King Abdulaziz University
Faculty of Engineering –Rabigh
Electrical Engineering Department

EEN271 Engineering Numerical Methods
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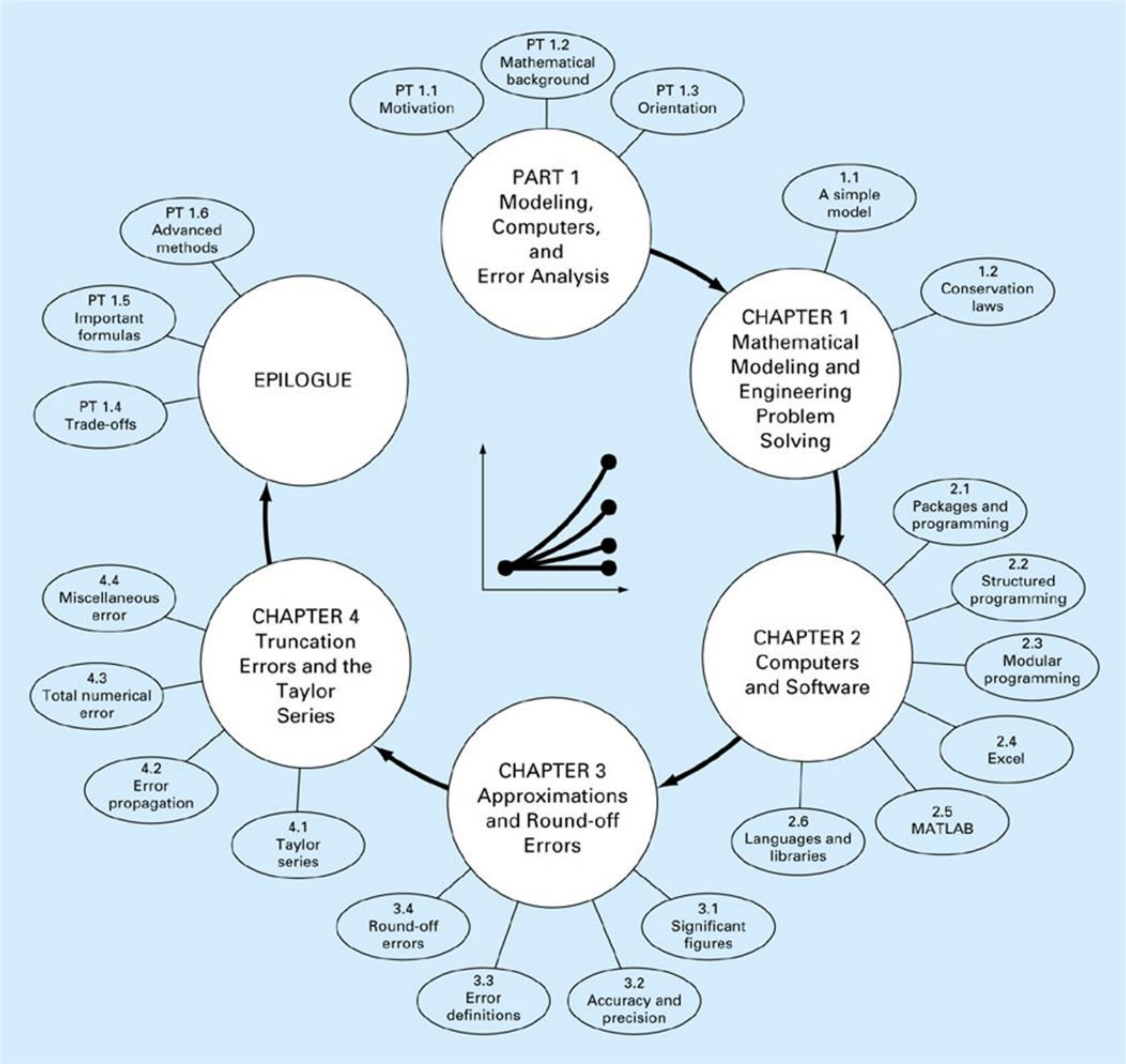
Textbook: Numerical Methods of Engineers
Chapra 7th edition 2014.

Part 1

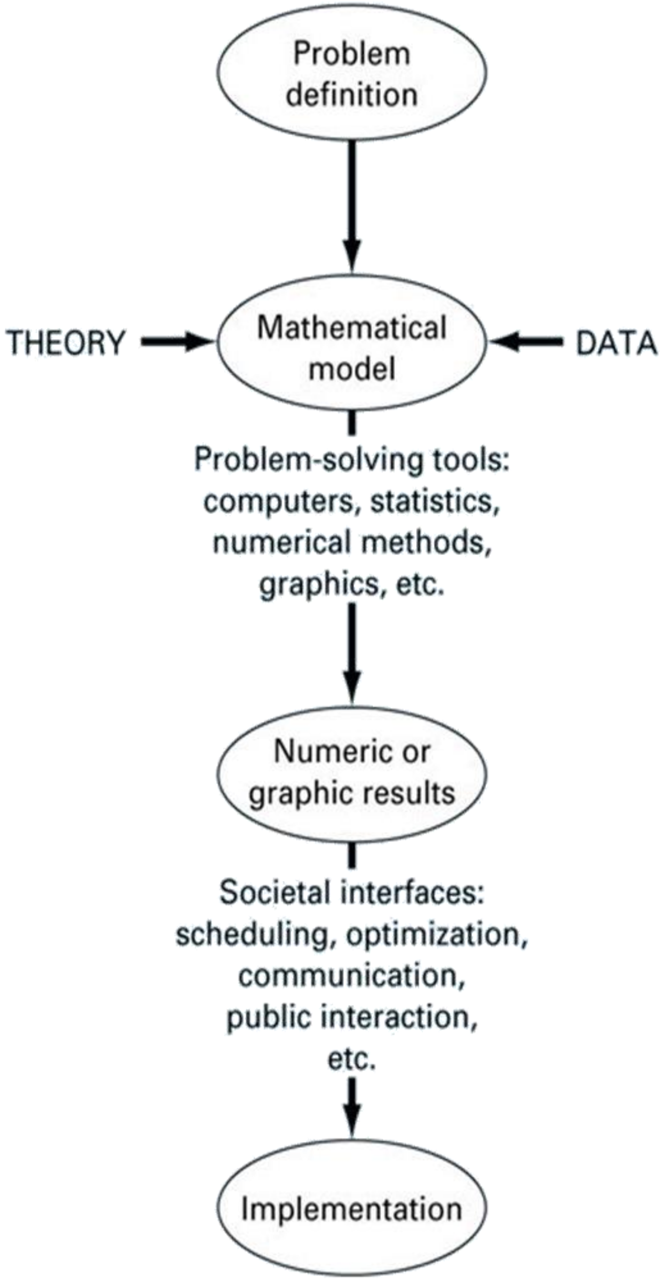
Chapter 1
Chapter 2
Chapter 3
Chapter 4



Mathematical Modeling and Engineering Problem solving

Chapter 1

- Requires understanding of engineering systems
  - By observation and experiment
  - Theoretical analysis and generalization
- Computers are great tools, however, without fundamental understanding of engineering problems, they will be useless.


