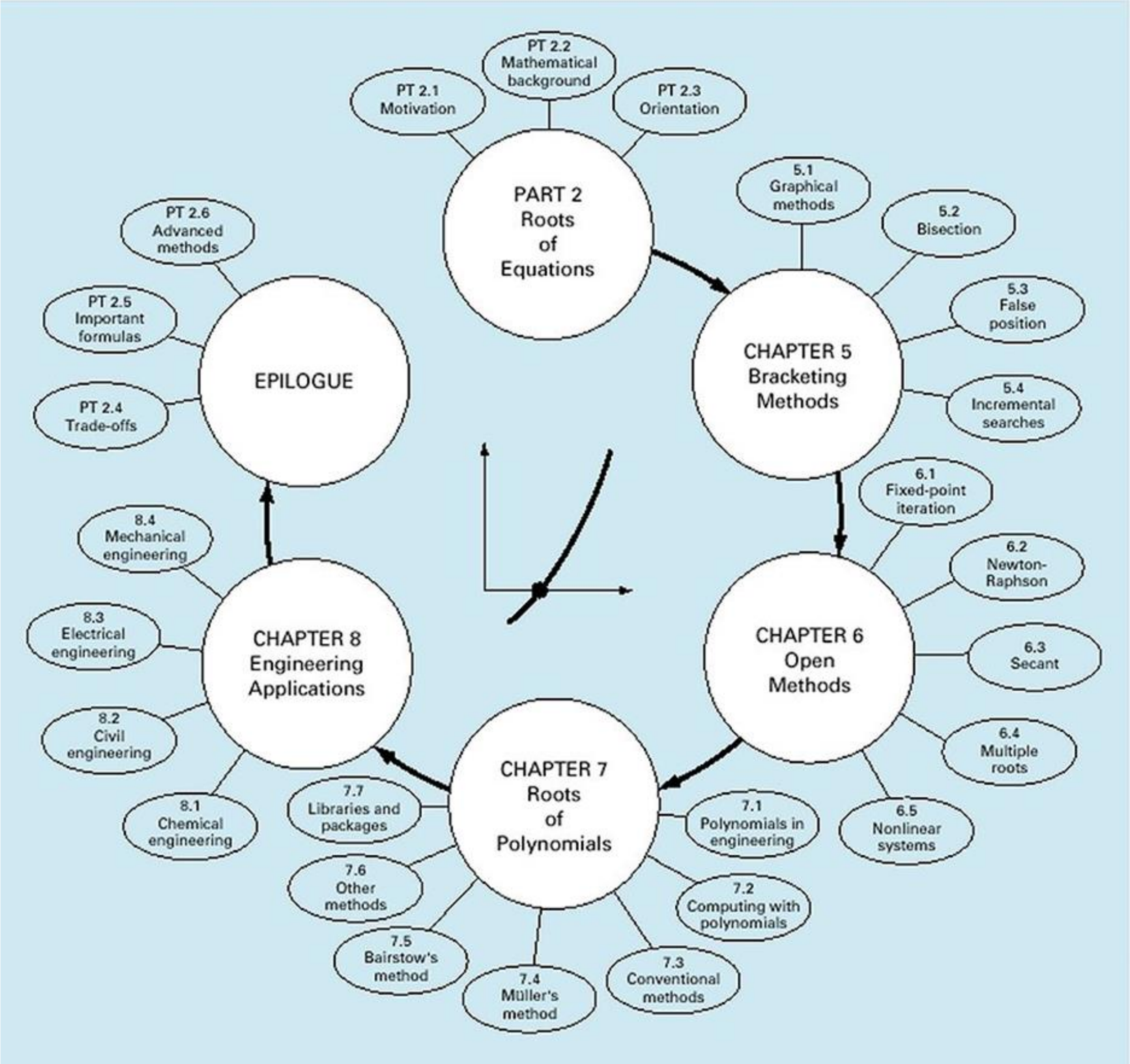
Part 2

Roots of Equations

Chapters 5,6, and 7.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Roots of Equations

Part 2

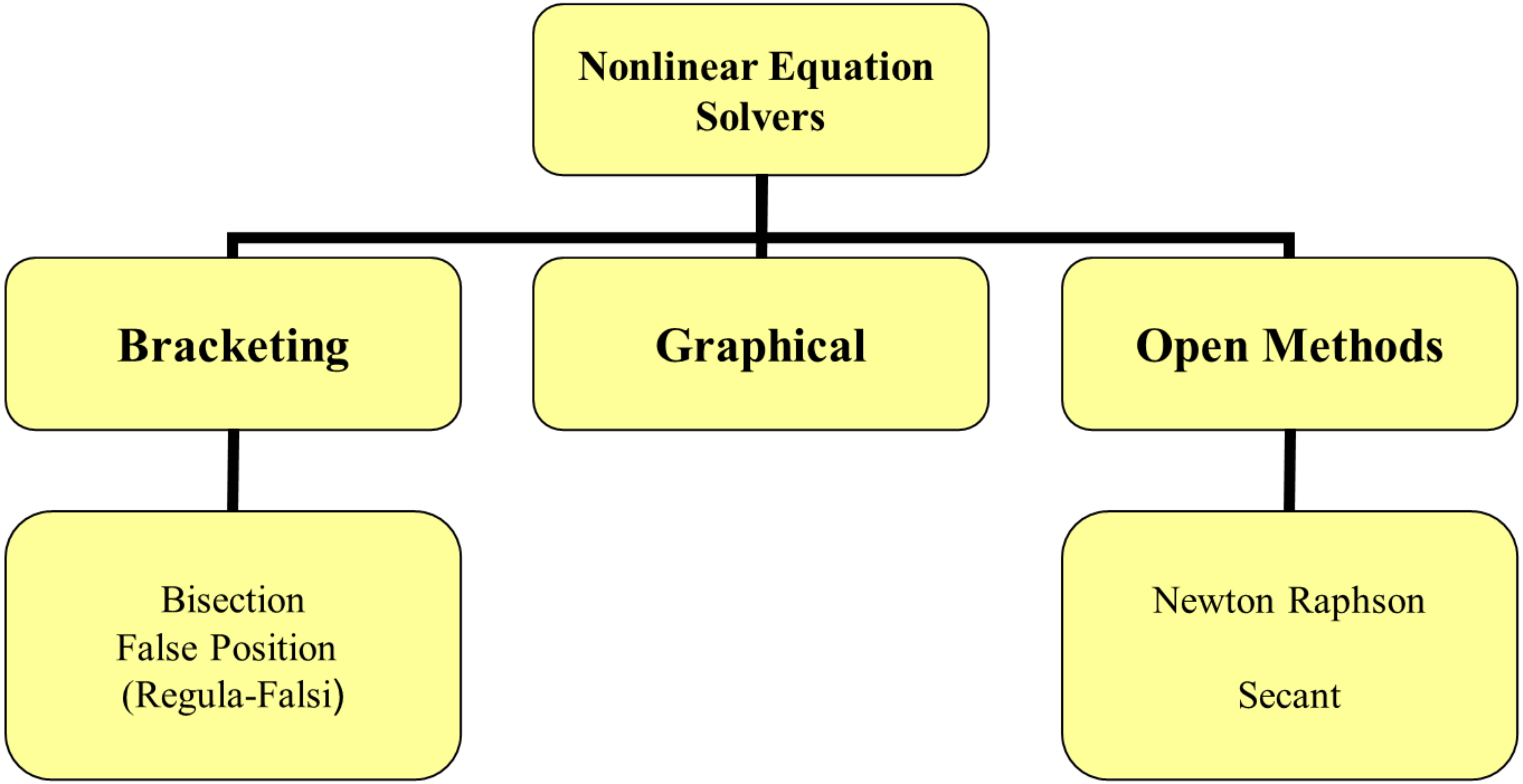
- Why?

