CHAPTER 2

Random Variables and Their Probability Distributions

2.1 INTRODUCTION

In Chapter 1 we dealt essentially with random experiments which can be described by finite sample spaces. We studied the assignment and computation of probabilities of events. In practice, one observes a function defined on the space of outcomes. Thus, if a coin is tossed n times, one is not interested in knowing which of the 2^n n-tuples in the sample space has occurred. Rather, one would like to know the number of heads in n tosses. In games of chance one is interested in the net gain or loss of a certain player. Actually in Chapter 1 we were concerned with such functions without defining the term *random variable*. Here we study the notion of a random variable and examine some of its properties.

In Section 2 we define a random variable, while in Section 3 we study the notion of probability distribution of a random variable. Section 4 deals with some special types of random variables, and Section 5 considers functions of a random variable and their induced distributions.

The fundamental difference between a random variable and a real-valued function of a real variable is the associated notion of a probability distribution. Nevertheless our knowledge of advanced calculus or real analysis is the basic tool in the study of random variables and their probability distributions.

2.2 RANDOM VARIABLES

In Chapter 1 we studied properties of a set function P defined on a sample

space $(\Omega, \mathcal{S})$. Since P is a set function, it is not very easy to handle. Moreover, in practice one frequently observes some function of elementary events. When a coin is tossed repeatedly, which replication resulted in heads is not of much interest. Rather one is interested in the number of heads, and consequently the number of tails, that appear in, say, n tossings of the coin. It is therefore desirable to introduce a point function on the sample space. We can then use our knowledge of advanced calculus.

Definition 1. Let $(\Omega, \mathcal{S})$ be a sample space. A finite, single-valued function which maps Ω into $\mathcal{R}$ is called a random variable (rv) if the inverse images under X of all Borel sets in $\mathcal{R}$ are events, that is, if

(1) $$X^{-1}(B) = \{\omega: X(\omega) \in B\} \in \mathcal{S} \qquad \text{for all } B \in \mathfrak{B}.$$

Let $x \in \mathcal{R}$, and consider the semiclosed interval $(-\infty, x]$. Since $(-\infty, x] \in \mathfrak{B}$, it follows that if X is an rv, then $X^{-1}(-\infty, x] = \{X(\omega) \le x\}$ is an event in $\mathcal{S}$. Also, if B is a Borel set in $\mathcal{R}$, then B can be obtained by a countable number of operations of unions, intersections, and differences of semiclosed intervals. The following result is obtained, using the properties of inverse images under X (see P. 2. 16).

Theorem 1. X is an rv if and only if for each $x \in \mathcal{R}$

(2) $$\{\omega: X(\omega) \le x\} = \{X \le x\} \in \mathcal{S}.$$

Remark 1. Note that the notion of probability does not enter into the definition of an rv.

Remark 2. If X is an rv, the sets $\{X = x\}$, $\{a < X \le b\}$, $\{X < x\}$, $\{a \le X < b\}$, $\{a < X < b\}$, $\{a \le X \le b\}$ are all events. Indeed, we could have defined an rv in the following equivalent manner: X is an rv if and only if

(3) $$\{\omega: X(\omega) < x\} \in \mathcal{S} \qquad \text{for all } x \in \mathcal{R}.$$

We have

(4) $$\{X < x\} = \bigcup_{n=1}^{\infty} \left\{X \le x - \frac{1}{n}\right\}$$

and

(5) $$\{X \le x\} = \bigcap_{n=1}^{\infty} \left\{X < x + \frac{1}{n}\right\}.$$

Theorem 2. Let X be an rv defined on $(\Omega, \mathscr{S})$, and a, b be constants. Then $aX + b$ is also an rv on $(\Omega, \mathscr{S})$.

Proof.

$$\{\omega: aX(\omega) + b \le x\} = \{aX \le x - b\}.$$

If $a > 0$, then

$$\{aX \le x - b\} = \left\{X \le \frac{x-b}{a}\right\} \in \mathscr{S}.$$

If $a < 0$, then

$$\{aX \le x - b\} = \left\{X \ge \frac{x-b}{a}\right\} = \left\{X < \frac{x-b}{a}\right\}^c \in \mathscr{S}.$$

If $a = 0$, then

$$\{aX \le x - b\} = \begin{cases} \Omega & \text{if } x - b \ge 0, \\ \phi & \text{if } x - b < 0. \end{cases}$$

The proof is complete.

Example 1. For any set $A \subseteq \Omega$, define

$$I_A(\omega) = \begin{cases} 0, & \omega \notin A, \\ 1, & \omega \in A. \end{cases}$$

$I_A(\omega)$ is called the *indicator function* of set A. I_A is an rv if and only if $A \in \mathscr{S}$.

Example 2. Let $\Omega = \{\text{H}, \text{T}\}$, and $\mathscr{S}$ be the class of all subsets of Ω. Define X by $X(\text{H}) = 1$, $X(\text{T}) = 0$. Then

$$X^{-1}(-\infty, x] = \begin{cases} \phi & \text{if } x < 0, \\ \{\text{T}\} & \text{if } 0 \le x < 1, \\ \{\text{H}, \text{T}\} & \text{if } 1 \le x, \end{cases}$$

and we see that X is an rv.

Example 3. Let $\Omega = \{\text{HH}, \text{TT}, \text{HT}, \text{TH}\}$, and $\mathscr{S}$ be the class of all subsets of Ω. Define X by

$$X(\omega) = \text{number of H's in } \omega.$$

Then $X(\text{HH}) = 2$, $X(\text{HT}) = X(TH) = 1$, and $X(\text{TT}) = 0$.

$$X^{-1}(-\infty, x] = \begin{cases} \phi, & x < 0, \\ \{\text{TT}\}, & 0 \le x < 1, \\ \{\text{TT}, \text{HT}, \text{TH}\}, & 1 \le x < 2, \\ \Omega, & 2 \le x. \end{cases}$$

Thus X is an rv.

Remark 3. Let $(\Omega, \mathscr{S})$ be a discrete sample space; that is, let Ω be a countable set of points, and $\mathscr{S}$ be the class of all subsets of Ω. Then every numerical valued function defined on $(\Omega, \mathscr{S})$ is an rv.

Example 4. Let $\Omega = [0, 1]$ and $\mathscr{S} = \mathfrak{B} \cap [0, 1]$, be the σ-field of Borel sets on $[0, 1]$. Define X on Ω by

$$X(\omega) = \omega, \qquad \omega \in [0, 1].$$

Clearly X is an rv. Any Borel subset of Ω is an event.

Remark 4. Let X be an rv. Then X^2 is an rv and $1/X$ is also an rv, provided that $\{X = 0\} = \phi$. For $\{X^2 \le x\} = \phi$ if $x < 0$ and if $x \ge 0$, then $\{X^2 \le x\} = \{-\sqrt{x} \le X \le \sqrt{x}\} \in \mathscr{S}$. Similarly,

$$\begin{aligned} \left\{\frac{1}{X} \le x\right\} &= \left\{\frac{1}{X} \le x, X < 0\right\} + \left\{\frac{1}{X} \le x, X > 0\right\} + \left\{\frac{1}{X} \le x, X = 0\right\} \\ &= \{xX \le 1\} \cap \{X < 0\} + \{xX \ge 1\} \cap \{X > 0\} \\ &= \begin{cases} \{X < 0\} & \text{if } x = 0, \\ \left\{X \le \frac{1}{x}\right\} \cap \{X < 0\} + \left\{X \ge \frac{1}{x}\right\} \cap \{X > 0\} & \text{if } x > 0, \\ \left\{X \ge \frac{1}{x}\right\} \cap \{X < 0\} + \left\{X \le \frac{1}{x}\right\} \cap \{X > 0\} & \text{if } x < 0. \end{cases} \end{aligned}$$

For a general result see Theorem 2.5.1.