Problem definition
THEORY
Mathematical model
DATA
Problem-solving tools: computers, statistics, numerical methods, graphics, etc.
Numeric or graphic results
Societal interfaces: scheduling, optimization, communication, public interaction, etc.
Implementation

- A mathematical model is represented as a functional relationship of the form

Dependent		independent		forcing
Variable	$= f$	variables,	parameters,	functions

- *Dependent variable*: Characteristic that usually reflects the state of the system
- *Independent variables*: Dimensions such as time and space along which the systems behavior is being determined
- *Parameters*: reflect the system's properties or composition
- *Forcing functions*: external influences acting upon the system

Newton's 2^{nd} law of Motion

- States that "*the time rate change of momentum of a body is equal to the resulting force acting on it.*"
- The model is formulated as

$$\mathbf{F = m\ a} \qquad (1.2)$$

F=net force acting on the body (N)

m=mass of the object (kg)

a=its acceleration (m/s^2)

- Formulation of Newton's 2nd law has several characteristics that are typical of mathematical models of the physical world:
  - It describes a natural process or system in mathematical terms
  - It represents an idealization and simplification of reality
  - Finally, it yields reproducible results, consequently, can be used for predictive purposes.

- Some mathematical models of physical phenomena may be much more complex.

- Complex models may not be solved exactly or require more sophisticated mathematical techniques than simple algebra for their solution

  - Example, modeling of a falling parachutist:

$$\frac{dv}{dt} = \frac{F}{m}$$

$$F = F_D + F_U$$

$$F_D = mg$$

$$F_U = -cv$$

$$\frac{dv}{dt} = \frac{mg - cv}{m}$$

$$\frac{dv}{dt} = g - \frac{c}{m} v$$

- This is a differential equation and is written in terms of the differential rate of change dv/dt of the variable that we are interested in predicting.
- If the parachutist is initially at rest (v=0 at t=0), using calculus

Independent variable

$$v(t) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right)$$

Dependent variable

Forcing function

Parameters

Conservation Laws and Engineering

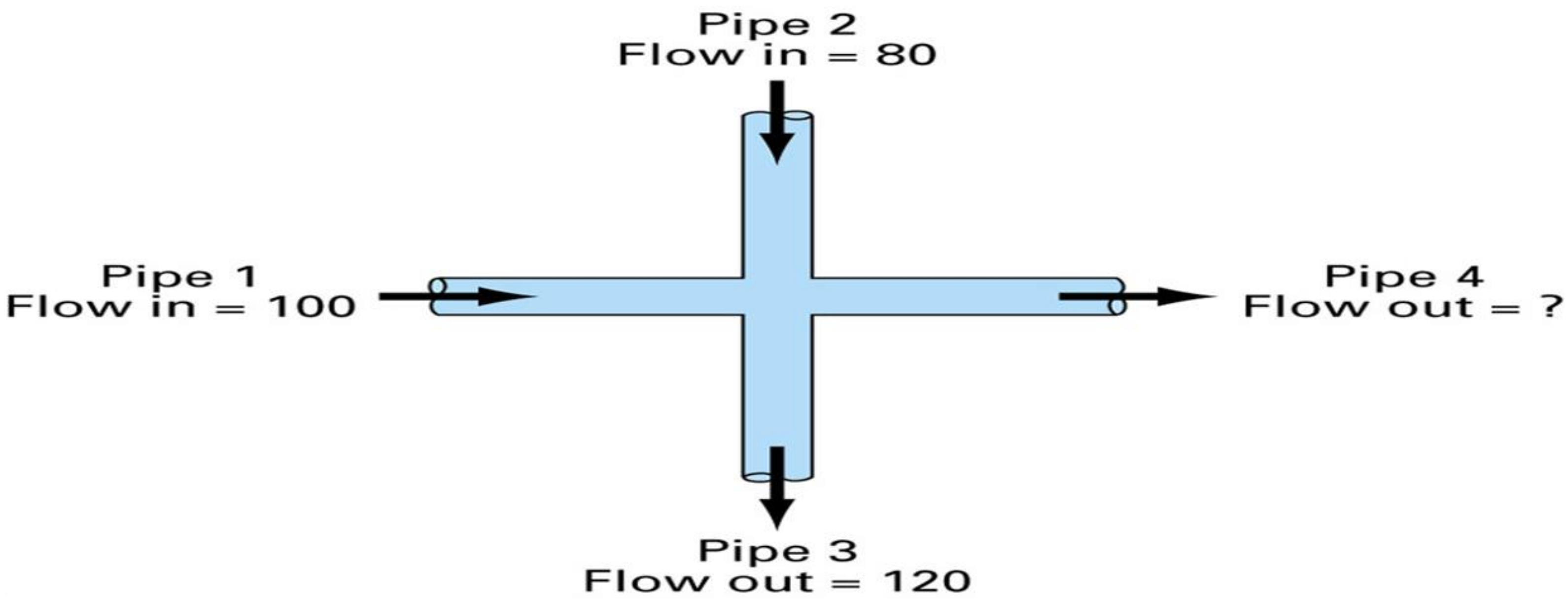
- Conservation laws are the most important and fundamental laws that are used in engineering.