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

- But

$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad \Rightarrow x = ?$$

$$\sin x + x = 0 \quad \Rightarrow x = ?$$



All Iterative

Bracketing Methods

(Or, two point methods for finding roots)

Chapter 5

- Two initial guesses for the root are required. These guesses must “bracket” or be on either side of the root.

== > Fig. 5.1

- If one root of a real and continuous function, $f(x)=0$, is bounded by values $x=x_l$, $x=x_u$ then
  $f(x_l) \cdot f(x_u) < 0$. (The function changes sign on opposite sides of the root)

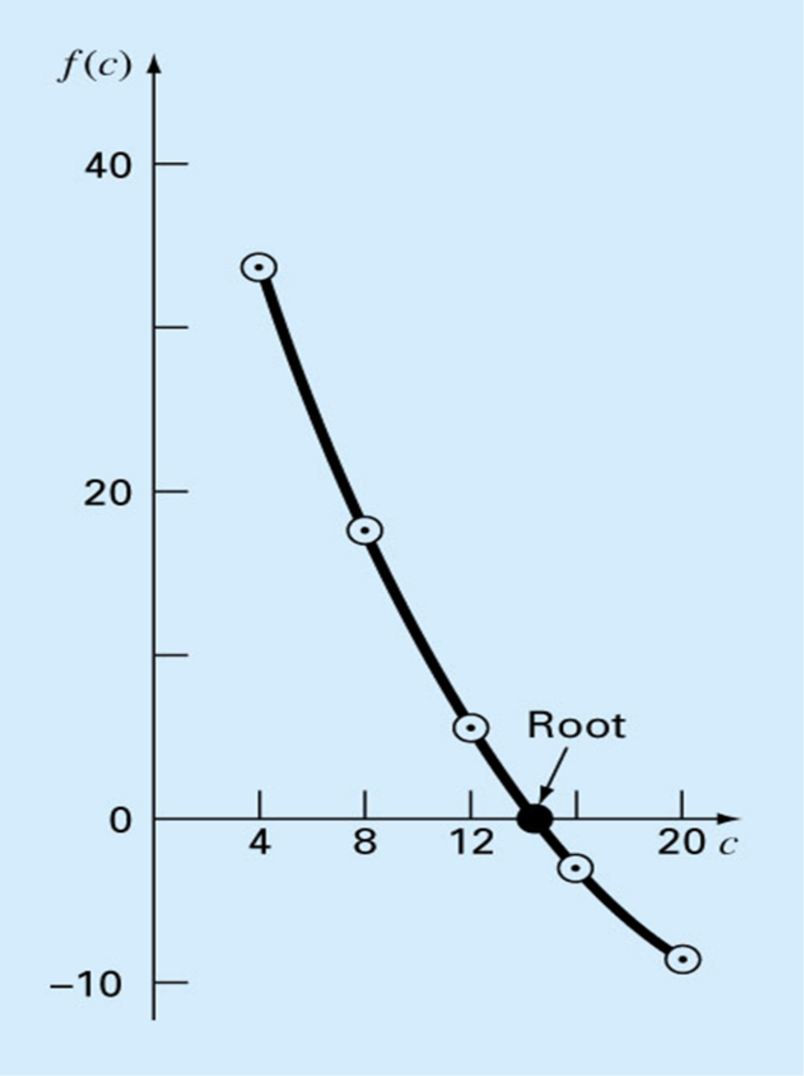
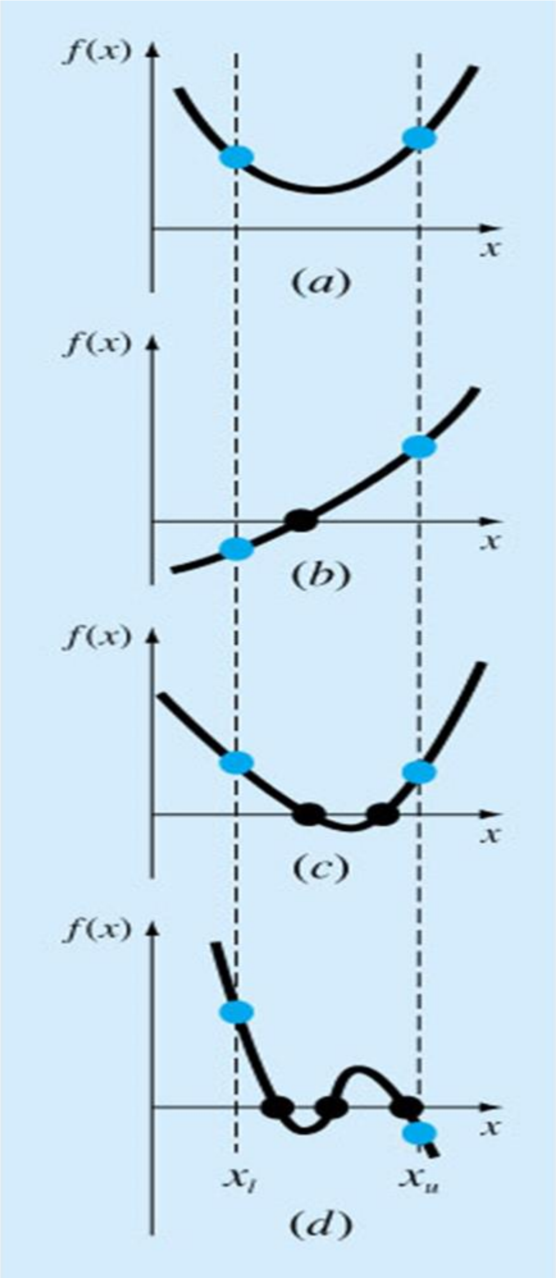


