Homework 6

6.1 Prove the following:

(a) $\lim_{n\to\infty}\left[\frac{2n}{3n^3+1}\right]=0$ (b) $\lim_{n\to\infty}\left[\frac{(-1)^{n+1}}{n^2}\right]=0$

6.2 Determine the limits of the following sequences and then prove your claim:

(a) $a_n=\frac{1}{n}\sin n$ (b) $a_n=\frac{(-1)^n}{2}$

6.3 Let (s_n) be a sequence of nonnegative real numbers and suppose that $\lim_{n\to\infty} s_n=0$. Prove that $\lim_{n\to\infty}\sqrt{s_n}=0$. This will complete the proof of example 3.7

6.4 (a)Consider three sequences (a_n), (b_n) and (s_n) such that $a_n \le s_n \le b_n$ for all n and $\lim_{n\to\infty} a_n=\lim_{n\to\infty} b_n = s$. Prove that $\lim_{n\to\infty} s_n = s$.

(b) Suppose that (s_n) and (t_n) are sequences such that $|s_n|\le t_n$ for all n and $\lim_{n\to\infty} t_n=0$. Prove that $\lim_{n\to\infty} s_n=0$.

6.5 Let (t_n) be a bounded sequence and let (s_n) be a convergent sequence such that $\lim_{n\to\infty} s_n=0$. Prove that $\lim_{n\to\infty}(t_n s_n)=0$

6.6 Let (s_n) be a convergent sequence and suppose that $\lim_{n\to\infty} s_n > a$. Prove that there exists a number N such that $n>N$ implies $s_n > a$.

6.7 (a) Let (s_n) be a convergent sequence of nonnegative real numbers. Prove that $\lim_{n\to\infty} s_n \ge 0$.

(b) Use (a) to prove that if $s_n \le t_n$ for all n, then $\lim_{n\to\infty} s_n \le \lim_{n\to\infty} t_n$

6.8 Show that if (s_n) and (t_n) are sequences such that (s_n) and (s_n+t_n) are convergent, then (t_n) is convergent.

6.9 Show that if (s_n) and (t_n) are sequences such that (s_n) converges to $s\neq 0$ and $(s_n t_n)$ converges, then (t_n) converges.

6.10 Show that the condition $s_n < t_n$ does not imply that $\lim_{n\to\infty} s_n < \lim_{n\to\infty} t_n$.

Homework 7

7.1 Let $s_1 = 1$ and for $n \geq 1$ let $s_{n+1} = \sqrt{s_n + 1}$.

(a) List the first four terms of (s_n).

(b) It turns about that (s_n) converges. Assume this fact and prove that the limit is $\dfrac{1+\sqrt{5}}{2}$.

7.2 Suppose that there exists N_o such that $s_n \leq t_n$ for all $n > N_o$

(a)Prove that if $\lim_{n\to\infty} s_n = +\infty$ then $\lim_{n\to\infty} t_n = +\infty$.

(b) Prove that if $\lim_{n\to\infty} t_n = -\infty$ then $\lim_{n\to\infty} s_n = -\infty$.

(c)Prove that if $\lim_{n\to\infty} s_n$ and $\lim_{n\to\infty} t_n$ exist, then $\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$.

7.3 (a) Show that if $\lim_{n\to\infty} s_n = +\infty$ and $\inf\{t_n : n \in N\} > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = +\infty$.

(b) Show that if $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n > -\infty$, then $\lim_{n\to\infty}(s_n + t_n) = +\infty$.

(c) Show that if $\lim_{n\to\infty} s_n = +\infty$ and if (t_n) is a bounded sequence, then $\lim_{n\to\infty}(s_n + t_n) = +\infty$

7.4 Show that if $\lim_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty}(s_n)^2 = +\infty$.

7.5 Prove theorem 3.26(ii).

Homework 8

8.1 Prove theorem 3.25 for bounded nonincreasing sequences.

8.2 Let S be a bounded nonempty subset of R and suppose $\sup S \notin S$. Prove that there is a nondecreasing sequence (s_n) of points in S such that $\lim_{n\to\infty} s_n = \sup S$.

8.3 Let $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1} s_n^2$ for $n \geq 1$.

(a) find s_2, s_3 and s_4.

(b) Use mathematical induction to show that $0 < s_{n+1} < s_n \leq 1$ for all n

(c) Show that $\lim_{n\to\infty} s_n$ exists and prove that $\lim_{n\to\infty} s_n = 0$.

8.4 Show directly from the definition that the sequence $\left(\frac{n+1}{n}\right)$ is a Cauchy sequence.

8.5 Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(|x_n - y_n|)$ and $(x_n \ y_n)$ are Cauchy sequences.

8.6 Find an example of a sequence of real numbers satisfying each set of properties if any (if not explain why): (a)

(a) Cauchy but not monotone

(b) Monotone but not Cauchy

(c) Bounded but not Cauchy

(d) Cauchy with a divergent subsequence

(e) Unbounded with a Cauchy subsequence

8.7 Show that the sequence $\left(1-(-1)^n + \frac{1}{n}\right)$ is divergent.