PROBLEMS 2.2

1. Let X be the number of heads in three tosses of a coin. What is Ω? What are the values that X assigns to points of Ω ? What are the events $\{X \le 2.75\}$, $\{.5 \le X \le 1.72\}$?

2. A die is tossed two times. Let X be the sum of face values on the two tosses, and Y be the absolute value of the difference in face values. What is Ω? What values do X and Y assign to points of Ω? Check to see whether X and Y are random variables.

3. Let X be an rv. Is $|X|$ also an rv? If X is an rv that takes only nonnegative values, is $\sqrt{X}$ also an rv?

4. A die is rolled five times. Let X be the sum of face values. Write the events $\{X = 4\}$, $\{X = 6\}$, $\{X = 30\}$, $\{X \geq 29\}$.

5. Let $\Omega = [0,1]$, and $\mathscr{S}$ be the Borel σ-field of subsets of Ω. Define X on Ω as follows: $X(\omega) = \omega$ if $0 \leq \omega \leq 1/2$, and $X(\omega) = \omega - 1/2$ if $1/2 < \omega \leq 1$. Is X an rv? If so, what is the event $\{\omega\colon X(\omega) \in (1/4, 1/2)\}$?

2.3 PROBABILITY DISTRIBUTION OF A RANDOM VARIABLE

In Section 2.2 we introduced the concept of an rv and noted that the concept of probability on the sample space was not involved in this definition. Let $(\Omega, \mathscr{S}, P)$ be a probability space, and let X be an rv defined on our probability space.

Theorem 1. The rv X defined on the probability space $(\Omega, \mathscr{S}, P)$ induces a probability space $(\mathscr{R}, \mathfrak{B}, Q)$ by means of the correspondence

$$Q(B) = P\{X^{-1}(B)\} = P\{\omega\colon X(\omega) \in B\} \qquad \text{for all } B \in \mathfrak{B}. \tag{1}$$

We write $Q = PX^{-1}$ and call Q or PX^{-1} the (probability) *distribution* of X.

Proof. Clearly $Q(B) \geq 0$ for all $B \in \mathfrak{B}$, and also $Q(\mathscr{R}) = P\{X \in \mathscr{R}\} = P(\Omega) = 1$.

Let $B_i \in \mathfrak{B}$, $i = 1, 2, \cdots$ with $B_i \cap B_j = \phi$, $i \neq j$. Since the inverse image of a disjoint union of Borel sets is the disjoint union of their inverse images, we have

$$\begin{aligned} Q(\sum_{i=1}^{\infty} B_i) &= P\{X^{-1}(\sum_{i=1}^{\infty} B_i)\} \\ &= P\{\sum_{i=1}^{\infty} X^{-1}(B_i)\} \\ &= \sum_{i=1}^{\infty} PX^{-1}(B_i) = \sum_{i=1}^{\infty} Q(B_i). \end{aligned}$$

It follows that $(\mathscr{R}, \mathfrak{B}, Q)$ is a probability space, and the proof is complete.

Since Q is a set function and set functions are not easy to handle, let us introduce a point function on $\mathscr{R}$.

Definition 1. A real-valued function F defined on $(-\infty, \infty)$ that is nondecreasing, right continous and satisfies

$$F(-\infty) = 0 \qquad \text{and} \qquad F(+\infty) = 1$$

is called a distribution function (df).

Remark 1. From our knowledge of calculus (see P.2. 10) we see that if F is a nondecreasing function on $\mathscr{R}$, then $F(x-) = \lim_{t \uparrow x} F(t)$, $F(x+) = \lim_{t \downarrow x} F(t)$ exist and are finite. Also, $F(+\infty)$ and $F(-\infty)$ exist as $\lim_{t \uparrow +\infty} F(t)$ and $\lim_{t \downarrow -\infty} F(t)$, respectively. In general,

$$F(x-) \leq F(x) \leq F(x+),$$

and x is a jump point of F if and only if $F(x+)$ and $F(x-)$ exist but are unequal. Thus a nondecreasing function F has only jump discontinuities. If we define

$$F^*(x) = F(x+) \qquad \text{for all } x,$$

we see that F^* is nondecreasing and right continuous on $\mathscr{R}$. Thus in Definition 1 the nondecreasing part is very important. Some authors demand left continuity in the definition of a df instead of right continuity.

Theorem 2. The set of discontinuity points of a df F is at most countable.

Proof. Let $(a, b]$ be a finite interval with at least n discontinuity points:

$$a < x_1 < x_2 < \cdots < x_n \leq b.$$

Then

$$F(a) \leq F(x_1-) < F(x_1) \leq \cdots \leq F(x_n-) < F(x_n) \leq F(b).$$

Let $p_k = F(x_k) - F(x_k-)$, $\quad k = 1, 2, \cdots, n$. Clearly

$$\sum_{k=1}^{n} p_k \leq F(b) - F(a),$$

and it follows that the number of points x in $(a, b]$ with jump $p(x) > \varepsilon > 0$ is at most $\varepsilon^{-1}\{F(b) - F(a)\}$. Thus, for every integer N, the number of discontinuity points with jump greater than $1/N$ is finite. It follows that there are no more than a countable number of discontinuity points in every finite interval $(a, b]$. Since $\mathscr{R}$ is a countable union of such intervals, the proof is complete.

Definition 2. Let X be an rv defined on $(\Omega, \mathscr{S}, P)$. Define a point function $F(.)$ on $\mathscr{R}$ by

$$F(x) = P\{\omega: X(\omega) \le x\} \quad \text{for all } x \in \mathscr{R}. \tag{2}$$

The function F is called the distribution function of rv X.

If there is no confusion, we will write

$$F(x) = P\{X \le x\}.$$

The following result justifies our calling F as defined by (2) a df.

Theorem 3. The function F defined in (2) is indeed a df.

Proof. Let $x_1 < x_2$. Then $(-\infty, x_1] \subset (-\infty, x_2]$, and we have

$$F(x_1) = P\{X \le x_1\} \le P\{X \le x_2\} = F(x_2).$$

Since F is nondecreasing, it is sufficient to show that for any sequence of numbers $x_n \downarrow x$, $x_1 > x_2 > \cdots > x_n > \cdots > x$, $F(x_n) \to F(x)$. Let $A_k = \{\omega: X(\omega) \in (x, x_k]\}$. Then $A_k \in \mathscr{S}$ and $A_k \downarrow$. Also,

$$\lim_{k\to\infty} A_k = \bigcap_{k=1}^{\infty} A_k = \phi,$$

since none of the intervals $(x, x_k]$ contains x. It follows that $\lim_{k\to\infty} P(A_k) = 0$. But

$$\begin{aligned} P(A_k) &= P\{X \le x_k\} - P\{X \le x\} \\ &= F(x_k) - F(x), \end{aligned}$$

so that

$$\lim_{k\to\infty} F(x_k) = F(x),$$

and F is right continuous.