Figure 5.2



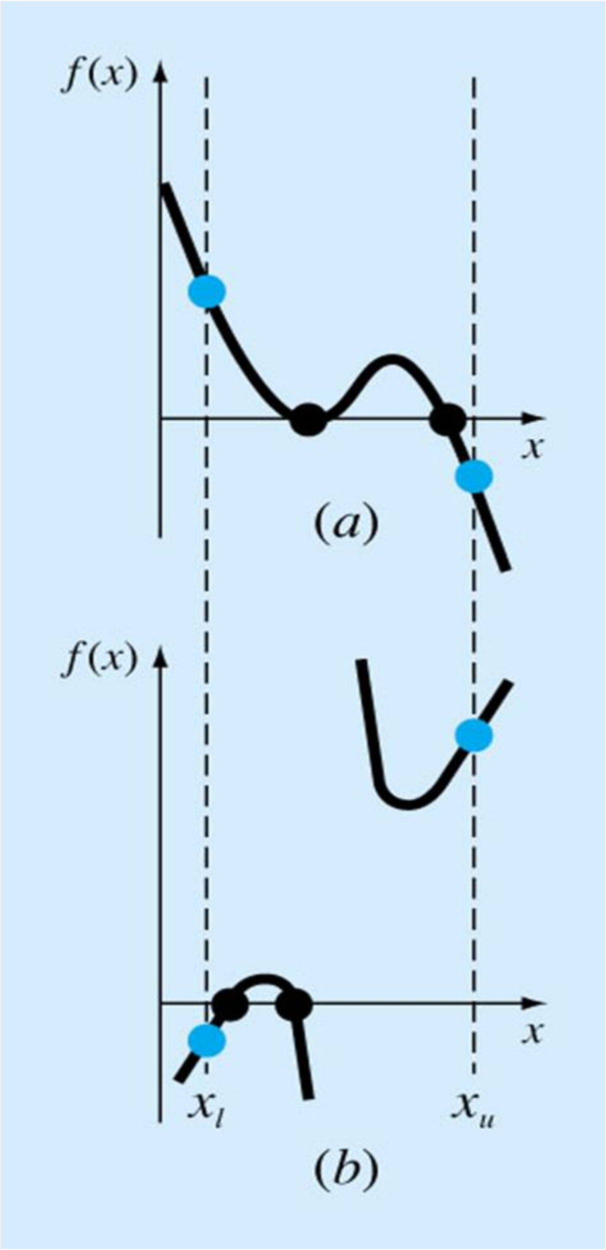
No answer (No root)

Nice case (one root)

Oops!! (two roots!!)

Three roots(Might work for a while!!)

Figure 5.3



Two roots(Might work for a while!!)

Discontinuous function. Need special method

MANY-MANY roots. What do we do?

Figure 5.4a

$f(x)=\sin 10x+\cos 3x$

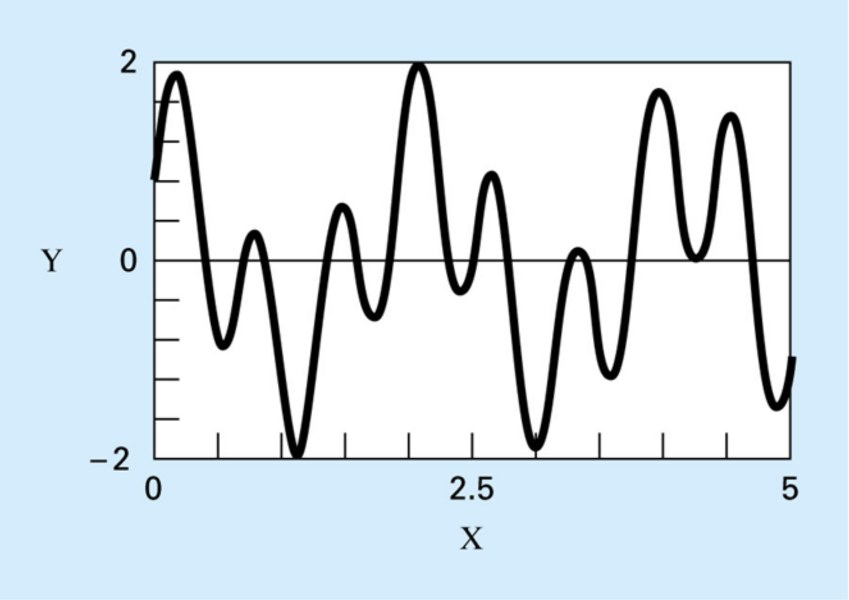


Figure 5.4b

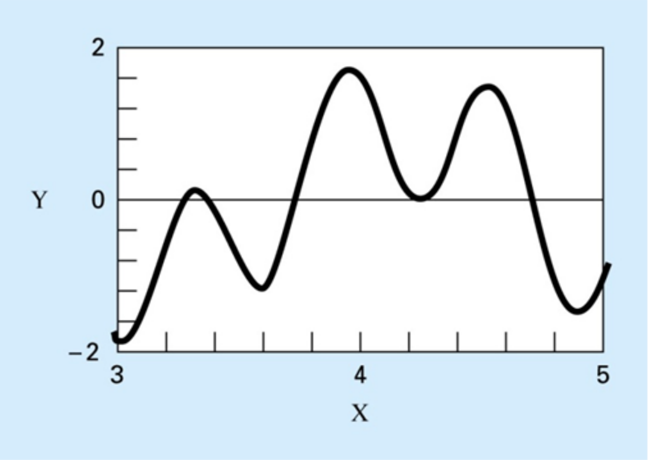
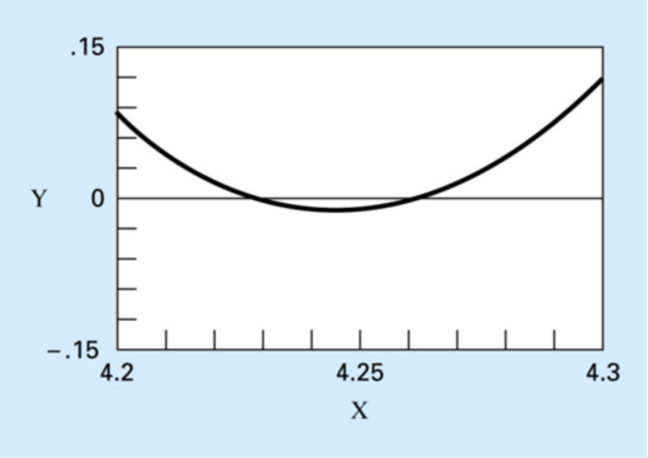


Figure 5.4c



The Bisection Method

For the arbitrary equation of one variable, $f(x)=0$

1. Pick x_l and x_u such that they bound the root of interest, check if $f(x_l).f(x_u) <0$.

2. Estimate the root by evaluating $f[(x_l+x_u)/2]$.

3. Find the pair
   - If $f(x_l)$. $f[(x_l+x_u)/2]<0$, root lies in the lower interval, then $x_u=(x_l+x_u)/2$ and go to step 2.

- If $f(x_l).\ f[(x_l+x_u)/2]>0$, root lies in the upper interval, then $x_l= [(x_l+x_u)/2$, go to step 2.

