Notes on Measure Theory

Math 414

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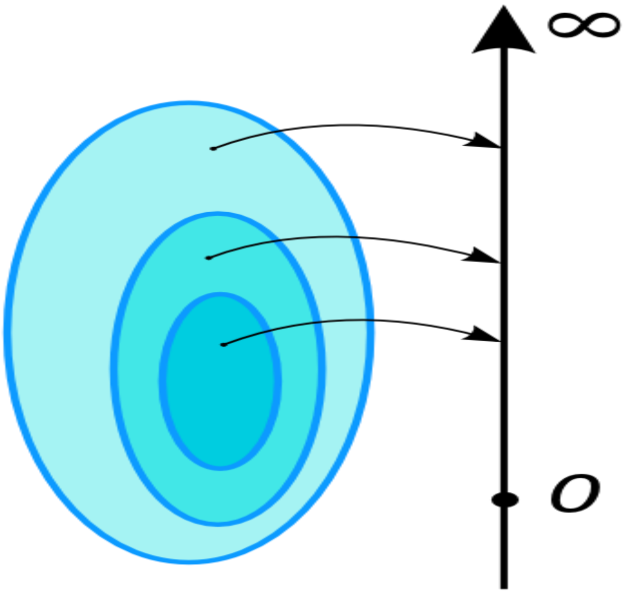

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Chapter I

Lebesgue Measure

1. Introduction

In mathematics, more specifically measure theory, a *measure* is a certain association between subsets of a given set X and the (extended set) of non-negative real numbers. Often, some subsets of a given set X are not required to be associated to a non-negative real number; the subsets which are required to be associated to a non-negative real number are known as the measurable subsets of X. The collection of all measurable subsets of X is required to form what is known as a sigma algebra; namely, a sigma algebra is a subcollection of the collection of all subsets of X that in addition, satisfies certain axioms.

Measures can be thought of as a generalization of the notions: 'length,' 'area' and 'volume.' The Lebesgue measure defines this for subsets of a Euclidean space, and an arbitrary measure generalizes this notion to subsets of any set. The original intent for measure was to define the Lebesgue integral, which increases the set of integrable functions considerably. It has since found numerous applications in probability theory, in addition to several other areas of academia, particularly in mathematical analysis. There is a related notion of volume form used in differential topology.

Definition 1.1:

The *length* $l(I)$ of an interval I is defined to be the difference of the endpoints of the interval.

i.e. if $I = [a,b]$ or $(a,b]$ or $[a,b)$ or (a,b) with $-\infty < a < b < \infty$, then $l(I) = b - a$.

If $a = -\infty$ or $b = \infty$, we define $l(I) = \infty$.

Definition 1.2:

If $E = \bigcup_{i=1}^{n} I_i$, where $I_i = [a_i, b_i]$ and $I_1, I_2, ...I_n$ are mutually disjoint, then $l(E) = \sum_{i=1}^{n} l(I_i)$.

Definition 1.3:

A *set function* is a function that associates an extended real number to each set in some collection of sets.

Example: The length $l(I)$ is a set function whose domain is the set of all intervals.

Is it possible to extend the notion of length to a more complicated sets then intervals?

For example what is the length of $E = \{x : 0 \le x \le 1, x \text{ is a rational number}\}$?

We could define the length of an open set since any open set O can be expressed as the union of a countable number of mutually disjoint open intervals.

Definition 1.4:

Let O be an open set, then $l(O)=\sum_{i=1}^{\infty} l(I_i)$ where $O=\bigcup_{i=1}^{\infty} I_i$ and I_1, I_2, ... are mutually disjoint open intervals.

The class of open sets is still too restricted, we would like to construct a set function $m: M \rightarrow \overline{R}$ *such* that $\forall E \in M$ we have $m(E) \geq 0$, and we call $m(E)$ the *measure* of E where M is a collection of sets of real numbers.

We should like m to have the following properties:

1. $m(E)$ is defined $\forall E \subset R$. i.e., $M=P(R)$
2. For any interval I, $m(I)=l(I)$
3. If $\{E_n\}_{n=1}^{\infty}$ is a sequence in M with $E_n \cap E_m=\phi$, then $m(\bigcup_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} m(E_n)$
4. m is translation invariant. i.e., if $E \in M$ then $m(E+y)=m(E)$

   where $E+y=\{x+y: x \in E\}$ is obtained by replacing each point x in E by the point $x+y$

Remark:

Unfortunately, it is impossible to construct a set function having all four of these properties. So we are going to construct m such that $m(E)$ is not defined for all sets E of real numbers.