$$\text{Change} = \text{increases} - \text{decreases} \qquad (1.13)$$

- Change implies changes with time (transient). If the change is nonexistent (steady-state), Eq. 1.13 becomes

$$\text{Increases} = \text{Decreases}$$

Fig 1.6



- For steady-state incompressible fluid flow in pipes:

$$\text{Flow in} = \text{Flow out}$$

or

$$100 + 80 = 120 + \text{Flow}_4$$

$$\text{Flow}_4 = 60$$

Refer to table 1.11

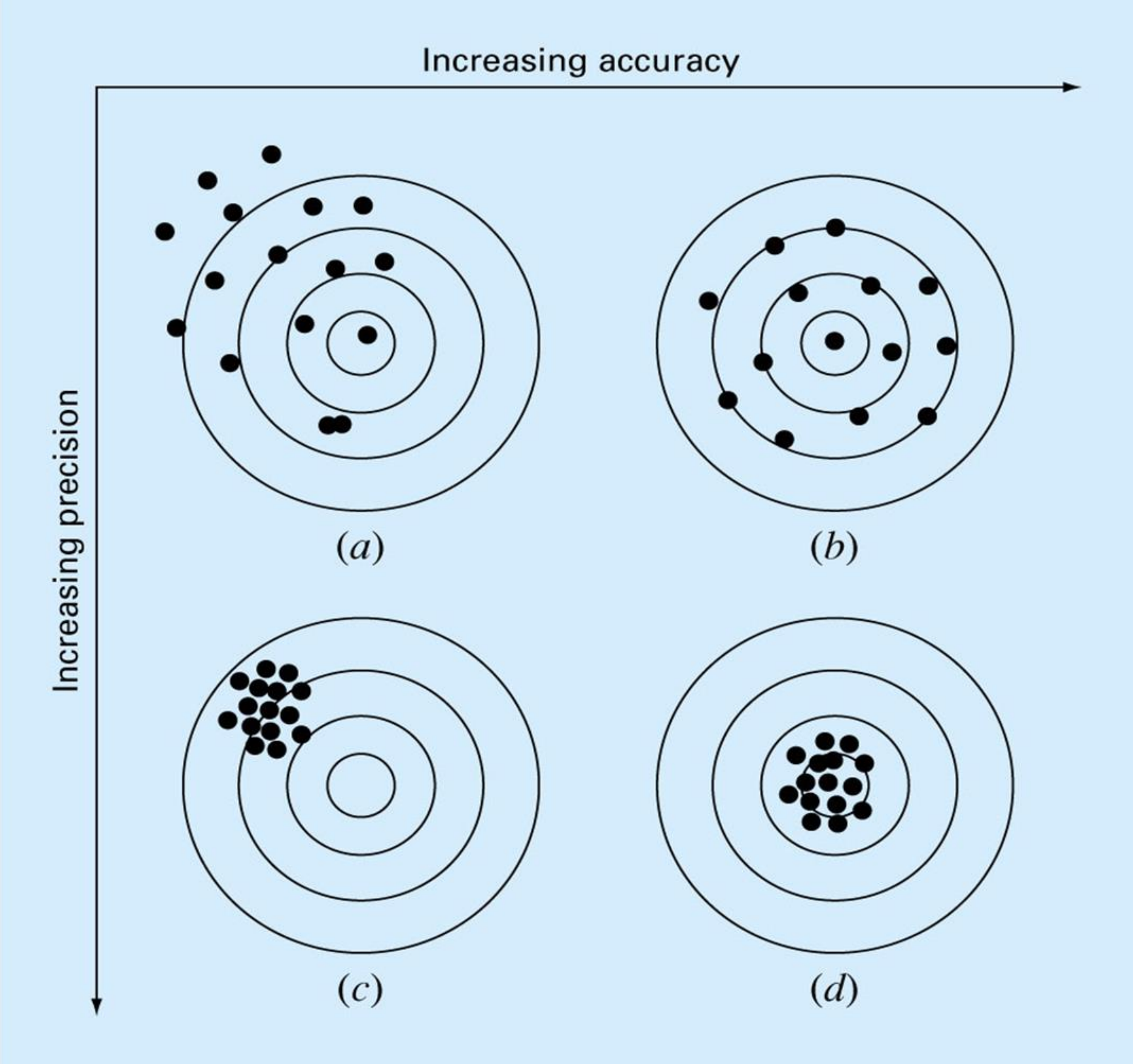
Approximations and Round-Off Errors

Chapter 3

- For many engineering problems, we cannot obtain analytical solutions.
- Numerical methods yield approximate results, results that are close to the exact analytical solution. We cannot exactly compute the errors associated with numerical methods.
  - Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.
  - Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc …
  - The output information will then contain error from both of these sources.
- How confident we are in our approximate result?
- The question is "*how much error is present in our calculation and is it tolerable?*"

- Accuracy. How close is a computed or measured value to the true value
- Precision (or *reproducibility*). How close is a computed or measured value to previously computed or measured values.
- Inaccuracy (or *bias*). A systematic deviation from the actual value.
- Imprecision (or *uncertainty*). Magnitude of scatter.

Fig. 3.2



Significant Figures

- Number of significant figures indicates precision. Significant digits of a number are those that can be *used* with *confidence,* e.g., the number of certain digits plus one estimated digit.

53,8<u>00</u> How many significant figures?

5.38×10^4	3
5.380×10^4	4
5.3800×10^4	5

Zeros are sometimes used to locate the decimal point not significant figures.

0.00001753	4
0.0001753	4
0.001753	4

Error Definitions

True Value = Approximation + Error

E_t = True value – Approximation (+/-)

True error

$$\text{True fractional relative error} = \frac{\text{true error}}{\text{true value}}$$

$$\text{True percent relative error, } \varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$$

- For numerical methods, the true value will be known only when we deal with functions that can be solved analytically (simple systems). In real world applications, we usually not know the answer a priori. Then

$$\varepsilon_a = \frac{\text{Approximate error}}{\text{Approximation}} \times 100\%$$

- *Iterative approach,* example Newton’s method

$$\varepsilon_a = \frac{\text{Current approximation - Previous approximation}}{\text{Current approximation}} \times 100\%$$

- Use absolute value.
- Computations are repeated until stopping criterion is satisfied.

$$\left|\varepsilon_a\right| < \varepsilon_s$$

Pre-specified % tolerance based on the knowledge of your solution

- If the following criterion is met

$$\varepsilon_s = (0.5 \times 10^{(2-n)})\%$$

you can be sure that the result is correct to at least n significant figures.