Finally, let x_n be a sequence of numbers decreasing to $-\infty$. Then

$$\{X \le x_n\} \supseteq \{X \le x_{n+1}\} \quad \text{for each } n$$

and

$$\lim_{n\to\infty} \{X \le x_n\} = \bigcap_{n=1}^{\infty} \{X \le x_n\} = \phi.$$

Therefore

$$F(-\infty) = \lim_{n\to\infty} P\{X \le x_n\} = P\{\lim_{n\to\infty} \{X \le x_n\}\} = 0.$$

Similarly,

$$F(+\infty) = \lim_{x_n\to\infty} P\{X \le x_n\} = 1,$$

and the proof is complete.

The next result, which we state without proof, establishes a correspondence between the induced probability Q on $(\mathscr{R}, \mathfrak{B})$ and a point function F defined on $\mathscr{R}$.

Theorem 4. Given a probability Q on $(\mathscr{R}, \mathfrak{B})$, there exists a distribution function F satisfying

$$Q(-\infty, x] = F(x) \quad \text{for all } x \in \mathscr{R}, \tag{3}$$

and, conversely, given a df F, there exists a unique probability Q defined on $(\mathscr{R}, \mathfrak{B})$ that satisfies (3).

For proof see Chung [15], pages 23–24.

Theorem 5. Every df is the df of an rv on some probability space.

Proof. Let F be a df. From Theorem 4 it follows that there exists a unique probability Q defined on $\mathscr{R}$ that satisfies

$$Q(-\infty, x] = F(x) \quad \text{for all } x \in \mathscr{R}.$$

Let $(\mathscr{R}, \mathfrak{B}, Q)$ be the probability space on which we define

$$X(\omega) = \omega, \quad \omega \in \mathscr{R}.$$

Then

$$Q\{\omega: X(\omega) \le x\} = Q(-\infty, x] = F(x),$$

and F is the df of rv X.

Remark 2. If X is an rv on $(\Omega, \mathscr{S}, P)$, we have seen (Theorem 3) that $F(x) = P\{X \le x\}$ is a df associated with X. Theorem 5 assures us that to every df F we can associate some rv. Thus, given an rv, there exists a df, and conversely. In this book when we speak of an rv we will assume that it is defined on some probability space.

Example 1. Let X be defined on $(\Omega, \mathscr{S}, P)$ by

$$X(\omega) = c \quad \text{for all } \omega \in \Omega.$$

Then

$$P\{X = c\} = 1,$$
$$F(x) = Q(-\infty, x] = P\{X^{-1}(-\infty, x]\} = 0 \quad \text{if} \quad x < c$$

and

$$F(x) = 1 \quad \text{if} \quad x \geq c.$$

Example 2. Let $\Omega = \{H, T\}$, and X be defined by

$$X(\mathrm{H}) = 1, \qquad X(\mathrm{T}) = 0.$$

If P assigns equal mass to $\{H\}$ and $\{T\}$, then

$$P\{X = 0\} = \tfrac{1}{2} = P\{X = 1\},$$

and

$$F(x) = Q(-\infty, x] = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \leq x < 1, \\ 1, & 1 \leq x. \end{cases}$$

Example 3. Let $\Omega = \{(i, j): i, j \in \{1, 2, 3, 4, 5, 6\}\}$, and $\mathscr{S}$ be the set of all subsets of Ω. Let $P\{(i, j)\} = 1/6^2$ for all 6^2 pairs (i, j) in Ω. Define

$$X(i, j) = i + j, \qquad 1 \leq i, \quad j \leq 6.$$

Then

$$F(x) = Q(-\infty, x] = P\{X \leq x\} = \begin{cases} 0, & x < 2, \\ \frac{1}{36}, & 2 \leq x < 3, \\ \frac{3}{36}, & 3 \leq x < 4, \\ \frac{6}{36}, & 4 \leq x < 5, \\ \vdots & \vdots \\ \frac{35}{36}, & 11 \leq x < 12, \\ 1, & 12 \leq x. \end{cases}$$

Example 4. We return to Example 2.2.4. For every subinterval I of $[0, 1]$ let $P(\mathrm{I})$ be the length of the interval. Then $(\Omega, \mathscr{S}, P)$ is a probability space, and the df of rv $X(\omega) = \omega$, $\omega \in \Omega$, is given by $F(x) = 0$ if $x < 0$, $F(x) = P\{\omega: X(\omega) \leq x\} = P([0, x]) = x$ if $x \in [0, 1]$, and $F(x) = 1$ if $x \geq 1$.

PROBLEMS 2.3

1. Write the df of rv X defined in Problem 2.2.1, assuming that the coin is fair.
2. What is the df of rv Y defined in Problem 2.2.2, assuming that the die is not loaded?

3. Do the following functions define df's?
   (a) $F(x) = 0$ if $x < 0$, $= x$ if $0 \leq x < 1/2$, and $= 1$ if $x \geq \frac{1}{2}$.
   (b) $F(x) = (1/\pi) \tan^{-1} x$, $-\infty < x < \infty$.
   (c) $F(x) = 0$ if $x \leq 1$, and $= 1 - (1/x)$ if $1 < x$.
   (d) $F(x) = 1 - e^{-x}$ if $x \geq 0$, and $= 0$ if $x < 0$.
4. Let X be an rv with df F.
   (a) If F is the df defined in Problem 3(a), find $P\{X > \frac{1}{4}\}$, $P\{\frac{1}{3} < X \leq \frac{3}{8}\}$.
   (b) If F is the df defined in Problem 3(d), find $P\{-\infty < X < 2\}$.

2.4 DISCRETE AND CONTINUOUS RANDOM VARIABLES

Let X be an rv defined on some fixed, but otherwise arbitrary, probability space $(\Omega, \mathscr{S}, P)$, and let F be the df of X. In this book we shall restrict ourselves to two types of rv's, namely, the case in which the rv assumes at most a countable number of values, and that in which the df F is absolutely continuous (see P.2.15).

Definition 1. An rv X defined on $(\Omega, \mathscr{S}, P)$ is said to be of the discrete type, or simply discrete, if there exists a countable set $E \subseteq \mathscr{R}$ such that $P\{X \in E\} = 1$. The points of E which have positive mass are called jump points or points of increase of the df of X, and their probabilities are called jumps of the df.

Note that $E \in \mathfrak{B}$ since every one-point set is in $\mathfrak{B}$. Indeed, if $x \in \mathscr{R}$, then

$$\{x\} = \bigcap_{n=1}^{\infty} \left\{\left(x - \frac{1}{n} < x \leq x + \frac{1}{n}\right)\right\}. \tag{1}$$

Thus $\{X \in E\}$ is an event. Let X take on the value x_i with probability $p_i (i = 1, 2, \cdots)$. We have

$$P\{\omega: X(\omega) = x_i\} = p_i, \qquad i = 1, 2, \cdots, \quad p_i \geq 0 \quad \text{for all } i.$$

Then $\sum_{i=1}^{\infty} p_i = 1$.