Definition 1.5:

A collection $\mathcal{A}$ of subsets of a set X is called an *algebra* of sets if:

i) If $A, B \in \mathcal{A}$, then $A \bigcup B \in \mathcal{A}$.

ii) If $A \in \mathcal{A}$, then $A^c = X - A \in \mathcal{A}$.

An algebra $\mathcal{A}$ of sets is called a *σ – algebra* if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$. i.e., if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

Note: We are going to require the family M for which m is defined to be a σ– algebra

Definition 1.6:

$m : M \to \overline{R}$ is said to be *countably additive measure* if:

i) $m(E) \geq 0$

ii) M is a σ– algebra of sets

iii) $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$ for each sequence $\{E_n\}_{n=1}^{\infty}$ of mutually disjoint sets in M.

Example 1.7:

Let $n : P(R) \to \overline{R}$ be defined by $n(E) = \infty$ if E is an infinite set and $n(E) =$ the number of elements in E if it is finite. Show that n is a countably additive measure that is translation invariant.

Solution: (i) Clearly $n(E) \geq 0$.

(ii) $M = P(R)$ which is a σ– algebra.

(iii) Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of mutually disjoint sets in R.

a) If $\exists i$ such that E_i is an infinite set, then $\bigcup_{n=1}^{\infty} E_n$ is also infinite, and so

$$n(E_i) = n(\bigcup_{n=1}^{\infty} E_n) = \infty$$

But $\sum_{n=1}^{\infty} n(E_n) = n(E_1) + n(E_2) + ... + n(E_i) + ... = \infty$

Therefore, $n(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} n(E_n)$

b) If E_n is a finite set for every n = 1,2,3,……, then $n(E_n) = r_n$, where r_n is the number of elements in E_n. So $\sum_{n=1}^{\infty} n(E_n) = r_1 + r_2 + ... + r_n + ... = \infty$.On the other hand $\bigcup_{n=1}^{\infty} E_n$ is infinite since the sets *En* are mutually disjoint and therefore $n(\bigcup_{n=1}^{\infty} E_n) = \infty = \sum_{n=1}^{\infty} n(E_n)$.

Hence, *n* is a countably additive measure. To prove that it is translation invariant:

a) If E is infinite then so is $E+y$. Therefore $n(E) = n(E+y) = \infty$.

b) If E is finite then the number of elements in E is the same as the number of elements in $E+y$. Therefore $n(E) = n(E+y)$.

Hence, *n* is a translation invariant.

Problem Set 1

1. Let $\mathcal{A}$ be an algebra of sets of X. Prove that if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$.

2. Let $\mathcal{A}$ be an σ– algebra of sets of X. Prove that if $\{A_i\}_{i=1}^{\infty} \subset \mathcal{A}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$.

3. Let $\mathcal{A}$ be an algebra of sets of a finite set X. Is $\mathcal{A}$ a topology on X? Is the converse true?

4. Give an example of an algebra $\mathcal{A}$ on $[0, \infty]$ that is not a σ– algebra.

5. Let m be a countably additive measure. If $A, B \in \mathrm{M}$ with $A \subset B$, prove that $mA \le mB$. This property is called *monotonicity*.

6. Let m be a countably additive measure. If $\exists A \in M$ such that $m(A) < \infty$, prove that $m(\phi) = 0$.

7. Let $M = \{E \subset R : E$ or E^c is countable$\}$. Define $m : M \to \overline{R}$ by

$$m(E) = \begin{cases} 0 & \text{if } E \text{ is countable} \\ 1 & \text{if } E^c \text{ is countable} \end{cases}$$

Show that m is a countably additive measure.

2. Outer Measure

Definition 2.1: Let $A \subset R$

(i) The set A is said to be *bounded above* if there exists a number $u \in R$ such that $x \leq u \quad \forall x \in A$. Each number u is called an upper bound of A.

The set A is said to be *bounded below* if there exists a number $w \in R$ such that $x \geq w$ $\forall x \in A$. Each number w is called a lower bound of A.

(ii) $a_1 = \sup A$ iff $a_1 = \min \{u : u$ is an upper bound of $A\}$. *i.e.,* $a_1 \leq u$ $\forall$ upper bound u of A.