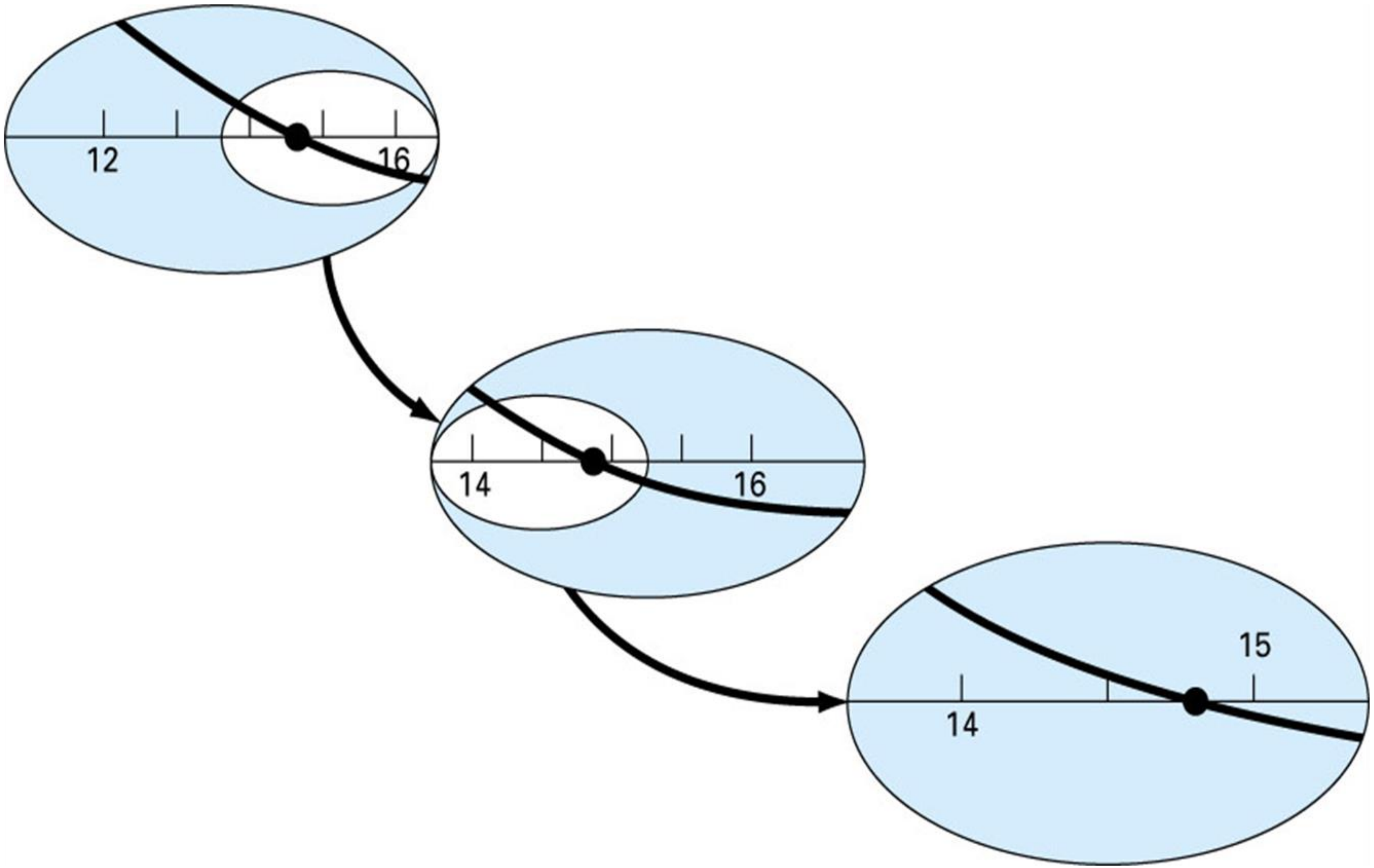
- If $f(x_l).\ f[(x_l+x_u)/2]=0$, then root is $(x_l+x_u)/2$ and terminate.

4. Compare ε_s with ε_a

5. If $\varepsilon_a<\varepsilon_s$, stop. Otherwise repeat the process.

$$\left\{ \begin{array}{l} \dfrac{\left| x_l - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \\ or \\ \dfrac{\left| x_u - \dfrac{x_l + x_u}{2} \right|}{\left| \dfrac{x_l + x_u}{2} \right|} \prec 100\% \end{array} \right\}$$

Figure 5.6



Evaluation of Method

Pros

- Easy
- Always find root
- Number of iterations required to attain an absolute error can be computed a priori.

Cons

- Slow
- Know a and b that bound root
- Multiple roots
- No account is taken of $f(x_l)$ and $f(x_u)$, if $f(x_l)$ is closer to zero, it is likely that root is closer to x_l .

How Many Iterations will It Take?

- Length of the first Interval $L_o = b - a$
- After 1 iteration $L_1 = L_o/2$
- After 2 iterations $L_2 = L_o/4$

- After k iterations $L_k = L_o/2^k$

$$\varepsilon_a \leq \frac{L_k}{x} \times 100\% \qquad \varepsilon_a \leq \varepsilon_s$$

- If the absolute magnitude of the error is

$$\frac{\varepsilon_s \cdot x}{100\%} = 10^{-4}$$

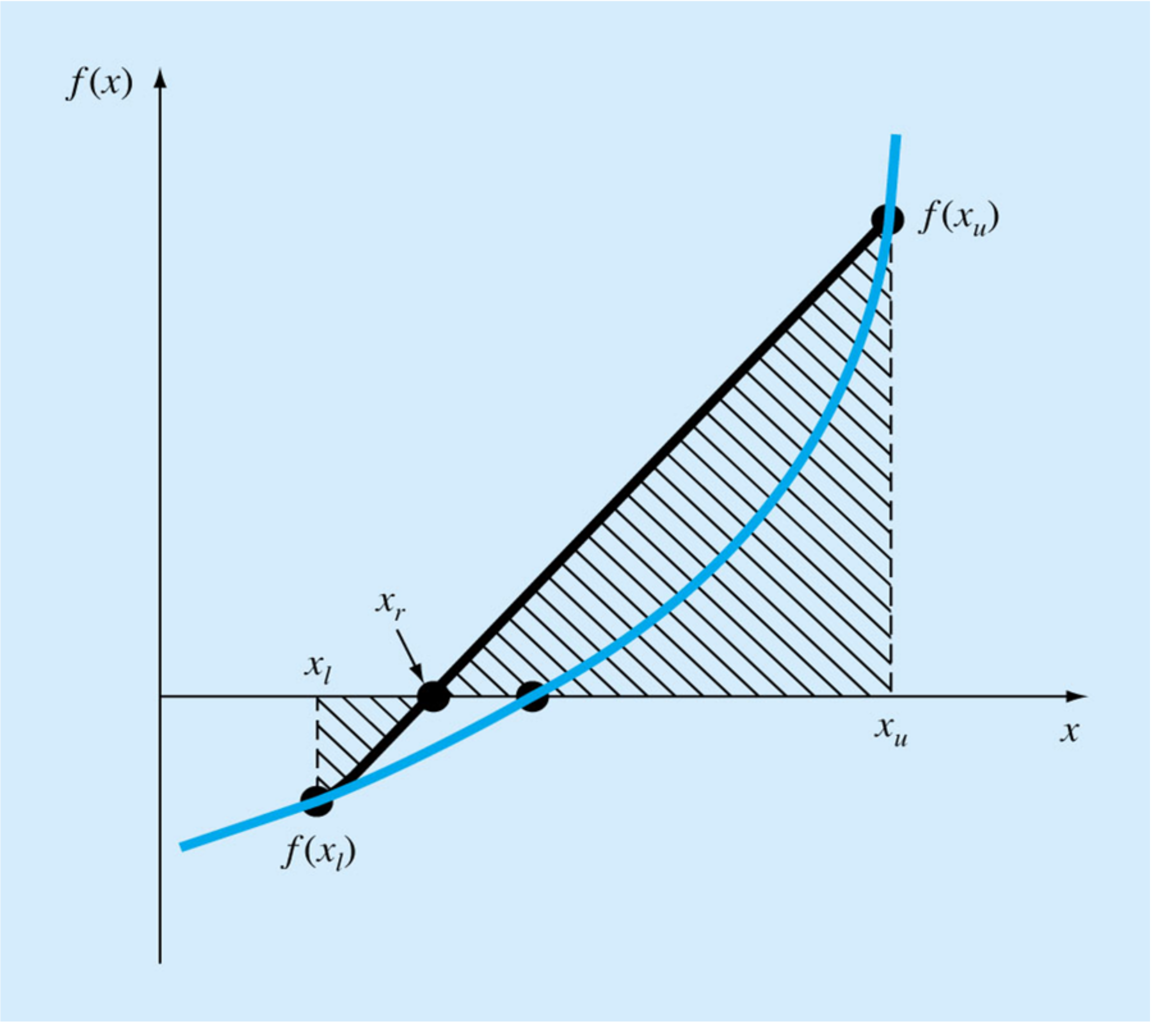
and L_o=2, how many iterations will you have to do to get the required accuracy in the solution?