Definition 2. The collection of numbers $\{p_i\}$ satisfying $P\{X = x_i\} = p_i \geq 0$, for all i and $\sum_{i=1}^{\infty} p_i = 1$, is called the probability mass function (pmf) of rv X.

The df F of X is given by

$$F(x) = P\{X \leq x\} = \sum_{x_i \leq x} p_i. \tag{2}$$

If I_A denotes the indicator function of the set A, we may write

$$X(\omega) = \sum_{i=1}^{\infty} x_i \, I_{[X = x_i]}(\omega). \tag{3}$$

Let us define a function $\varepsilon(x)$ as follows:

$$\varepsilon(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Then we have

$$F(x) = \sum_{i=1}^{\infty} p_i \varepsilon(x - x_i). \tag{4}$$

Example 1. The simplest example is that of an rv X degenerate at c, $P\{X = c\} = 1$:

$$F(x) = \varepsilon(x-c) = \begin{cases} 0, & x < c, \\ 1, & x \geq c. \end{cases}$$

Example 2. A box contains good and defective items. If an item drawn is good, we assign the number 1 to the drawing; otherwise, the number 0. Let p be the probability of drawing at random a good item. Then

$$P\left\{X = \begin{matrix} 0 \\ 1 \end{matrix}\right\} = \begin{cases} 1 - p, \\ p, \end{cases}$$

and

$$F(x) = P\{X \leq x\} = \begin{cases} 0, & x < 0, \\ 1 - p, & 0 \leq x < 1, \\ 1, & 1 \leq x. \end{cases}$$

Example 3. Let X be an rv with pmf

$$P\{X = k\} = \frac{6}{\pi^2} \cdot \frac{1}{k^2}, \qquad k = 1, 2, \cdots.$$

Then

$$F(x) = \frac{6}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \varepsilon(x - k).$$

The following result is obvious.

Theorem 1. Let $\{p_k\}$ be a collection of nonnegative real numbers such that $\sum_{k=1}^{\infty} p_k = 1$. Then $\{p_k\}$ is the pmf of some rv X.

We next consider rv's associated with df's that have no jump points. The

df of such rv's is continuous. We shall restrict our attention to a special subclass of such rv's.

Definition 3. Let X be an rv defined on $(\Omega, \mathscr{S}, P)$ with df F. Then X is said to be of the continuous type (or, simply, continuous) if F is absolutely continuous, that is, if there exists a nonnegative function $f(x)$ such that for every real number x we have

$$F(x) = \int_{-\infty}^{x} f(t)\, dt. \tag{5}$$

The function f is called the probability density function (pdf) of the rv X.

Note that $f \geq 0$ and satisfies $\lim_{x \to +\infty} F(x) = F(+\infty) = \int_{-\infty}^{\infty} f(t)\, dt = 1$. Let a and b be any two real numbers with $a < b$. Then

$$\begin{aligned} P\{a < X \leq b\} &= F(b) - F(a) \\ &= \int_a^b f(t)\, dt. \end{aligned}$$

Let B be a Borel set of the real line. Since B can be obtained by a countable number of operations of unions, intersections, and differences on intervals, the following result holds.

Theorem 2. Let X be an rv of the continuous type with pdf f. Then for every Borel set $B \in \mathfrak{B}$

$$P(B) = \int_B f(t)\, dt. \tag{6}$$

If F is absolutely continuous and f is continuous at x, we have

$$F'(x) = \frac{dF(x)}{dx} = f(x). \tag{7}$$

Theorem 3. Every nonnegative real function f that is integrable over $\mathscr{R}$ and satisfies

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

is the pdf of some continuous type rv X.

Proof. In view of Theorem 2.3.5 it suffices to show that there corresponds a df F to f. Define

$$F(x) = \int_{-\infty}^{x} f(t)\, dt, \qquad x \in \mathscr{R}.$$

Then $F(-\infty) = 0$, $F(+\infty) = 1$, and, if $x_2 > x_1$,

$$F(x_2) = \left(\int_{-\infty}^{x_1} + \int_{x_1}^{x_2}\right) f(t)\,dt \geq \int_{-\infty}^{x_1} f(t)\,dt = F(x_1).$$

Finally, F is (absolutely) continuous and hence continuous from the right.

Remark 1. In the discrete case, $P\{X = a\}$ is the probability that X takes the value a. In the continuous case, $f(a)$ is not the probability that X takes the value a. Indeed, if X is of the continuous type, it assumes every value with probability 0.

Theorem 4 Let X be any rv. Then

$$P\{X = a\} = \lim_{\substack{t \to a \\ t < a}} P\{t < X \leq a\}. \tag{8}$$

Proof. Let $t_1 < t_2 < \cdots < a$, $t_n \to a$, and write

$$A_n = \{t_n < X \leq a\}.$$

Then A_n is a nonincreasing sequence of events which converges to $\bigcap_{n=1}^{\infty} A_n = \{X = a\}$. It follows that $\lim_{n\to\infty} PA_n = P\{X = a\}$.

Remark 2. Since $P\{t < X \leq a\} = F(a) - F(t)$, it follows that

$$\lim_{\substack{t \to a \\ t < a}} P\{t < X \leq a\} = P\{X = a\} = F(a) - \lim_{\substack{t \to a \\ t < a}} F(t)$$
$$= F(a) - F(a-).$$

Thus F has a jump disontinuity at a if and only if $P\{X = a\} > 0$, that is, F is continuous at a if and only if $P\{X = a\} = 0$. If X is an rv of the continuous type, $P\{X = a\} = 0$ for all $a \in \mathcal{R}$. Moreover,

$$P\{X \in \mathcal{R} - \{a\}\} = 1.$$

This justifies Remark 4 in Section 1.3.

Example 4. Let X be an rv with df F given by (Fig. 1)

$$F(x) = \begin{cases} 0, & x \leq 0, \\ x, & 0 < x \leq 1, \\ 1, & 1 < x. \end{cases}$$

Differentiating F with respect to x at continuity points of f, we get

$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \text{ or } x > 1, \\ 1, & 0 < x < 1. \end{cases}$$

The function f is not continuous at $x = 0$, or at $x = 1$ (Fig. 2). We may define $f(0)$ and $f(1)$ in any manner. Choosing $f(0) = f(1) = 0$, we have

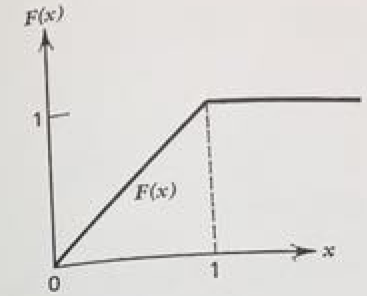

Fig. 1

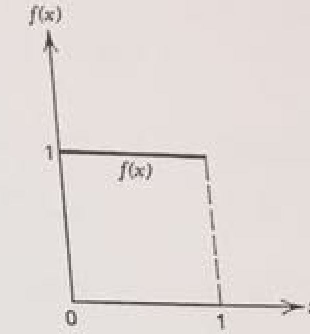

Fig. 2

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$P\{.4 < X \leq .6\} = F\{.6\} - F\{.4\} = .2.$$