Also $a_o = \inf A$ iff $a_o = \max \{w : w$ is a lower bound of $A\}$. *i.e.,* $a_o \geq w$ $\forall$ lower bound w of A.

Definition 2.2:

Let $A \subset R$. Consider the countable collections of open intervals $\{I_n\}_{n=1}^{\infty}$ for which $A \subset \bigcup_{n=1}^{\infty} I_n$, and for each such collection consider the sum of the length of the intervals in the collection $\sum_{n=1}^{\infty} l(I_n)$. We define the *outer measure* (Lebesgue outer measure) m^*A to be:

$$m^*A = \inf \left\{ \sum_{n=1}^{\infty} l(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n \right\}$$

Proposition 2.3:

i) If $A \subset B$, then $m^* A \leq m^* B$.

ii) $m^* \phi = 0$.

iii) $m^* \{x\} = 0$.

Proof: i) Let $\varepsilon > 0$, then $m^* B + \varepsilon > m^* B$ and hence $m^* B + \varepsilon$ is not a lower bound of the set $\left\{ \sum_{n=1}^{\infty} l(I_n) : B \subset \bigcup_{n=1}^{\infty} I_n \right\}$. Therefore, there exists a covering $\{I_n\}_{n=1}^{\infty}$ of B such that $m^* B + \varepsilon > \sum_{n=1}^{\infty} l(I_n)$.

But $A \subset B \subset \bigcup_{n=1}^{\infty} I_n \Rightarrow \{I_n\}_{n=1}^{\infty}$ is a covering of A also $\Rightarrow m^* A \leq \sum_{n=1}^{\infty} l(I_n)$.

Therefore, $m^* A < m^* B + \varepsilon \ \ \forall \varepsilon > 0$, and hence $m^* A \leq m^* B$.

ii) $\phi \subset (0, \frac{1}{n}) \Rightarrow m^* \phi \leq l(0, \frac{1}{n}) = \frac{1}{n}$.

Therefore $0 \leq m^* \phi \leq \frac{1}{n} \ \ \forall n \in N$, and hence $m^* \phi = 0$.

iii) $\{x\} \subset (x - \frac{1}{n}, x + \frac{1}{n}) \Rightarrow m^* \{x\} \leq l(x - \frac{1}{n}, x + \frac{1}{n}) = \frac{2}{n}$.

Therefore $0 \le m^*\{x\} \le \frac{2}{n} \quad \forall n \in N$, and hence $m^*\{x\} = 0$.

Proposition 2.4:

The outer measure of an interval is its length, i.e., $m^* I = l(I)$ for any interval *I*.

Proof: **Case (1)**: *I* is a closed finite interval. i.e., $I = [a, b]$ with $-\infty < a < b < \infty$.

Let $\varepsilon > 0$. Then $[a,b] \subset (a-\varepsilon, b+\varepsilon) \Rightarrow m^*[a,b] \le l(a-\varepsilon, b+\varepsilon) = b-a+2\varepsilon$.

Therefore, $m^*[a,b] \le b-a+2\varepsilon \quad \forall \varepsilon > 0$, and hence $m^*[a,b] \le b-a$ (1)

Conversely, we want to show that $m^*[a,b] \ge b-a$.

If we could prove that for any countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covering $[a, b]$ we have $\sum_{n=1}^{\infty} l(I_n) > b-a$, then we are done because this means that $b-a$ is a lower bound of the set $\left\{ \sum_{n=1}^{\infty} l(I_n) : [a,b] \subset \bigcup_{n=1}^{\infty} I_n \right\}$ which implies that $m^*[a,b] \ge b-a$.

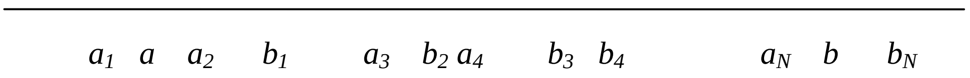
So we are going to prove that $\sum_{n=1}^{\infty} l(I_n) > b-a$ for any countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ covering $[a, b]$. Now $[a, b]$ is compact, therefore any open cover has a finite subcover, $\{I_n\}_{n=1}^{N}$, i.e., $[a,b] \subset \bigcup_{n=1}^{N} I_n$ and $\sum_{n=1}^{N} l(I_n) \le \sum_{n=1}^{\infty} l(I_n)$.

Now $a \in \bigcup_{n=1}^{N} I_n \Rightarrow \exists (a_1, b_1) \in \{I_n\}_{n=1}^{\infty}$ such that $a \in (a_1, b_1)$ and we have $a_1 < a < b_1$.