Round-off Errors

- Numbers such as π, e, or $\sqrt{7}$ cannot be expressed by a fixed number of significant figures.
- Computers use a base-2 representation, they cannot precisely represent certain exact base-10 numbers.
- Fractional quantities are typically represented in computer using “floating point” form, e.g.,

Integer part

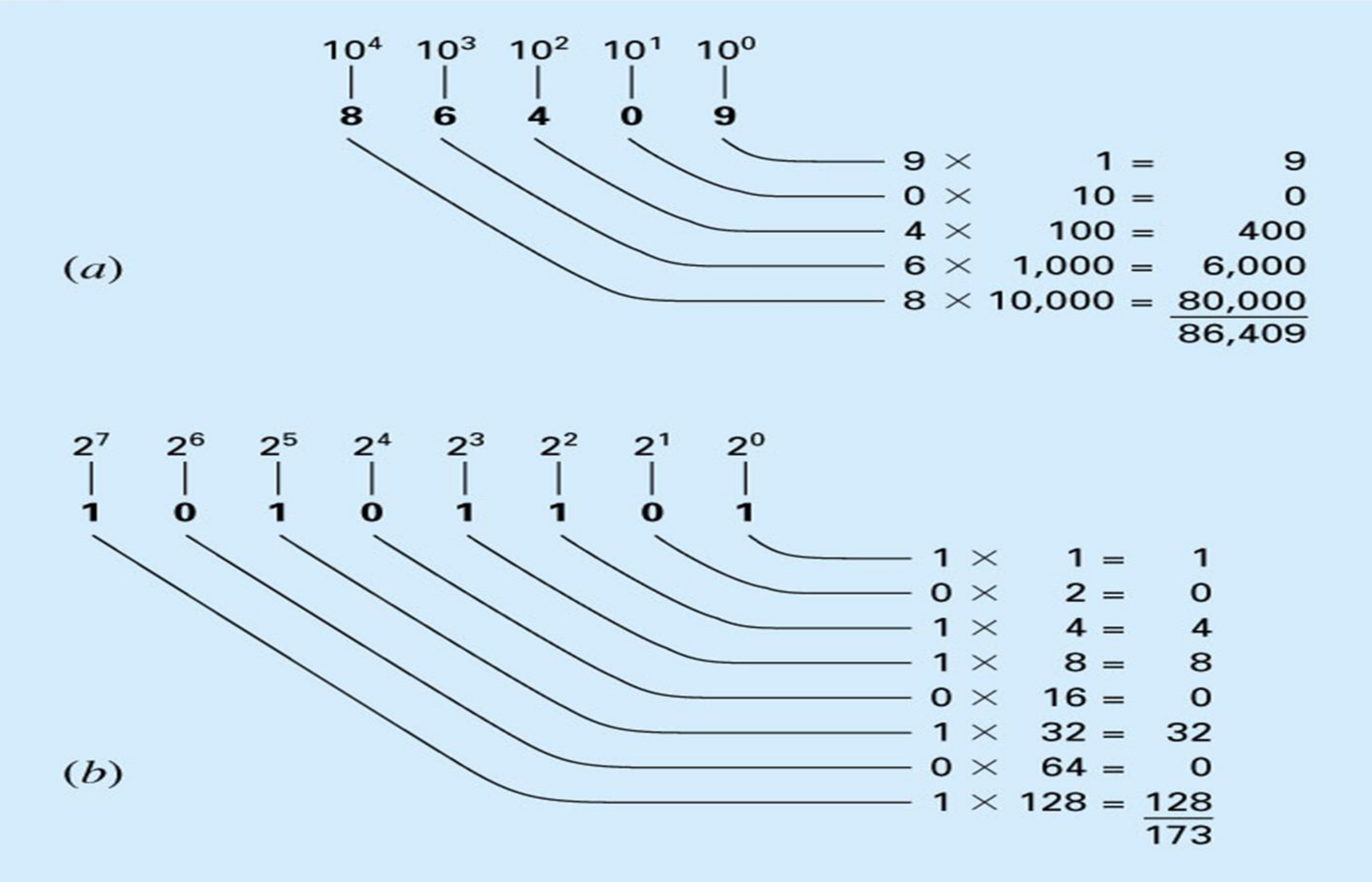
exponent

$$m.b^e$$

mantissa

Base of the number system used

Figure 3.3



10⁴ 10³ 10² 10¹ 10⁰
8 6 4 0 9
9 × 1 = 9
0 × 10 = 0
4 × 100 = 400
6 × 1,000 = 6,000
8 × 10,000 = 80,000
86,409
(a)
2⁷ 2⁶ 2⁵ 2⁴ 2³ 2² 2¹ 2⁰
1 0 1 0 1 1 0 1
1 × 1 = 1
0 × 2 = 0
1 × 4 = 4
1 × 8 = 8
0 × 16 = 0
1 × 32 = 32
0 × 64 = 0
1 × 128 = 128
173
(b)