Example 5. Let X have the *triangular* pdf (Fig. 3)

$$f(x) = \begin{cases} x, & 0 < x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that f is a pdf. For the df F of X we have (Fig. 4)

$$F(x) = 0 \qquad \text{if } x \leq 0,$$
$$F(x) = \int_0^x t\,dt = \frac{x^2}{2} \qquad \text{if } 0 < x \leq 1,$$
$$F(x) = \int_0^1 t\,dt + \int_1^x (2 - t)\,dt = 2x - \frac{x^2}{2} - 1 \qquad \text{if } 1 < x \leq 2,$$

and

$$F(x) = 1 \qquad \text{if } x \geq 2.$$

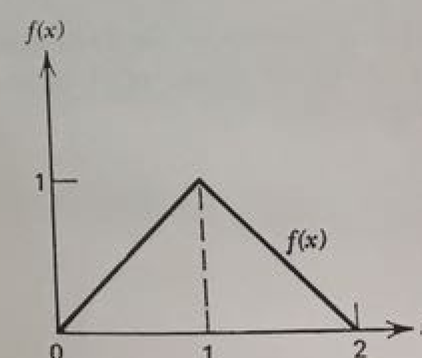

Fig. 3 Graph of f.

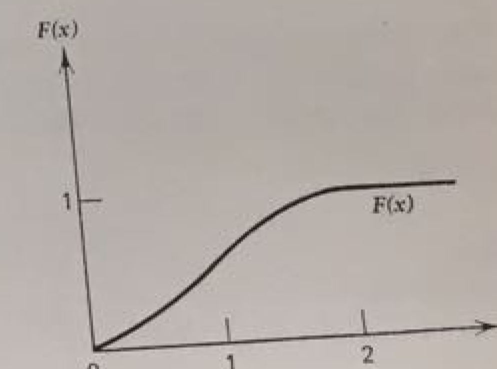

Fig. 4 Graph of F.

Then

$$P\{.3 < X \le 1.5\} = P\{X \le 1.5\} - P\{X \le .3\}$$
$$= .83.$$

Example 6. Let $k > 0$ be a constant, and

$$f(x) = \begin{cases} kx(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\int_0^1 f(x)\,dx = k/6$. It follows that $f(x)$ defines a pdf if $k = 6$. We have

$$P\{X > .3\} = 1 - 6\int_0^{.3} x(1-x)\,dx = .784.$$

We conclude this discussion by emphasizing that the two types of rv's considered above form only a small part of the class of all rv's. These two classes, however, contain practically all the random variables that arise in practice. We note without proof (see Chung [15], 9) that every df F can be decomposed into two parts according to

$$(9) \qquad F(x) = aF_d(x) + (1-a)F_c(x).$$

Here F_d and F_c are both df's; F_d is the df of a discrete rv, while F_c is a continuous (not necessarily absolutely continuous) df. In fact, F_c can be further decomposed, but we will not go into that. (See Chung [15], 11.)

Example 7. Let X be an rv with df

$$F(x) = \begin{cases} 0, & x < 0, \\ \dfrac{1}{2}, & x = 0, \\ \frac{1}{2} + \dfrac{x}{2}, & 0 < x < 1, \\ 1, & 1 \le x. \end{cases}$$

Note that the df F has a jump at $x = 0$ and F is continuous (in fact, absolutely continuous) in the interval (0, 1). F is the df of an rv X that is neither discrete nor continuous. We can write

$$F(x) = \frac{1}{2}F_d(x) + \frac{1}{2}F_c(x),$$

where

$$F_d(x) = \begin{cases} 0, & x = 0, \\ 1, & x \ge 0; \end{cases}$$

and

$$F_c(x) = \begin{cases} 0, & x \le 0, \\ x, & 0 < x < 1, \\ 1, & 1 \le x. \end{cases}$$

Here $F_d(x)$ is the df of the rv degenerate at $x = 0$, and $F_c(x)$ is the df with pdf

$$f_c(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

PROBLEMS 2.4

1. Let

$$p_k = p(1-p)^k, \qquad k = 0, 1, 2, \cdots, \quad 0 < p < 1.$$

Does $\{p_k\}$ define the pmf of some rv? What is the df of this rv? If X is an rv with pmf $\{p_k\}$, what is $P\{n \le X \le N\}$, where n, $N (N > n)$ are positive integers?

2. In Problem 2.3.3, find the pdf associated with the df's of parts (a),(c), and (d).

3. Does the function $f_\theta(x) = \theta^2 x e^{-\theta x}$ if $x > 0$, and $= 0$ if $x \le 0$, where $\theta > 0$, define a pdf? Find the df associated with $f_\theta(x)$; if X is an rv with pdf $f_\theta(x)$, find $P\{X \ge 1\}$.

4. Does the function $f_\theta(x) = (x+1)/[\theta(\theta+1)]e^{-x/\theta}$ if $x > 0$, and $= 0$ otherwise, where $\theta > 0$ define a pdf? Find the corresponding df.

5. For what values of K do the following functions define the pmf of some rv?
(a) $f(x) = K(\lambda^x/x!)$, $x = 0, 1, 2, \cdots, \lambda > 0$.
(b) $f(x) = K/N$, $x = 1, 2, \cdots, N$.

6. Show that the function

$$f(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty,$$

is a pdf.

7. For the pdf $f(x) = x$ if $0 \le x < 1$, and $= 2 - x$ if $1 \le x < 2$, find $P\{1/6 < X \le 7/4\}$.

8. Prove Theorems 1 and 2.

2.5 FUNCTIONS OF A RANDOM VARIABLE

Let X be an rv with a known distribution, and let g be a function defined on the real line. We seek the distribution of $Y = g(X)$, provided that Y is also an rv. We first prove the following result.

Theorem 1. Let X be an rv defined on $(\Omega, \mathscr{S}, P)$. Also, let g be a Borel-measurable function on $\mathscr{R}$. Then $g(X)$ is also an rv.

Proof.

$$\{g(X) \le y\} = \{X \in g^{-1}(-\infty, y]\},$$

and since g is Borel-measurable, $g^{-1}(-\infty, y]$ is a Borel set. It follows that $\{g(X) \le y\} \in \mathscr{S}$, and the proof is complete.

Theorem 2. Given an rv X with a known df, the distribution of the rv $Y = g(X)$, where g is a Borel-measurable function, is determined.

Proof. We have

$$P\{Y \le y\} = P\{X \in g^{-1}(-\infty, y]\}. \tag{1}$$

In what follows, we will always assume that the functions under consideration are Borel-measurable.

