Homework 9

9.1 Show that $\lim_{x\to c}\sqrt{x} = \sqrt{c}$ for $c > 0$.

9.2 Use the ε-δ definition of limit and the sequential criterion for limits to establish that $\lim_{x\to c} x^3 = c^3$.

9.3 For each of the following limits determine if the limit exist or not. Explain your answer:

$$\lim_{x\to 0}\frac{1}{x^2} \quad \text{for } x > 0$$

$$\lim_{x\to 0}\cos\left(\frac{1}{\sqrt{x}}\right)$$

$$\lim_{x\to 0} x\cos\left(\frac{1}{\sqrt{x}}\right)$$

9.4 Let $c \in R$ and let $f : R \to R$ be such that $\lim_{x\to c}\left(f(x)\right)^2 = L$.

(a) Show that if $L = 0$, then $\lim_{x\to c} f(x) = L$.

(b) Show by example that if $L \neq 0$, then f may not have a limit at c.

Homework 10

10.1 Let $A \subseteq B \subseteq R$, let $f : B \to R$ and let g be the restriction of f to A (that is, $g(x) = f(x)$ for $x \in S$).

(a) If f is continuous at $c \in A$, show that g is continuous at c.

(b) Show by example that if g is continuous at c, it need not follow that f is continuous at c.

10.2 Let $K > 0$ and let $f : R \to R$ satiafy the condition $|f(x) - f(y)| \leq K|x - y|$ For all $x, y \in R$. Show that f is continuous at every point in R.

10.3 Give examples of functions f and g that are both discontinuous at a point $c \in R$ such that

(a) the sum $f + g$ is continuous at c, (b) the product fg is continuous at c.

10.4 Give an example of a function $f:[0,1]\to R$ that is discontinuous at every point of [0,1] but such that $|f|$ is continuous on [0,1].

10.5 If *f* and *g* are continuous on *R*, let $S=\{x\in R: f(x)\ge g(x)\}$. If $(s_n)\subseteq S$ and $\lim_{n\to\infty} s_n = s$, show that $s\in S$.

10.6 Let *f* and *g* be real-valued functions.

(a) Show that $\min(f,g)=\frac{1}{2}(f+g)-\frac{1}{2}|f-g|$.

(b) Prove that if *f* and *g* are continuous at x_o in *R*, then min (*f* , *g*) is continuous at x_o.

10.7 Use the ε-δ definition to prove that the following functions are continuous at x_o

(a) $f(x)=\begin{cases} x\sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ $\qquad x_o=0$

10.8 Prove that the following functions are discontinuous at the indicated points. You may use the ε-δ definition or the discontinuity criterion.

(a) $f(x)=\begin{cases} 1 & x>0 \\ 0 & x\le 0 \end{cases}$ $\qquad x_o=0$

(b) $g(x)=\begin{cases} \sin\frac{1}{x} & x\neq 0 \\ 0 & x=0 \end{cases}$ $\qquad x_o=0$

(c) $\operatorname{sgn}(x)=\begin{cases} -1 & x<0 \\ 1 & x>0 \\ 0 & x=0 \end{cases}$ $\qquad x_o=0$

Homework 11

11.1 If $f:[0,1]\to R$ is continuous and has only rational values, must *f* be constant? Prove your assertion.

11.2 Let *f* and *g* be continuous functions on [*a*,*b*] such that let $f(a)\ge g(a)$ and $f(b)\le g(b)$. Prove that $f(x_o)=g(x_o)$ for at least one x_o in [*a*,*b*].

11.3 Prove that there is at least one $x \in R$ such that $e^x = \cos x + 1$.

11.4 Suppose that f is a real-valued continuous function on R and that $f(a)f(b) < 0$ for some $a, b \in R$. Prove that there exists x between a and b such that $f(x) = 0$

Homework 12

12.1 Show that the function $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on $A = [1, \infty]$, but that it is not uniformly continuous on $B = (0, \infty)$.

12.2 Show that if f and g are uniformly continuous on a subset A of R and if they are both bounded on A, then $f\,g$ is uniformly continuous on A.

12.3 If f is uniformly continuous on a subset A of R, and $|f(x)| \geq k > 0$ for all $x \in A$, show that $\dfrac{1}{f}$ is uniformly continuous on A.

12.4 Prove that if f is uniformly continuous on a bounded subset A of R, then f is bounded on A.

12.5 Which of the following continuous functions are uniformly continuous on the indicated set? Justify your answer.
(a) $f(x) = x^3$ on $[0,1]$
(b) $f(x) = x^3$ on $(0,1)$

12.6 Use the ε-δ definition to prove that the following functions are uniformly continuous on the indicated set.
(a) $f(x) = 3x + 11$ on R
(b) $f(x) = x^2$ on $[0,3]$