Figure 3.4

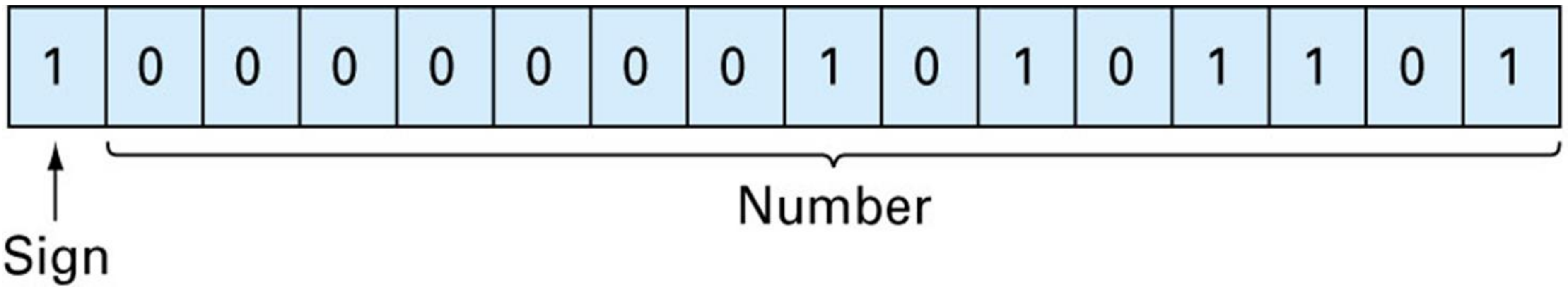
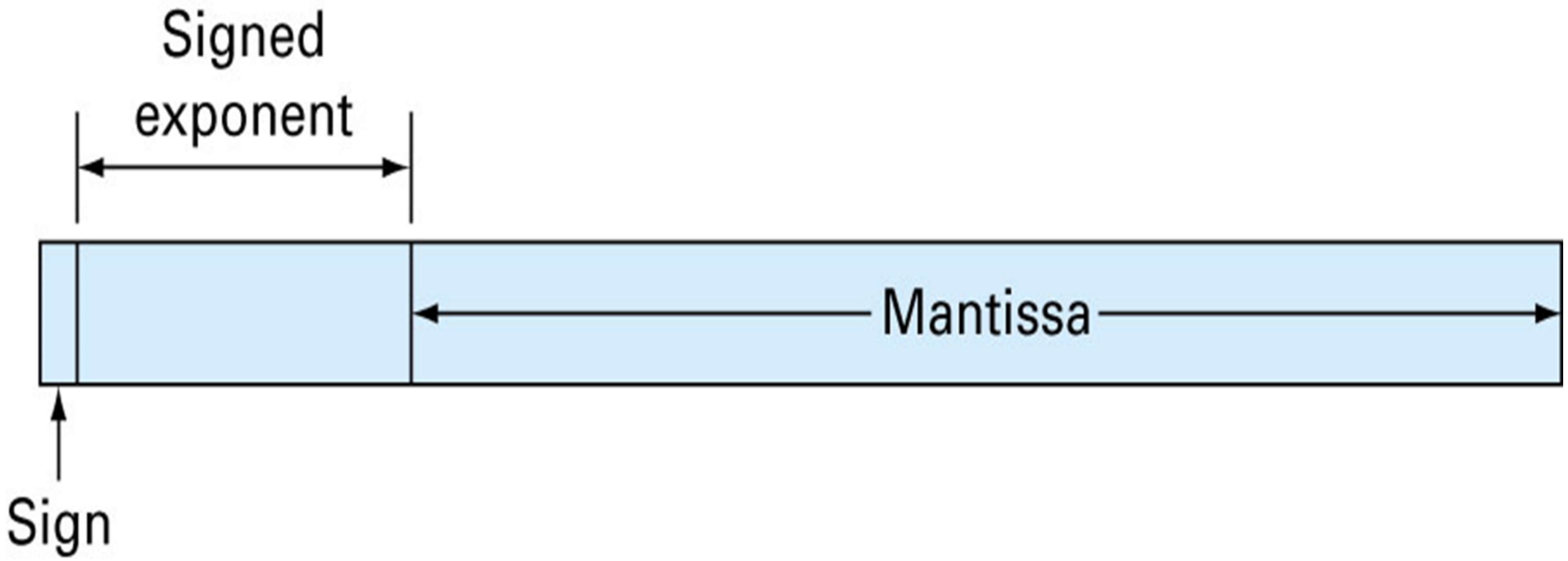


Figure 3.5



156.78 ⏩ 0.15678×10^3 in a floating point base-10 system

$$\frac{1}{34} = 0.029411765$$

Suppose only 4 decimal places to be stored

$$0.0294 \times 10^0 \qquad \frac{1}{2} \le |m| < 1$$

- Normalized to remove the leading zeroes. Multiply the mantissa by 10 and lower the exponent by 1

$0.294\underline{1} \times 10^{-1}$

Additional significant figure is retained

$$\frac{1}{b} \le |m < 1|$$

Therefore

for a base-10 system $0.1 \le m < 1$

for a base-2 system $0.5 \le m < 1$

- Floating point representation allows both fractions and very large numbers to be expressed on the computer. However,
  - Floating point numbers take up more room.
  - Take longer to process than integer numbers.
  - Round-off errors are introduced because mantissa holds only a finite number of significant figures.

Chopping

Example:

π=3.14159265358 to be stored on a base-10 system carrying 7 significant digits.

π=3.141592 chopping error $\varepsilon_t = 0.00000065$

If rounded

π=3.141593 $\varepsilon_t = 0.00000035$

- Some machines use chopping, because rounding adds to the computational overhead. Since number of significant figures is large enough, resulting chopping error is negligible.

Chapter 4

Truncation Errors and the Taylor Series

n^{th} order approximation

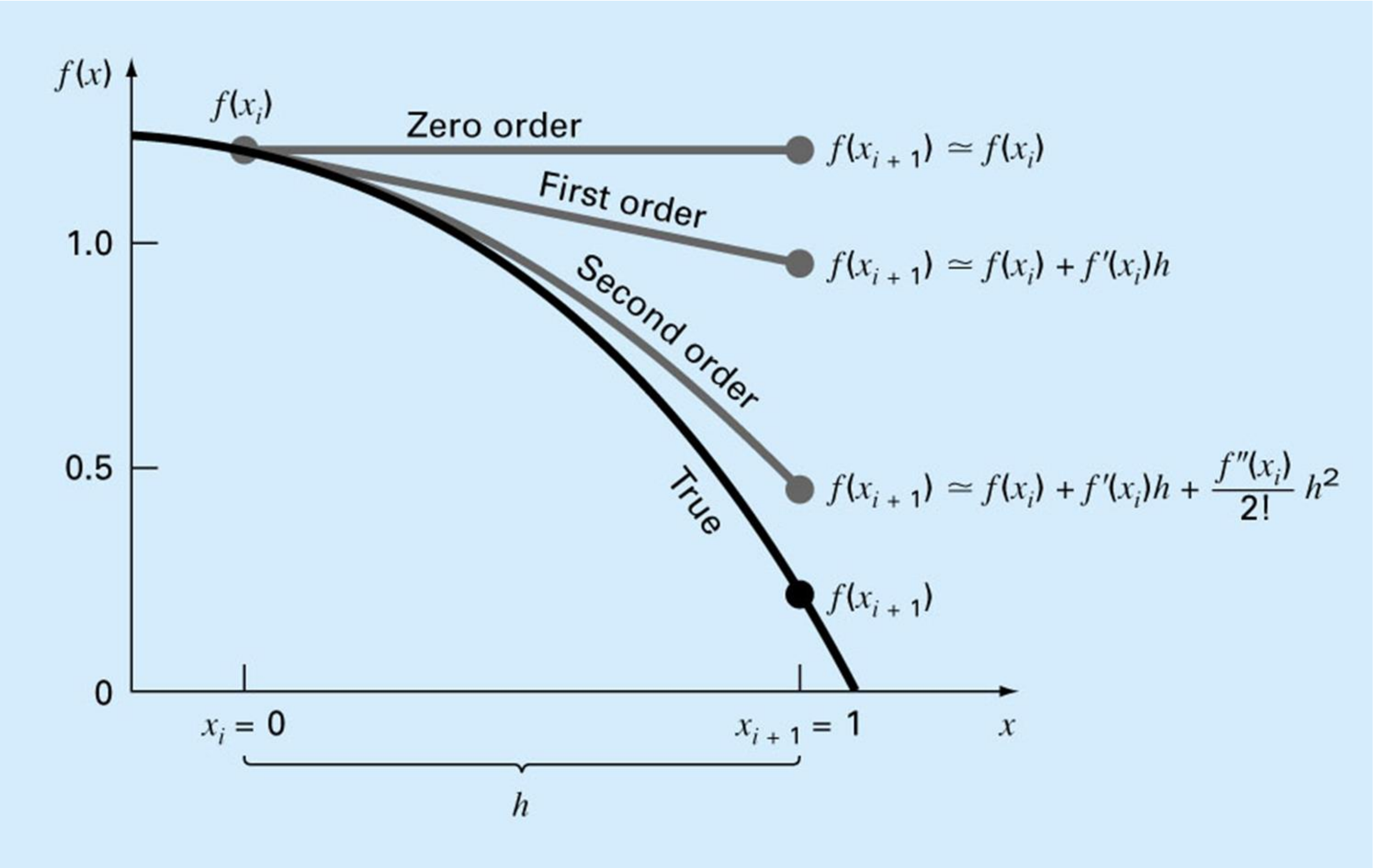
$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \dots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1}-x_i)= h$ *step size* (define first)

