 2253.5/2 = 11.28 \qquad (s_{\bar{p}})_1 = 0 \qquad (s_{\bar{p}})_2 = \frac{167.21}{20^{1/2}} = 37.4 \qquad (b_{\bar{p}})_2 = 0$$

Temperature

$$(b_{\bar{T}})_1 = (B_T)_1/2 = 0.3R \qquad (s_{\bar{T}})_1 = 0 \qquad (s_{\bar{T}})_2 = \frac{3}{10^{1/2}} = 0.9R \qquad (b_{\bar{T}})_2 = 0$$

$$b_{\bar{p}} = \left[11.28^2 + 0^2\right]^{1/2} = 11.28 \qquad s_{\bar{p}} = \left[0^2 + 37.4^2\right]^{1/2} = 37.4 \qquad b_T = \left[0.3^2 + 0^2\right]^{1/2} = 0.3 \qquad s_T = \left[0^2 + 0.9^2\right]^{1/2} = 0.9$$

$$s_{\bar{\rho}} = \left[\left(\frac{\partial \rho}{\partial T} s_{\bar{T}} \right)^2 + \left(\frac{\partial \rho}{\partial P} s_{\bar{P}} \right)^2 \right]^{1/2} = 0.0012 \quad \text{lbm/ft}^3$$

$$b_{\bar{\rho}} = \left[\left(\frac{\partial \rho}{\partial T} b_{\bar{T}} \right)^2 + \left(\frac{\partial \rho}{\partial P} b_{\bar{P}} \right)^2 \right]^{1/2} = 0.0004 \ \text{lbm/ft}^3$$

$$v = \frac{\left[\left(\frac{\partial \rho}{\partial T} s_{\bar{T}} \right)^2 + \left(\frac{\partial \rho}{\partial P} s_{\bar{P}} \right)^2 \right]^2}{\left(\frac{\partial \rho}{\partial T} s_{\bar{T}} \right)^4 / v_T + \left(\frac{\partial \rho}{\partial P} s_{\bar{P}} / v_P \right)^2} = 23$$

$$t_{v,P} = t_{23,95} = 2.06$$

$$u_\rho = t_{v,P} [b_{\bar{\rho}}{}^2 + s_{\bar{\rho}}{}^2]^{1/2} = \pm 0.0026 \, lbm / \, ft^3$$

$$\rho' = 0.074 \pm 0.0026 \ \ lbm / \ ft^3$$

Sections 5.9 & 5.10 are excluded.