$$10^{-4} = \frac{2}{2^k} \quad \Rightarrow 2^k = 2 \times 10^4 \quad \Rightarrow k \cong 14.3 = 15$$

The False-Position Method
(Regula-Falsi)

- If a real root is bounded by x_l and x_u of $f(x)=0$, then we can approximate the solution by doing a linear interpolation between the points $[x_l, f(x_l)]$ and $[x_u, f(x_u)]$ to find the x_r value such that $l(x_r)=0$, $l(x)$ is the linear approximation of $f(x)$.

==> Fig. 5.12



Procedure

1. Find a pair of values of x, x_l and x_u such that $f_l = f(x_l) < 0$ and $f_u = f(x_u) > 0$.
2. Estimate the value of the root from the following formula (Refer to Box 5.1)

$$x_r = \frac{x_l f_u - x_u f_l}{f_u - f_l}$$

and evaluate $f(x_r)$.

3. Use the new point to replace one of the original points, keeping the two points on opposite sides of the x axis.

If $f(x_r)<0$ then $x_l=x_r$ ==> $f_l=f(x_r)$

If $f(x_r)>0$ then $x_u=x_r$ ==> $f_u=f(x_r)$

If $f(x_r)=0$ then you have found the root and need go no further!

4. See if the new x_l and x_u are close enough for convergence to be declared. If they are not go back to step 2.

- Why this method?
  - Faster
  - Always converges for a single root.

➔See Sec.5.3.1, Pitfalls of the False-Position Method

Note: Always check by substituting estimated root in the original equation to determine whether $f(x_r) \approx 0$.

Simple Fixed-point Iteration

- Rearrange the function so that x is on the left side of the equation:

$$f(x) = 0 \quad \Rightarrow \quad g(x) = x$$

$$x_k = g(x_{k-1}) \qquad x_o \text{ given, k} = 1, 2, \ldots$$

- Bracketing methods are “convergent”.
- Fixed-point methods may sometime “diverge”, depending on the stating point (initial guess) and how the function behaves.

Example:

$$f(x) = x^2 - x - 2 \qquad\qquad x \succ 0$$

$$g(x) = x^2 - 2$$

$$or$$

$$g(x) = \sqrt{x+2}$$

$$or$$

$$g(x) = 1 + \frac{2}{x}$$

$$\vdots$$

Convergence

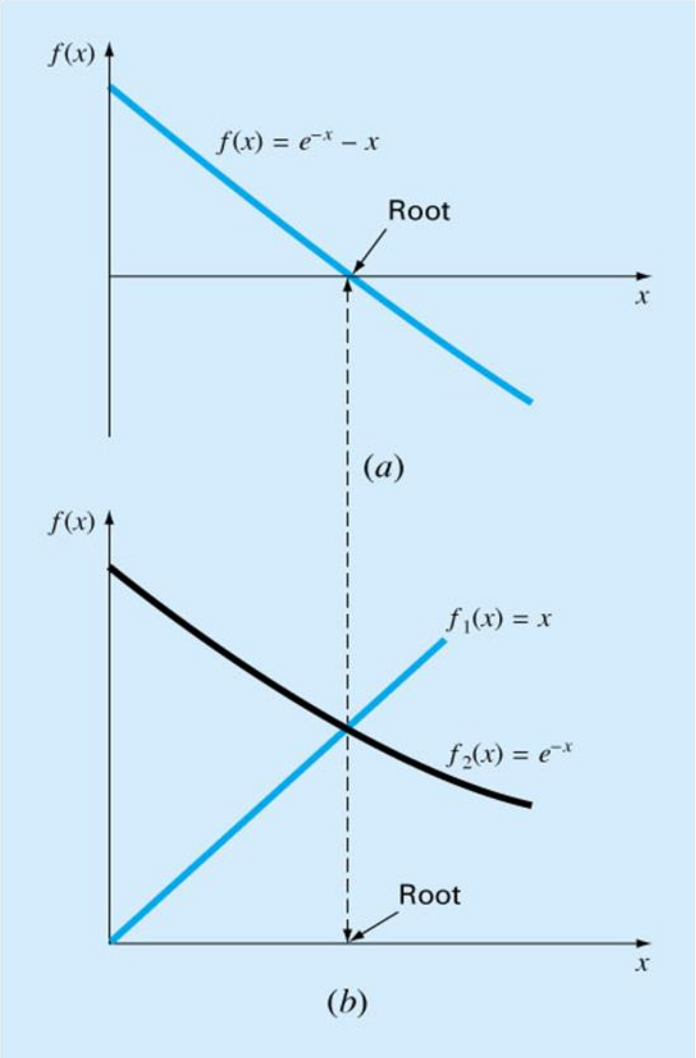
Figure 6.2

- x=g(x) can be expressed as a pair of equations:

  $y_1=x$

  $y_2=g(x)$ (component equations)

- Plot them separately.



Conclusion

- Fixed-point iteration converges if

$$|g'(x)| \prec 1 \quad \text{(slope of the line f(x) = x)}$$

•When the method converges, the error is roughly proportional to or less than the error of the previous step, therefore it is called “linearly convergent.”