Example 1. Let X be an rv with df F. Then $|X|$, $aX + b$ (where $a \ne 0$ and, b are constants), X^k (where $k \ge 0$ is an integer), and $|X|^\alpha$ $(\alpha > 0)$ are all rv's. Define

$$X^+ = \begin{cases} X, & X \ge 0, \\ 0, & X < 0, \end{cases}$$

and

$$X^- = \begin{cases} X, & X \le 0, \\ 0, & X > 0. \end{cases}$$

Then X^+, X^- are also rv's. We have

$$\begin{aligned} P\{|X| \le y\} &= P\{-y \le X \le y\} = P\{X \le y\} - P\{X < -y\} \\ &= F(y) - F(-y) + P\{X = -y\}, \qquad y > 0; \end{aligned}$$

$$\begin{aligned} P\{aX + b \le y\} &= P\{aX \le y - b\} \\ &= \begin{cases} P\left\{X \le \dfrac{y-b}{a}\right\} & \text{if } a > 0, \\ P\left\{X \ge \dfrac{y-b}{a}\right\} & \text{if } a < 0; \end{cases} \end{aligned}$$

$$P\{X^+ \le y\} = \begin{cases} 0 & \text{if } y < 0, \\ P\{X \le 0\} & \text{if } y = 0, \\ P\{X < 0\} + P\{0 \le X \le y\} & \text{if } y > 0. \end{cases}$$

Similarly,

$$P\{X^- \le y\} = \begin{cases} 1 & \text{if } y \ge 0, \\ P\{X \le y\} & \text{if } y < 0. \end{cases}$$

Let X be an rv of the discrete type, and A be the countable set such that $P\{X \in A\} = 1$ and $P\{X = x\} > 0$ for $x \in A$. Let $Y = g(X)$ be a one-to-one mapping from A onto some set B. Then the inverse map, g^{-1}, is a single-valued function of y. To find $P\{Y = y\}$, we note that

$$\begin{aligned} P\{Y = y\} &= P\{g(X) = y\} = P\{X = g^{-1}(y)\}, \quad y \in B, \\ \text{and} \quad P\{Y = y\} &= 0, \quad y \in B^c. \end{aligned}$$

Example 2. Let X be a *Poisson* rv with pmf

$$P\{X = k\} = \begin{cases} e^{-\lambda} \dfrac{\lambda^k}{k!}, & k = 0, 1, 2, \cdots; \lambda > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = X^2 + 3$. Then $y = x^2 + 3$ maps $A = \{0, 1, 2, \cdots\}$ onto $B = \{3, 4, 7, 12, 19, 28, \cdots\}$. The inverse map is $x = \sqrt{(y-3)}$, and since there are no negative values in A we take the positive square root of $y - 3$. We have

$$P\{Y = y\} = P\{X = \sqrt{y-3}\} = \frac{e^{-\lambda} \lambda^{\sqrt{y-3}}}{\sqrt{(y-3)}!}, \qquad y \in B,$$

and $P\{Y = y\} = 0$ elsewhere.

Actually the restriction of a single-valued inverse on g is not necessary. If g has a finite (or even a countable) number of inverses for each y, from countable additivity of P we have

$$\begin{aligned} P\{Y = y\} &= P\{g(X) = y\} = P\{\bigcup_a [X = a, g(a) = y]\} \\ &= \sum_a P\{X = a, g(a) = y\}. \end{aligned}$$

Example 3. Let X be an rv with pmf

$$P\{X = -2\} = \tfrac{1}{5}, \quad P\{X = -1\} = \tfrac{1}{6}, \quad P\{X = 0\} = \tfrac{1}{5},$$
$$P\{X = 1\} = \tfrac{1}{15}, \quad \text{and} \quad P\{X = 2\} = \tfrac{11}{30}.$$

Let $Y = X^2$. Then

$$A = \{-2, -1, 0, 1, 2\}, \quad \text{and} \quad B = \{0, 1, 4\}.$$

We have

$$P\{Y=y\}=\begin{cases}\frac{1}{5} & y=0,\\ \frac{1}{6}+\frac{1}{15}=\frac{7}{30}, & y=1,\\ \frac{1}{5}+\frac{11}{30}=\frac{17}{30}, & y=4.\end{cases}$$

The case in which X is an rv of the continuous type is not as simple. First we note that, if X is a continuous type rv and g is some Borel-measurable function, $Y=g(X)$ may not be an rv of the continuous type.

Example 4. Let X be an rv with *uniform* distribution on $[-1, 1]$, that is, the pdf of X is $f(x)=1/2$, $-1 \le x \le 1$, and $=0$ elsewhere. Let $Y=X^+$. Then, from Example 1,

$$P\{Y \le y\}=\begin{cases}0, & y<0,\\ \frac{1}{2}, & y=0,\\ \frac{1}{2}+\frac{1}{2}y, & 1 \ge y > 0,\\ 1, & y>1.\end{cases}$$

We see that the df of Y has a jump at $y=0$ and that Y is neither discrete nor continuous. Note that all we require is that $P\{X<0\}>0$.

Example 4 shows that we need some conditions on g to ensure that $g(X)$ is also an rv of the continuous type whenever X is continuous. This will be the case when g is a continuous monotonic function (see P. 2.10). A sufficient condition is given in the following theorem.

Theorem 3. Let X be an rv of the continuous type with pdf f. Let $y=g(x)$ be differentiable for all x and either $g'(x)>0$ for all x or $g'(x)<0$ for all x. Then $Y=g(X)$ is also an rv of of the continuous type with pdf given by

$$h(y)=\begin{cases} f[g^{-1}(y)]\left|\frac{d}{dy}g^{-1}(y)\right|, & \alpha<y<\beta,\\ 0, & \text{otherwise,}\end{cases} \tag{2}$$

where $\alpha=\min\{g(-\infty), g(+\infty)\}$ and $\beta=\max\{g(-\infty), g(+\infty)\}$.

Proof. If g is differentiable for all x and $g'(x)>0$ for all x, then g is continuous and strictly increasing, the limits α, β exist (may be infinite), and the inverse function $x=g^{-1}(y)$ exists, is strictly increasing, and is differentiable (see P.2.10). The df of Y for $\alpha<y<\beta$ is given by

$$P\{Y\le y\}=P\{X \le g^{-1}(y)\}.$$

The pdf of g is obtained on differentiation. We have

$$\begin{aligned}h(y)&=\frac{d}{dy}P\{Y\le y\}\\ &=f[g^{-1}(y)]\frac{d}{dy}g^{-1}(y).\end{aligned}$$

Similarly, if $g'<0$, then g is strictly decreasing and we have

$$\begin{aligned}P\{Y\le y\}&=P\{X\ge g^{-1}(y)\}\\ &=1-P\{X\le g^{-1}(y)\} \qquad (X \text{ is a continuous type rv})\end{aligned}$$

so that

$$h(y)=-f[g^{-1}(y)]\cdot\frac{d}{dy}g^{-1}(y).$$

Since g and g^{-1} are both strictly decreasing, $d/dy\, g^{-1}(y)$ is negative and (2) follows.

Note that (see Theorem P.2.15)

$$\frac{d}{dy}g^{-1}(y)=\frac{1}{d\,g(x)/dx}\bigg|_{x=g^{-1}(y)},$$

so that (2) may be rewritten as

$$h(y)=\frac{f(x)}{|d\,g(x)/dx|}\bigg|_{x=g^{-1}(y)}, \qquad \alpha<y<\beta. \tag{3}$$

Remark 1. The key to computation of the induced distribution of $Y=g(X)$ from the distribution of X is (1). If the conditions of Theorem 3 are satisfied, we are able to identify the set $\{X\in g^{-1}(-\infty, y]\}$ as $\{X\le g^{-1}(y)\}$ or $\{X \ge g^{-1}(y)\}$, according to whether g is increasing or decreasing. In practice Theorem 3 is quite useful, but whenever the conditions are violated one should return to (1) to compute the induced distribution. This is the case, for example, in Examples 7 and 8 and Theorem 4 below.