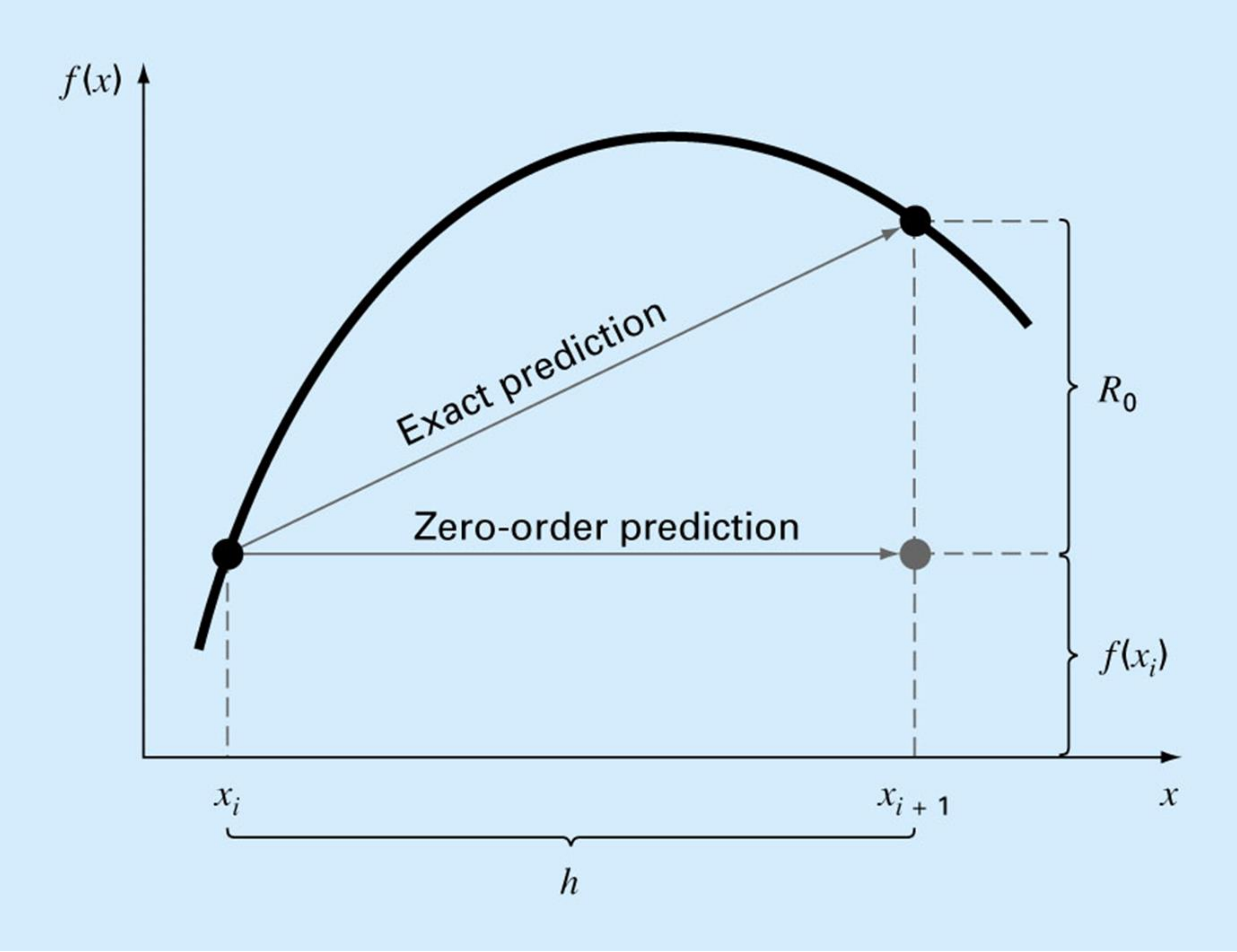
$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

- Reminder term, R_n, accounts for all terms from (n+1) to infinity.