Newton-Raphson Method

- Most widely used method.
- Based on Taylor series expansion:

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + f''(x_i)\frac{\Delta x^2}{2!} + O\Delta x^3$$

The root is the value of x_{i+1} when $f(x_{i+1}) = 0$

Rearranging,

$$0 = f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Newton-Raphson formula

- A convenient method for functions whose derivatives can be evaluated analytically. It may not be convenient for functions whose derivatives cannot be evaluated analytically.

Fig. 6.5

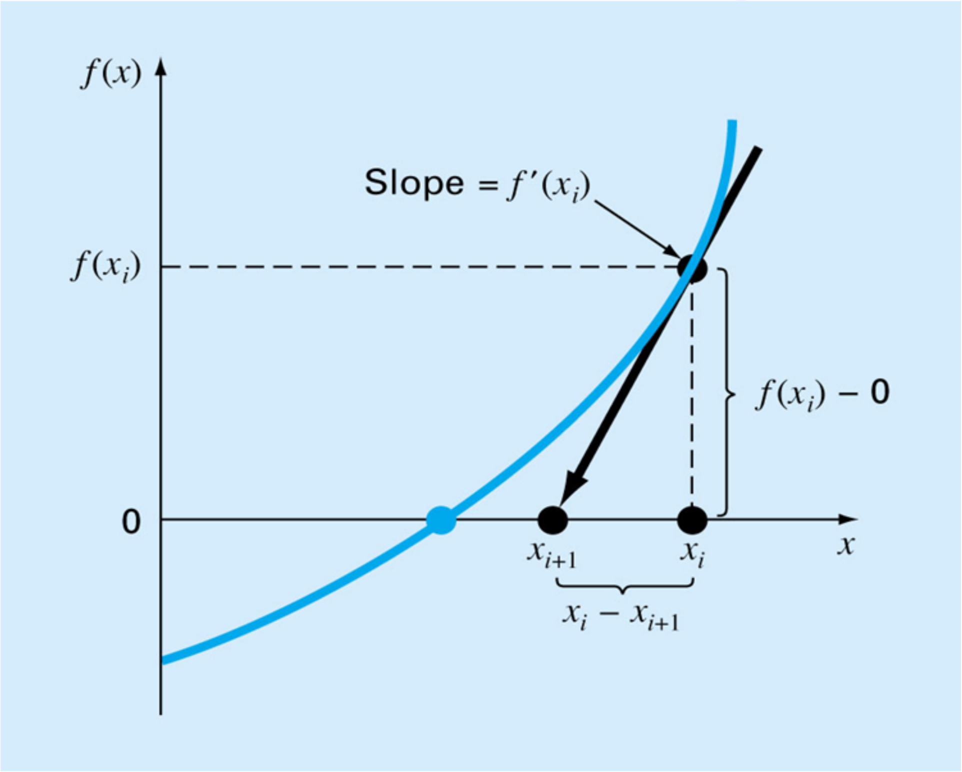
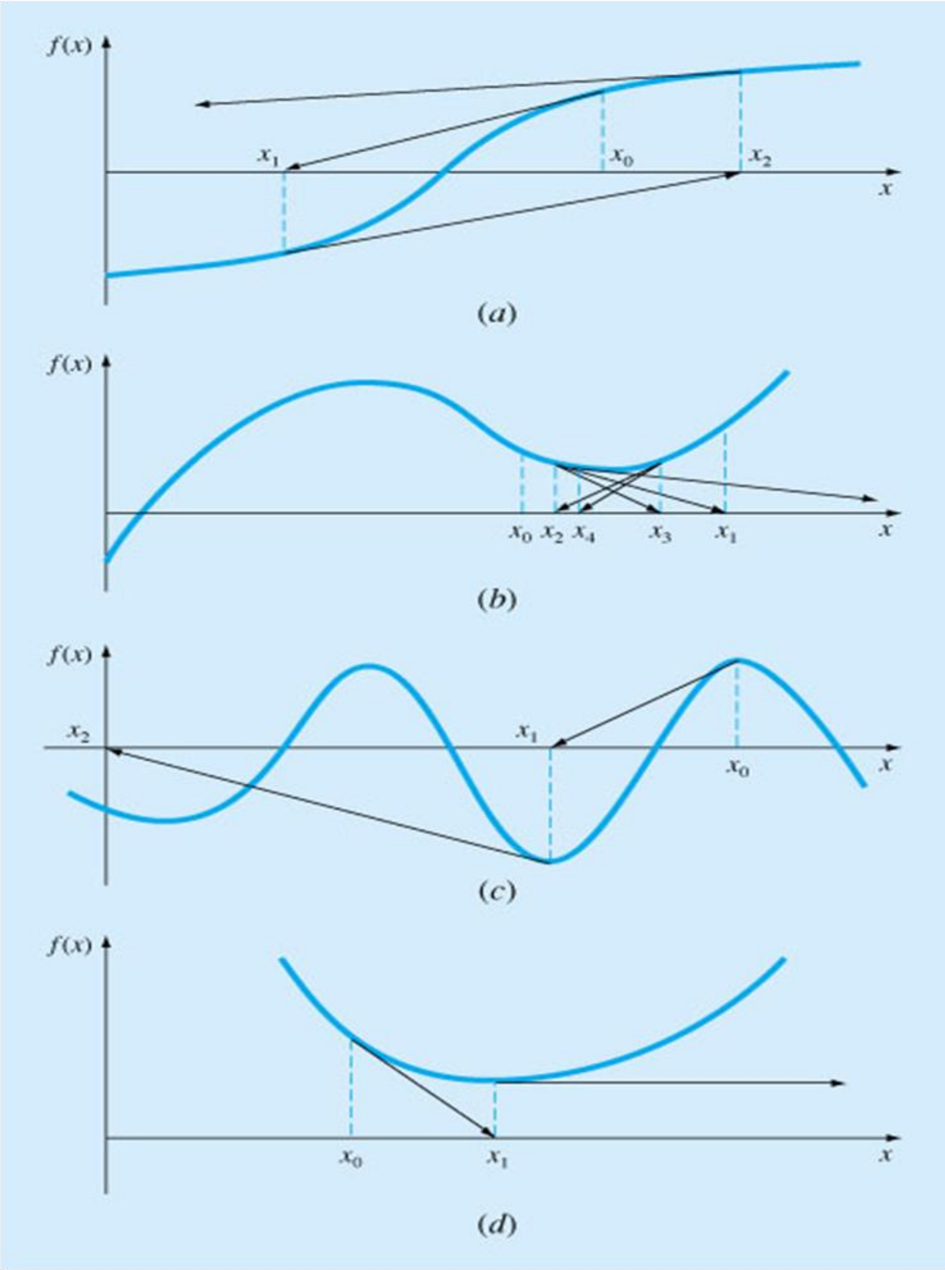


Fig. 6.6



The Secant Method

- A slight variation of Newton's method for functions whose derivatives are difficult to evaluate. For these cases the derivative can be approximated by a backward finite divided difference.

$$\frac{1}{f'(x_i)} \cong \frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})}$$

$$x_{i+1} = x_i - f(x_i)\frac{x_i - x_{i-1}}{f(x_i) - f(x_{i-1})} \qquad i = 1,2,3,\ldots$$

Fig. 6.7

- Requires two initial estimates of x , e.g, x_o, x_1. However, because f(x) is not required to change signs between estimates, it is not classified as a "bracketing" method.
- The secant method has the same properties as Newton's method. Convergence is not guaranteed for all x_o, f(x).