If $b_1 < b$, then $b_1 \in [a, b] \subset \bigcup_{n=1}^{N} I_n \Rightarrow b_1 \in \bigcup_{n=1}^{N} I_n \Rightarrow \exists (a_2, b_2) \in \{I_n\}_{n=1}^{\infty}$ such that $b_1 \in (a_2, b_2)$ and we have $a_2 < b_1 < b_2 \Rightarrow a_2 - b_1 < 0$.

Again If $b_2 < b$, then $b_2 \in \bigcup_{n=1}^{N} I_n \Rightarrow \exists (a_3, b_3) \in \{I_n\}_{n=1}^{\infty}$ such that $b_2 \in (a_3, b_3)$ and we have $a_3 < b_2 < b_3 \Rightarrow a_3 - b_2 < 0$.

We continue until we reach an interval (a_N, b_N) such that $b \in (a_N, b_N) \Rightarrow a_N < b < b_N$



Thus, $\sum_{n=1}^{N} l(I_n) = \sum_{n=1}^{N} l(a_n, b_n) = \sum_{n=1}^{N} b_n - a_n$

$$= (b_N - a_N) + (b_{N-1} - a_{N-1}) + \ldots\ldots\ldots\ldots + (b_1 - a_1)$$

$$= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \ldots\ldots\ldots\ldots - (a_2 - b_1) - a_1$$

$$> b_N - a_1 > b - a$$

Therefore $\sum_{n=1}^{\infty} l(I_n) \geq \sum_{n=1}^{N} l(I_n) > b - a \Rightarrow m^*[a, b] \geq b - a$. (2)

From (1) and (2) we get that $m^*[a,b] = b - a = l\,(I)$.

Case (2): *I* is any finite interval (open or half open, i.e., *I* = (*a* ,*b*) or [*a* , *b*) or (*a* ,*b*]).

Let $\varepsilon > 0$, then there exists a closed interval *J* such that $J \subset I$ and $l(J) > l(I) - \varepsilon$.

Hence, $l(I) - \varepsilon < l(J) = m^*J \le m^*I \le m^*\bar{I} = l(\bar{I}) = l(I) \;\; \forall \varepsilon > 0$.

i.e., $l(I) - \varepsilon < m^*I \le l(I) \;\; \forall \varepsilon > 0$, and so $m^*I = l(I)$.

Case (3): *I* is an infinite interval. In this case $l(I) = \infty$, so we want to prove that $m^*I = \infty$.

Let $\delta > 0$, then there exists a closed interval *J* such that $J \subset I$ with $l(J) = \delta$. Therefore, $m^*I \ge m^*J = l(J) = \delta$. i.e., $m^*I \ge \delta \;\; \forall \delta > 0$. Hence, $m^*I = \infty = l(I)$.

Proposition 2.5:

Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets of real numbers. Then

$$m^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} m^*A_n$$

Proof: Consider the collection $\{A_n\}_{n=1}^{\infty}$, and let $\varepsilon > 0$. Then for each A_n there exists a countable collection $\{I_{n,i}\}_{i=1}^{\infty}$ of open intervals such that $A_n \subset \bigcup_{i=1}^{\infty} I_{n,i}$ and

$$m^* A_n + \frac{\varepsilon}{2^n} > \sum_{i=1}^{\infty} l(I_{n,i}).$$

But $\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} I_{n,i}$, which means that $\{I_{n,i}\}_{n,i=1}^{\infty}$ covers $\bigcup_{n=1}^{\infty} A_n$. Therefore,

$$m^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n,i=1}^{\infty} l(I_{n,i}) = \sum_{n=1}^{\infty}\sum_{i=1}^{\infty} l(I_{n,i}) < \sum_{n=1}^{\infty}(m^* A_n + \frac{\varepsilon}{2^n}) = \sum_{n=1}^{\infty} m^* A_n + \varepsilon.$$

So we have $m^*(\bigcup_{n=1}^{\infty} A_n) < \sum_{n=1}^{\infty} m^* A_n + \varepsilon \quad \forall \varepsilon > 0$. Hence, $m^*(\bigcup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} m^* A_n$

Proposition 2.6:

If A is countable, then $m^* A = 0$.

Proof: Let $A = \{a_1, a_2, a_3, \ldots\ldots\ldots.\}$ and let $\varepsilon > 0$.