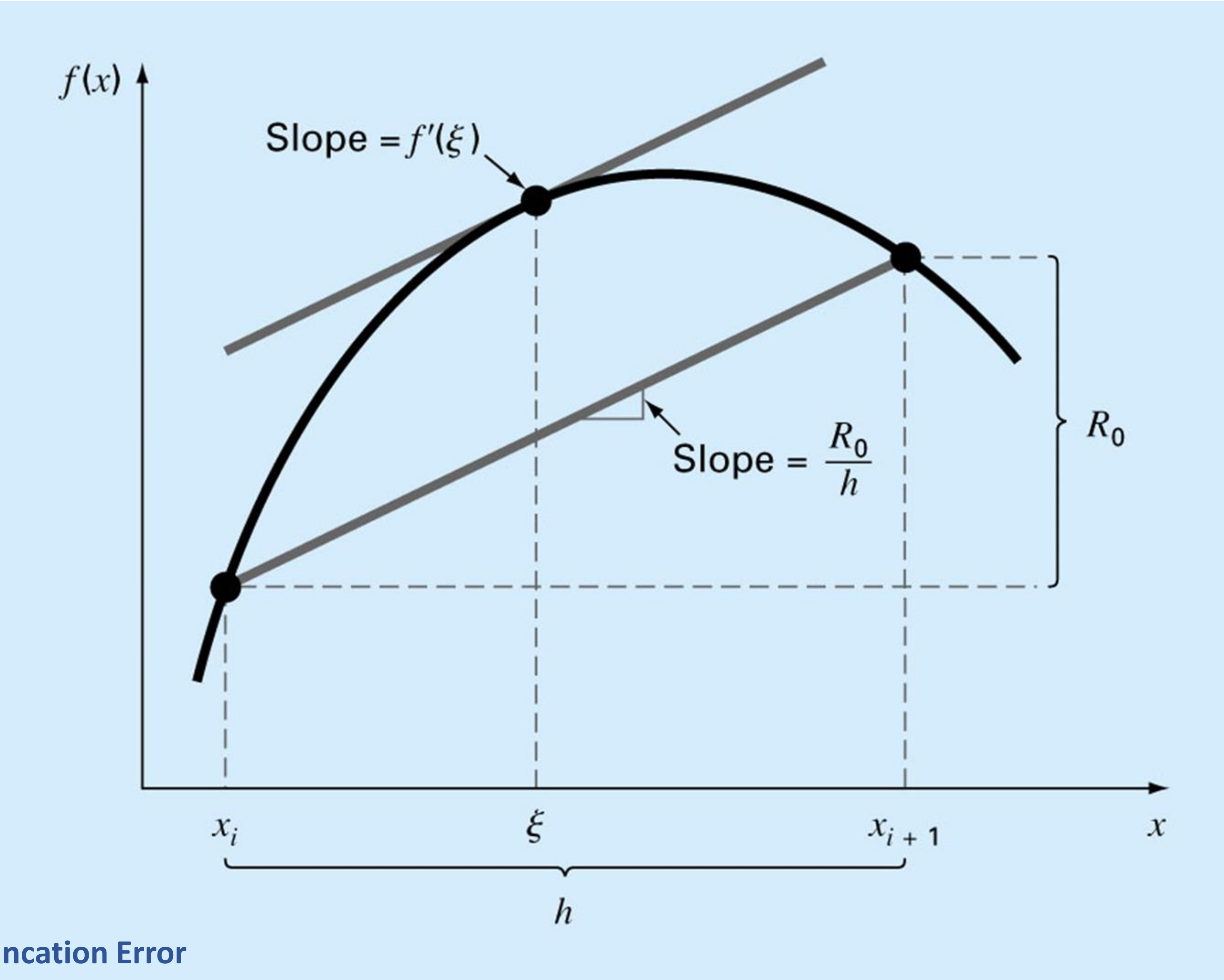


$f(x)$
$f(x_i)$
Zero order
$f(x_{i+1}) \simeq f(x_i)$
First order
$f(x_{i+1}) \simeq f(x_i) + f'(x_i)h$
Second order
$f(x_{i+1}) \simeq f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$
True
$f(x_{i+1})$
1.0
0.5
0
$x_i = 0$
$x_{i+1} = 1$
x
h

- ε is not known exactly, lies somewhere between $x_{i+1} > \varepsilon > x_i$.
- Need to determine $f^{n+1}(x)$, to do this you need $f'(x)$.
- If we knew f(x), there wouldn't be any need to perform the Taylor series expansion.
- However, $R = O(h^{n+1})$, $(n+1)^{th}$ order, the order of truncation error is h^{n+1}.
- $O(h)$, halving the step size will halve the error.
- $O(h^2)$, halving the step size will quarter the error.



$f(x)$
Exact prediction
Zero-order prediction
R_0
$f(x_i)$
x_i
x_{i+1}
x
h



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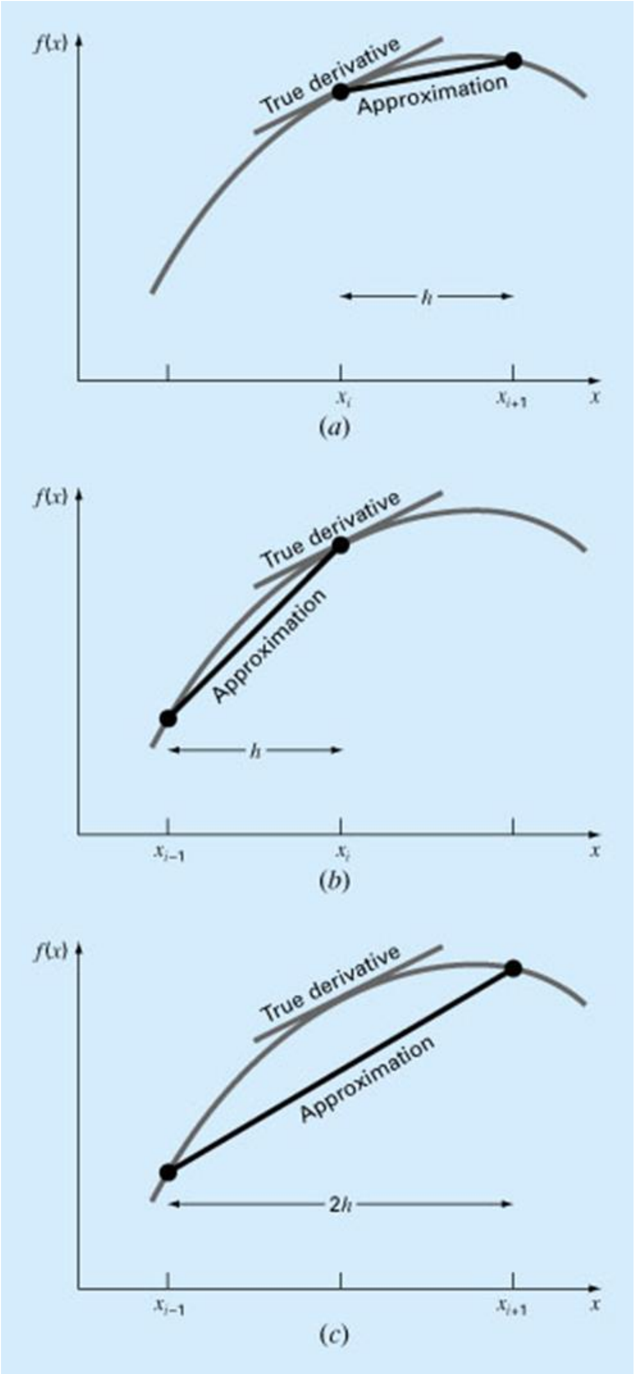
Problem 4.5
Determine truncation Error



Example 4.4
problem 4.5.

