Math 311
The Final Exam
Name:
Student number:

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(I) For each of the following decide whether $(i) \Rightarrow (ii)$ **or** $(ii) \Rightarrow (i)$ **or** $(ii) \Leftrightarrow (i)$ **or** none of the above:

(1) (i) S has an infimum
(ii) S has a minimum

(2) (i) A is a countable set
(ii) A is an open set

(3) (i) A is a closed and bounded set
(ii) A is a compact set

(4) (i) (x_n) is a monotone sequence
(ii) (x_n) is a convergent sequence

(5) (i) (x_n) is a convergent sequence
(ii) (x_n) has a convergent subsequence

(6) (i) f is a continuous function on the set A
(ii) f is a bounded function on the set A

(7) (i) f is continuous on the set A
(ii) f is uniformly continuous on the set A

(8) (i) f is continuous at a
(ii) f is differentiable at a

(II) Give the conditions that make each of the following statements hold:

(1) $|a+b| = |a| + |b|$

(2) The set S has a supremum.

(3) $\liminf x_n = \limsup x_n$

(4) The function f has an absolute maximum and an absolute minimum.

(5) The function f has a fixed point in [0 , 1]

(6) The function f is constant on (a , b).

(7) $\exists c \in (a,b)$ such that $f'(c) = 0$.

(8) $\exists c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$.

(III) Give an example for each of the following <u>(if any)</u>. If there is no example <u>explain why</u>:

(1) A rational number that is not an algebraic number.

(2) A finite set with no supremum.

(3) A set G with $G \bigcap G' = \varphi$.

(4) A set G with $\overset{o}{G} = \varphi$ and $\overline{G} = R$.

(5) A function f and two sets $B \subset S$ such that f is uniformly continuous on B but not uniformly continuous on S.

(6) A function that is uniformly continuous on R.

(7) A bounded continuous function f on a set A that has no absolute maximum.

(8) A function that is differentiable at a but not continuous at a.

(IV) Answer the following questions:

(1) Prove that between any two real numbers there is a rational number (assume that $0 < x < y$)

(2)Prove that the intersection of an arbitrary collection of compact sets is compact.

(3) If (x_n) converges to x, where $x \neq 0$ and $\forall n \; x_n \neq 0$, prove that

$$\lim_{n\to\infty}\left(\frac{1}{x_n}\right) = \frac{1}{x}$$

(4) Let $f : A \to R$ be a uniformly continuous function on A. Prove that if (x_n) is a Cauchy sequence in A then $(f(x_n))$ is a Cauchy sequence in R.

(5) (a) State and prove the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to prove that $f(x) = x^3 + 4x + 4$ has a root.

(6) Let $f(x) = \begin{Bmatrix} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{Bmatrix}$

What is $\lim_{x\to 0} f(x)$?

Is f continuous at $x = 0$? Why?

Is f differentiable at $x = 0$? Why?

(7) Show that $\sin x \leq x \quad \forall x \geq 0$

(8) Let f be defined on an open interval containing x_o . If f assumes its maximum at x_o and is differentiable at x_o, prove that $f'(x_o) = 0$.