Remark 2. If the pdf f of X vanishes outside an interval $[a, b]$ of finite length, we need only to assume that g is differentiable in (a, b), and either $g'(x)>0$ or $g'(x)<0$ throughout the interval. Then we take

$$\alpha=\min\{g(a), g(b)\} \quad \text{and} \quad \beta=\max\{g(a), g(b)\}$$

in Theorem 3.

Example 5. Let X have the density $f(x) = 1$, $0 < x < 1$, and $= 0$ otherwise. Let $Y = e^X$. Then $X = \log Y$, and we have

$$h(y) = \left|\frac{1}{y}\right| \cdot 1, \qquad 0 < \log y < 1,$$

that is,

$$h(y) = \begin{cases} \dfrac{1}{y}, & 1 < y < e, \\ 0, & \text{otherwise.} \end{cases}$$

If $Y = -2\log X$, then $x = e^{-y/2}$ and

$$\begin{aligned} h(y) &= \left|-\tfrac{1}{2} e^{-y/2}\right| \cdot 1, \qquad 0 < e^{-y/2} < 1, \\ &= \begin{cases} \tfrac{1}{2} e^{-y/2}, & 0 < y < \infty, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Example 6. Let X be a nonnegative rv of the continuous type with pdf f, and let $\alpha > 0$. Let $Y = X^\alpha$. Then

$$P\{X^\alpha \le y\} = \begin{cases} P\{X \le y^{1/\alpha}\} & \text{if } y \ge 0, \\ 0 & \text{if } y < 0. \end{cases}$$

The pdf of Y is given by

$$\begin{aligned} h(y) &= f(y^{1/\alpha}) \left|\frac{d}{dy} y^{1/\alpha}\right| \\ &= \begin{cases} \dfrac{1}{\alpha} y^{1/\alpha - 1} f(y^{1/\alpha}), & y > 0, \\ 0, & y \le 0. \end{cases} \end{aligned}$$

Example 7. Let X be an rv with pdf

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad -\infty < x < \infty.$$

Let $Y = X^2$. In this case, $g'(x) = 2x$ which is > 0 for $x > 0$, and < 0 for $x < 0$, so that the conditions of Theorem 3 are not satisfied. But for $y > 0$

$$\begin{aligned} P\{Y \le y\} &= P\{-\sqrt{y} \le X \le \sqrt{y}\} \\ &= F(\sqrt{y}) - F(-\sqrt{y}), \end{aligned}$$

where F is the df of X. Thus the pdf of Y is given by

$$h(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} \{f(\sqrt{y}) + f(-\sqrt{y})\}, & y > 0, \\ 0, & y \le 0. \end{cases}$$

Thus

$$h(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} e^{-y/2}, & 0 < y, \\ 0, & y \le 0. \end{cases}$$

Example 8. Let X be an rv with pdf

$$f(x) = \begin{cases} \dfrac{2x}{\pi^2}, & 0 < x < \pi, \\ 0, & \text{otherwise.} \end{cases}$$

Let $Y = \sin X$. In this case $g'(x) = \cos x > 0$ for x in $(0, \pi/2)$ and < 0 for x in $(\pi/2, \pi)$, so that the conditions of Theorem 3 are not satisfied. To compute the pdf of Y we return to (1) and see that the df of Y is given by

$$\begin{aligned} P\{Y \le y\} &= P\{\sin X \le y\}, \qquad 0 < y < 1, \\ &= P\{[0 \le X \le x_1] \cup [x_2 \le X \le \pi]\}, \end{aligned}$$

where $x_1 = \sin^{-1} y$ and $x_2 = \pi - \sin^{-1} y$. Thus

$$\begin{aligned} P\{Y \le y\} &= \int_0^{x_1} f(x)\,dx + \int_{x_2}^{\pi} f(x)\,dx \\ &= \left(\frac{x_1}{\pi}\right)^2 + 1 - \left(\frac{x_2}{\pi}\right)^2, \end{aligned}$$

and the pdf of Y is given by

$$\begin{aligned} h(y) &= \frac{d}{dy}\left(\frac{\sin^{-1} y}{\pi}\right)^2 + \frac{d}{dy}\left[1 - \left(\frac{\pi - \sin^{-1} y}{\pi}\right)^2\right] \\ &= \begin{cases} \dfrac{2}{\pi\sqrt{1 - y^2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In Examples 7 and 8 the function $y = g(x)$ can be written as the sum of two monotone functions. We applied Theorem 3 to each of these monotonic summands. These two examples are special cases of the following result, the proof of which the reader is asked to construct.

Theorem 4. Let X be an rv of the continuous type with pdf f. Let $y = g(x)$ be differentiable for all x, and assume that $g'(x)$ is continuous and nonzero at all but a finite number of values of x. Then, for every real number y,

(a) there exist a positive integer $n = n(y)$ and real numbers (inverses) $x_1(y), x_2(y), \cdots, x_n(y)$ such that

$$g[x_k(y)] = y, \qquad g'[x_k(y)] \ne 0, \quad k = 1, 2, \cdots, n(y),$$

or

(b) there does not exist any x such that $g(x) = y$, $g'(x) \neq 0$, in which case we write $n(y) = 0$.

Then Y is a continuous rv with pdf given by

$$h(y) = \begin{cases} \sum_{k=1}^{n} f[x_k(y)] \, |g'[x_k(y)]|^{-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Example 9. Let X be an rv with pdf f, and let $Y = |X|$. Here $n(y) = 2$, $x_1(y) = y$, $x_2(y) = -y$ for $y > 0$, and

$$h(y) = \begin{cases} f(y) + f(-y), & y > 0, \\ 0, & y \leq 0. \end{cases}$$

Thus, if $f(x) = 1/2$, $-1 \leq x \leq 1$, and $= 0$ otherwise, then

$$h(y) = \begin{cases} 1, & 0 \leq y \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $f(x) = (1/\sqrt{2\pi})\, e^{-(x^2/2)}$, $-\infty < x < \infty$, then

$$h(y) = \begin{cases} \dfrac{2}{\sqrt{2\pi}} e^{-(y^2/2)}, & y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 10. Let X be an rv of the continuous type with pdf f, and let $Y = X^{2m}$, where m is a positive integer. In this case $g(x) = x^{2m}$, $g'(x) = 2mx^{2m-1} > 0$ for $x > 0$ and $g'(x) < 0$ for $x < 0$. Writing $n = 2m$, we see that, for any $y > 0$, $n(y) = 2$, $x_1(y) = -y^{1/n}$, $x_2(y) = y^{1/n}$. It follows that

$$h(y) = f[x_1(y)] \cdot \frac{1}{ny^{1-1/n}} + f[x_2(y)] \frac{1}{ny^{1-1/n}}$$

$$= \begin{cases} \dfrac{1}{ny^{1-1/n}} \{f(y^{1/n}) + f(-y^{1/n})\} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