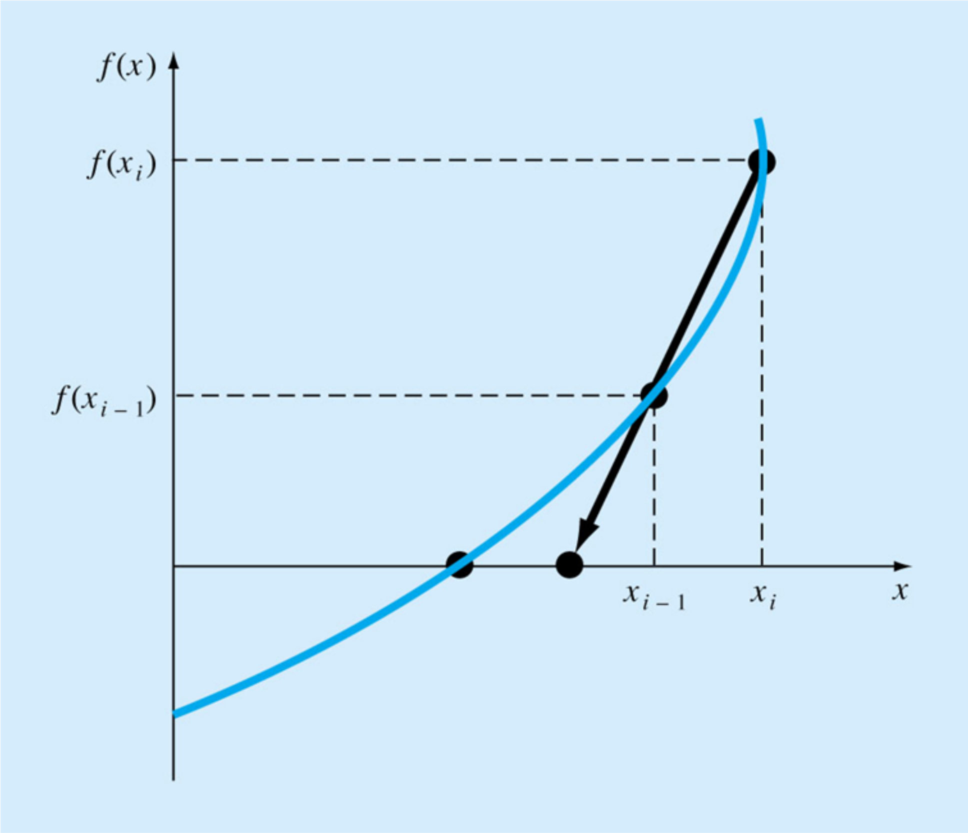
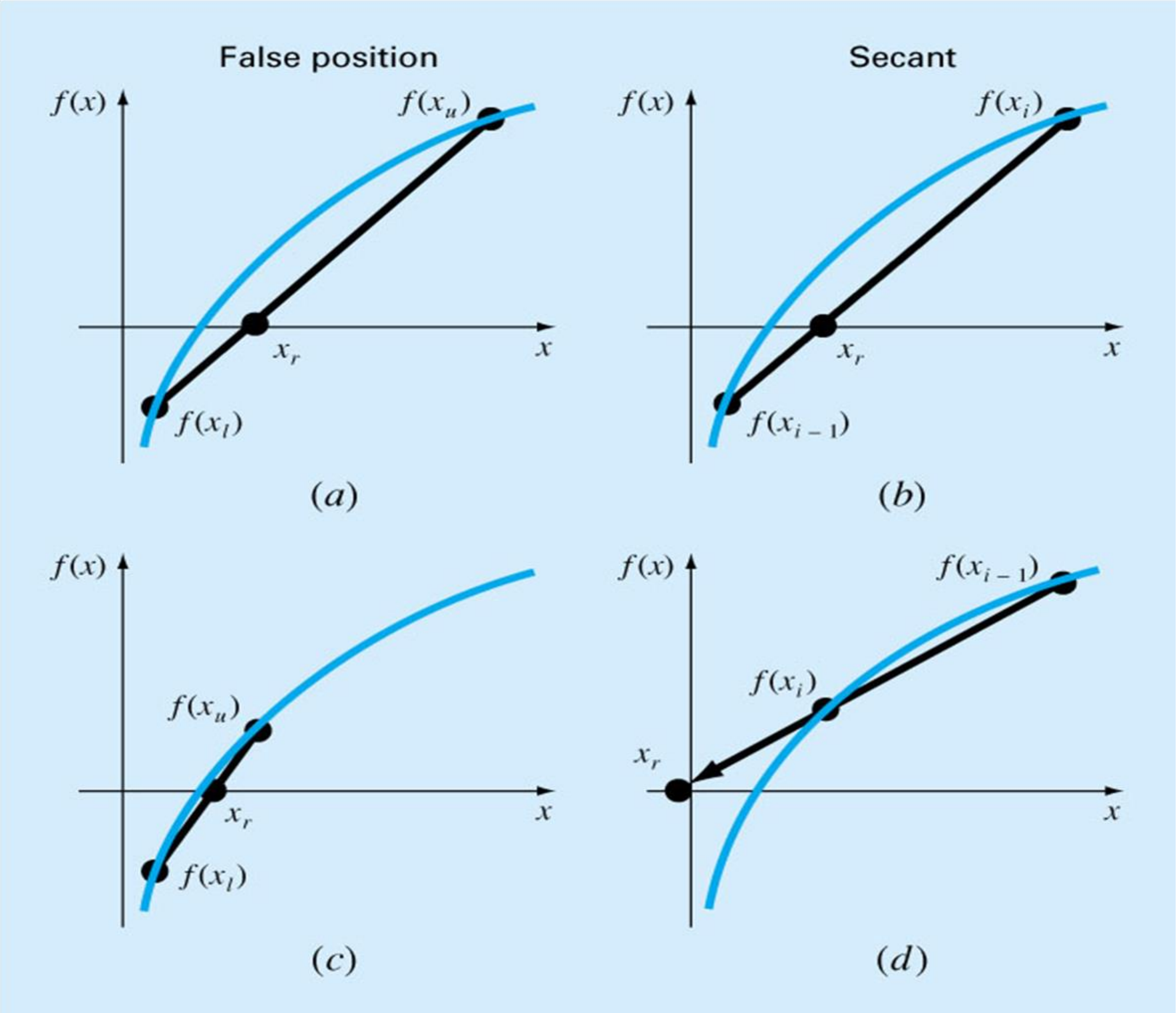


Fig. 6.8


False position
Secant
f(x)
f(x_u)
x_r
x
f(x_l)
(a)
f(x)
f(x_i)
x_r
x
f(x_{i − 1})
(b)
f(x)
f(x_u)
x_r
x
f(x_l)
(c)
f(x)
f(x_{i − 1})
f(x_i)
x_r
x
(d)

Roots of Polynomials

Chapter 7

$$f_n(x) = a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

Follow these rules:

1. For an nth order equation, there are n real or complex roots.
2. If n is odd, there is at least one real root.
3. If complex root exist in conjugate pairs (that is, $\lambda+\mu i$ and $\lambda-\mu i$), where $i=sqrt(-1)$.