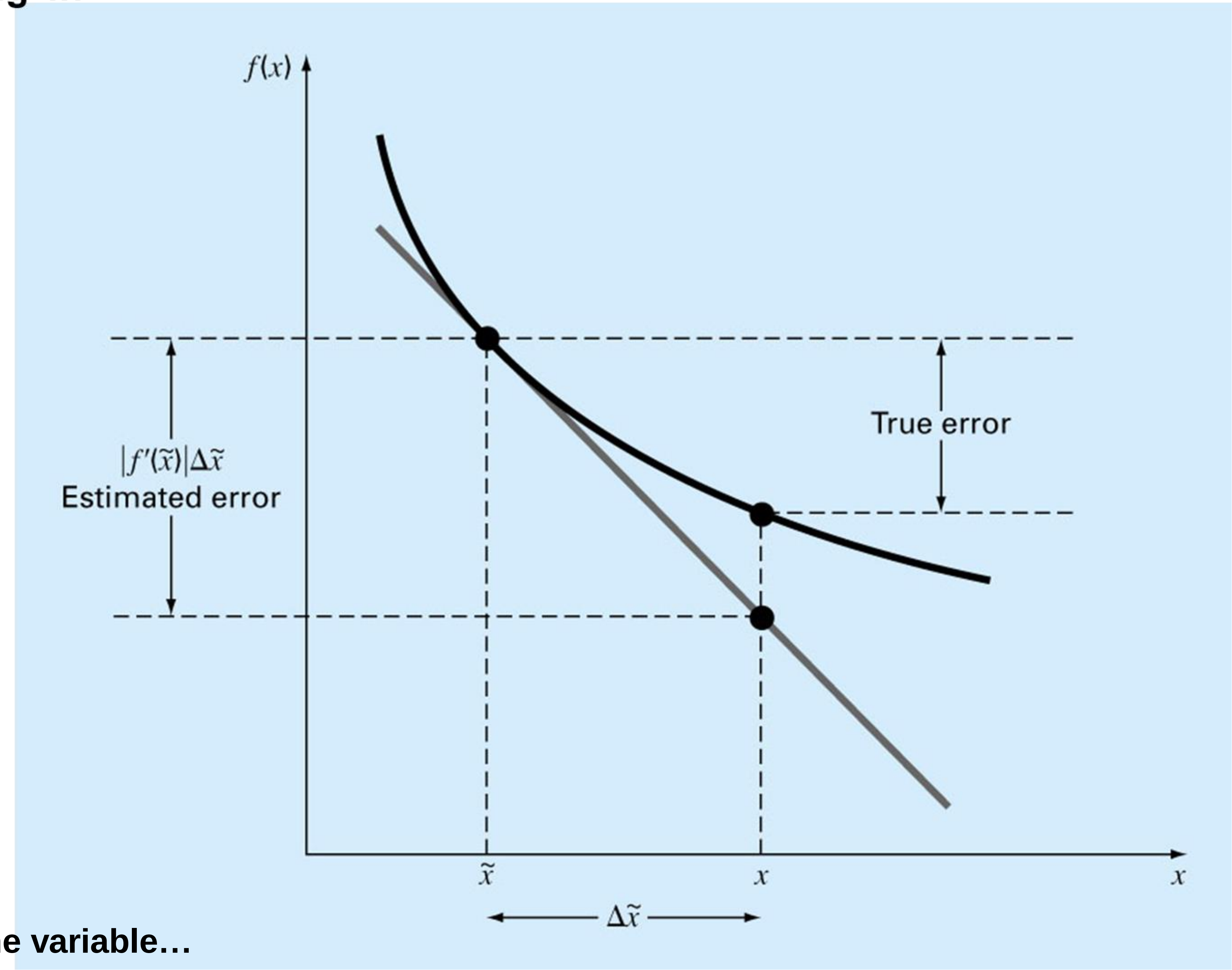
- Suppose that we have a function f(x) that is dependent on a single independent variable x. fl(x) is an approximation of x and we would like to estimate the effect of discrepancy between x and fl(x) on the value of the function:

$\Delta f(x_{fl}) = \left| f(x) - f(x_{fl}) \right|$ both f(x) and x_{fl} are unknown

Employ Taylor series to compute f(x) near f(x_{fl}), dropping the second and higher order terms

$$f(x) - f(x_{fl}) \cong f'(x_{fl})(x - x_{fl})$$

Fig 4.7



Example 4.5

Derive formula
for more than one variable…

- Addition of x_1 and x_2 with associated errors ε_{t1} and ε_{t2} yields the following result:

  $fl(x_1)=x_1(1+\varepsilon_{t1})$

  $fl(x_2)=x_2(1+\varepsilon_{t2})$

  $fl(x_1)+fl(x_2)=\varepsilon_{t1}\, x_1+\varepsilon_{t2}\, x_2+x_1+x_2$

$$\frac{\varepsilon_t}{100\%}=\frac{fl(x_1)+fl(x_2)-(x_1+x_2)}{x_1+x_2}=\frac{\varepsilon_{t1}x_1+\varepsilon_{t2}x_2}{x_1+x_2}$$

•A large error could result from addition if x_1 and x_2 are almost equal magnitude but opposite sign, therefore one should avoid subtracting nearly equal numbers.

- Multiplication of x_1 and x_2 with associated errors e_{t1} and e_{t2} results in:

$$fl(x_1)fl(x_2) = x_1(1+\varepsilon_{t1})\, x_2(1+\varepsilon_{t2})$$

$$fl(x_1)fl(x_2) = x_1x_2(\varepsilon_{t1}\varepsilon_{t2} + \varepsilon_{t1} + \varepsilon_{t2} + 1)$$

$$\frac{\varepsilon_t}{100\%} = \frac{fl(x_1)fl(x_2) - x_1x_2}{x_1x_2} = \varepsilon_{t1}\varepsilon_{t2} + \varepsilon_{t1} + \varepsilon_{t2}$$

Example 4-6

problem 4-6
problem 4-8
problem 4-15