In particular, if f is the pdf given in Example 7, then

$$h(y) = \begin{cases} \dfrac{2}{\sqrt{2\pi}\, ny^{1-1/n}} \exp\left\{-\dfrac{y^{2/n}}{2}\right\} & \text{if } y > 0, \\ 0 & \text{if } y \leq 0. \end{cases}$$

Remark 3. The basic formula (1) and the countable additivity of probability allow us to compute the distribution of $Y = g(X)$ in some instances even if g has a countable number of inverses. Let $A \subseteq \mathscr{R}$ and g map A into $B \subseteq \mathscr{R}$. Suppose that A can be represented as a countable union of disjoint

sets A_k, $k = 1, 2, \cdots$. Then the df of Y is given by

$$\begin{aligned} P\{Y \leq y\} &= P\{X \in g^{-1}(-\infty, y]\} \\ &= P\left\{X \in \sum_{k=1}^{\infty} [\{g^{-1}(-\infty, y]\} \cap A_k]\right\} \\ &= \sum_{k=1}^{\infty} P\{X \in A_k \cap \{g^{-1}(-\infty, y]\}\}. \end{aligned}$$

If the conditions of Theorem 3 are satisfied by the restriction of g to each A_k, we may obtain the pdf of Y on differentiating the df of Y. We remind the reader that term-by-term differentiation is permissible if the differentiated series is uniformly convergent (see Theorem P.2.19).

Example 11. Let X be an rv with pdf

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0, \\ 0, & x \leq 0, \end{cases} \qquad \theta > 0.$$

Let $Y = \sin X$, and let $\sin^{-1} y$ be the principal value. Then, for $0 < y < 1$,

$$\begin{aligned} &P\{\sin X \leq y\} \\ &= P\{0 < X \leq \sin^{-1} y \text{ or } (2n-1)\pi - \sin^{-1} y \leq X \leq 2n\pi + \sin^{-1} y \\ &\qquad \text{for all integers } n \geq 1\} \\ &= P\{0 < X \leq \sin^{-1} y\} + \sum_{n=1}^{\infty} P\{(2n-1)\pi - \sin^{-1} y \leq X \leq 2n\pi + \sin^{-1} y\} \\ &= 1 - e^{-\theta \sin^{-1} y} + \sum_{n=1}^{\infty} [e^{-\theta((2n-1)\pi - \sin^{-1} y)} - e^{-\theta(2n\pi + \sin^{-1} y)}] \\ &= 1 - e^{-\theta \sin^{-1} y} + (e^{\theta\pi + \theta \sin^{-1} y} - e^{-\theta \sin^{-1} y}) \sum_{n=1}^{\infty} e^{-(2\theta\pi)n} \\ &= 1 - e^{-\theta \sin^{-1} y} + (e^{\theta\pi + \theta \sin^{-1} y} - e^{-\theta \sin^{-1} y}) \left(\frac{e^{-2\theta\pi}}{1 - e^{-2\theta\pi}}\right) \\ &= 1 + \frac{e^{-\theta\pi + \theta \sin^{-1} y} - e^{-\theta \sin^{-1} y}}{1 - e^{-2\pi\theta}}. \end{aligned}$$

A similar computation can be made for $y < 0$. If follows that the pdf of Y is given by

$$h(y) = \begin{cases} \theta e^{-\theta\pi} (1 - e^{-2\theta\pi})^{-1} (1 - y^2)^{-1/2} [e^{\theta \sin^{-1} y} + e^{-\theta\pi - \theta \sin^{-1} y}] & \text{if } -1 < y < 0, \\ \theta (1 - e^{-2\theta\pi})^{-1} (1 - y^2)^{-1/2} [e^{-\theta \sin^{-1} y} + e^{-\theta\pi + \theta \sin^{-1} y}] & \text{if } 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

PROBLEMS 2.5

1. Let X be a random variable with probability mass function

$$P\{X = r\} = \binom{n}{r} p^r (1 - p)^{n-r}, \qquad r = 0, 1, 2, \cdots, n, \quad 0 \leq p \leq 1.$$

Find the pmf's of the rv's (a) $Y = aX + b$, (b) $Y = X^2$, (c) $Y = \sqrt{X}$.

2. Let X be an rv with pdf

$$f(x) = \begin{cases} 0 & \text{if } x \le 0, \\ \dfrac{1}{2} & \text{if } 0 < x \le 1, \\ \dfrac{1}{2x^2} & \text{if } 1 < x < \infty. \end{cases}$$

Find the pdf of the rv $1/X$.

3. Let X be a positive rv of the continuous type with pdf $f(\cdot)$. Find the pdf of the rv $U = X/(1 + X)$. If, in particular, X has the pdf

$$f(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

what is the pdf of U?

4. Let X be an rv with pdf f defined in Example 11. Let $Y = \cos X$, and $Z = \tan X$. Find the df's of Y and Z.

5. Let X be an rv with pdf

$$f_\theta(x) = \begin{cases} \theta e^{-\theta x} & \text{if } x \ge 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = [X - 1/\theta]^2$. Find the pdf of Y.

6. A point is chosen at random on the circumference of a circle of radius r with center at the origin, that is, the polar angle θ of the point chosen has the pdf

$$f(\theta) = \frac{1}{2\pi}, \qquad \theta \in (-\pi, \pi).$$

Find the pdf of the abscissa of the point selected.

7. For the rv X of Example 7 find the pdf of the following rv's: (a) $Y_1 = e^X$, (b) $Y_2 = 2X^2 + 1$, (c) $Y_3 = g(X)$, where $g(x) = 1$ if $x > 0$, $= 1/2$ if $x = 0$, and $= -1$ if $x < 0$.

8. Suppose that a projectile is fired at an angle θ above the earth with a velocity V. Assuming that θ is an rv with pdf

$$f(\theta) = \begin{cases} \dfrac{12}{\pi} & \text{if } \dfrac{\pi}{6} < \theta < \dfrac{\pi}{4}, \\ 0 & \text{otherwise,} \end{cases}$$

find the pdf of the range R of the projectile, where $R = V^2 \sin 2\theta/g$, g being the gravitational constant.

9. Let X be an rv with pdf $f(x) = 1/(2\pi)$ if $0 < x < 2\pi$, and $= 0$ otherwise. Let $Y = \sin X$. Find the df and pdf of Y.

10. Let X be an rv with pdf $f(x) = 1/3$ if $-1 < x < 2$, and $= 0$ otherwise. Let $Y = |X|$. Find the pdf of Y.

11. Let X be an rv with pdf $f(x) = (1/2\theta)$ if $-\theta \le x \le \theta$, and $= 0$ otherwise. Let $Y = 1/X^2$. Find the pdf of Y.

12. Let X be an rv of the continuous type, and let $Y = g(X)$ be defined as follows:
(a) $g(x) = 1$ if $x > 0$, and $= -1$ if $x \le 0$.
(b) $g(x) = b$ if $x \ge b$, $= x$ if $|x| < b$, and $= -b$ if $x \le -b$.
(c) $g(x) = x$ if $|x| \ge b$, and $= 0$ if $|x| < b$.
Find the distribution of Y in each case.

13. Prove Theorem 4.