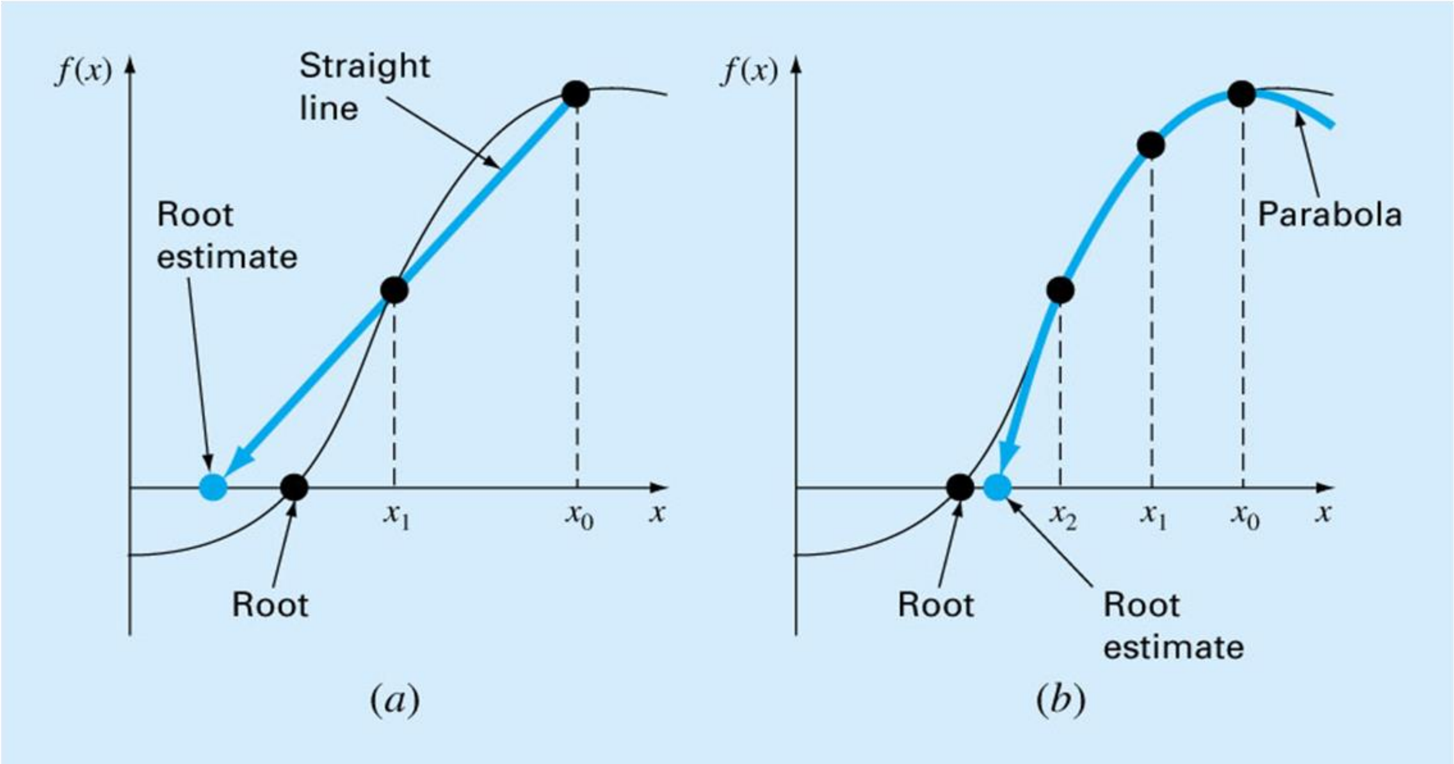
Conventional Methods

- The efficacy of bracketing and open methods depends on whether the problem being solved involves complex roots. If only real roots exist, these methods could be used. However,
  - Finding good initial guesses complicates both the open and bracketing methods, also the open methods could be susceptible to divergence.
- Special methods have been developed to find the real and complex roots of polynomials – Müller and Bairstow methods.

Müller Method

- Müller's method obtains a root estimate by projecting a parabola to the x axis through three function values.

Figure 7.3



Müller Method

- The method consists of deriving the coefficients of parabola that goes through the three points:

1. Write the equation in a convenient form:

$$f_2(x) = a(x - x_2)^2 + b(x - x_2) + c$$

2. The parabola should intersect the three points $[x_o, f(x_o)]$, $[x_1, f(x_1)]$, $[x_2, f(x_2)]$. The coefficients of the polynomial can be estimated by substituting three points to give

$$f(x_o) = a(x_o - x_2)^2 + b(x_o - x_2) + c$$

$$f(x_1) = a(x_1 - x_2)^2 + b(x_1 - x_2) + c$$

$$f(x_2) = a(x_2 - x_2)^2 + b(x_2 - x_2) + c$$

3. Three equations can be solved for three unknowns, *a, b, c*. Since two of the terms in the 3rd equation are zero, it can be immediately solved for $c=f(x_2)$.

$$f(x_o) - f(x_2) = a(x_o - x_2)^2 + b(x_o - x_2)$$

$$f(x_1) - f(x_2) = a(x_1 - x_2)^2 + b(x_1 - x_2)$$

If

$$h_o = x_1 - x_o \qquad h_1 = x_2 - x_1$$

$$\delta_o = \frac{f(x_1) - f(x_o)}{x_1 - x_o} \qquad \delta_1 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$(h_o + h_1)b - (h_o + h_1)^2 a = h_o\delta_o + h_1\delta_1$$

$$h_1 b - h_1^2 a = h_1\delta_1$$

Solved for a and *b*

$$a = \frac{\delta_1 - \delta_o}{h_1 + h_o} \qquad b = ah_1 + \delta_1 \qquad c = f(x_2)$$

- Roots can be found by applying an alternative form of quadratic formula:

$$x_3 = x_2 + \frac{-2c}{b \pm \sqrt{b^2 - 4ac}}$$

- The error can be calculated as

$$\varepsilon_a = \left| \frac{x_3 - x_2}{x_3} \right| 100\%$$

- ±term yields two roots, the sign is chosen to agree with *b*. This will result in a largest denominator, and will give root estimate that is closest to x_2.

- Once x_3 is determined, the process is repeated using the following guidelines:

1. If only real roots are being located, choose the two original points that are nearest the new root estimate, x_3.
2. If both real and complex roots are estimated, employ a sequential approach just like in secant method, x_1, x_2, and x_3 to replace x_0, x_1, and x_2.