Let $I_1 = (a_1 - \frac{\varepsilon}{2^2}, a_1 + \frac{\varepsilon}{2^2})$, then $l(I_1) = \frac{\varepsilon}{2}$.

Let $I_2 = (a_2 - \frac{\varepsilon}{2^3}, a_2 + \frac{\varepsilon}{2^3})$, then $l(I_2) = \frac{\varepsilon}{2^2}$.

In general, let $I_n = (a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}})$, then $l(I_n) = \frac{\varepsilon}{2^n}$.

Now $\{I_n\}_{n=1}^{\infty}$ covers *A*. Therefore $m^* A \leq \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$.

i.e., $0 \leq m^* A \leq \varepsilon \ \ \forall \varepsilon > 0$. Hence, $m^* A = 0$.

Corollary 2.7:

The set [0 , 1] is not countable.

Proof: If [0 , 1] is countable, then by proposition 2.6 $m^*[0 , 1] = 0$.

But we know that $m^*[0 , 1] = l[0 , 1] = 1 \neq 0$. Therefore, [0 , 1] is not countable.

Problem Set 2

1. Prove that if $m^* A = 0$, then $m^*(A \cup B) = m^* B$.
2. Prove that m^* is translation invariant.
3. For $A \subset R$ define

$$\overline{m}^* A = \inf \left\{ \sum_{n=1}^{\infty} l(J_n) : A \subset \bigcup_{n=1}^{\infty} J_n \right\}$$

where J_n is an interval not necessarily open. Prove that $\overline{m}^* A = m^* A$.

3. Measurable Sets and Lebesgue Measure

While the outer measure has the advantage that it is defined for all sets, it is not countably additive. To make our outer measure countably additive, something has to give. We decide to restrict the domain to gain countable additivity. There are several ways to restrict an outer measure. In our course we will use an approach due to Caratheodory to define measurable sets.

Definition 3.1: A set E is said to be *measurable* if for each set A we have

$$m^*A = m^*(A \cap E) + m^*(A \cap E^c).$$

Proposition 3.2:

i) E is measurable if and only if for each A we have

$$m^*A \geq m^*(A \cap E) + m^*(A \cap E^c)$$

ii) E is measurable if and only if E^c is measurable.

iii) ϕ and R are measurable.

Proof: i) $A = A \cap R = A \cap (E \cup E^c) = (A \cap E) \cup (A \cap E^c)$.

Therefore, $m^* A = m^*[(A \cap E) \cup (A \cap E^c)] \leq m^*(A \cap E) + m^*(A \cap E^c)$ by prop. 2.5

So $m^* A = m^*(A \cap E) + m^*(A \cap E^c)$ if and only if $m^* A \geq m^*(A \cap E) + m^*(A \cap E^c)$.

Hence, *E* is measurable if and only if $m^* A \geq m^*(A \cap E) + m^*(A \cap E^c)$.

Lemma 3.3:

If $m^*E = 0$, then *E* is measurable.

Proof: Let *A* be any set. Then $A \cap E \subset E \Rightarrow m^*(A \cap E) \leq m^* E = 0$.

So $0 \leq m^*(A \cap E) \leq 0$ which means that $m^*(A \cap E) = 0$.

Also, $A \supset A \cap E^c \Rightarrow m^* A \geq m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c)$

i.e., $m^* A \geq m^*(A \cap E) + m^*(A \cap E^c)$ and hence *E* is measurable.

Lemma 3.4:

If E_1 and E_2 are measurable then $E_1 \cup E_2$ is measurable.

Proof: We want to show that for any set *A* we have

$$m^* A \geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c).$$

E_2 is measurable, so for any set T we have: $m^*T \geq m^*(T \bigcap E_2) + m^*(T \bigcap E_2{}^c)$.

Let $T = A \bigcap E_1^c$, then $m^*(A \bigcap E_1{}^c) \geq m^*(A \bigcap E_1{}^c \bigcap E_2) + m^*(A \bigcap E_1{}^c \bigcap E_2{}^c)$ (1)

Now $E_1 \bigcup E_2 = (E_1 \bigcup E_2) \bigcap R = (E_1 \bigcup E_2) \bigcap (E_1 \bigcup E_1{}^c) = E_1 \bigcup (E_2 \bigcap E_1{}^c)$.

Therefore, $A \bigcap (E_1 \bigcup E_2) = A \bigcap [E_1 \bigcup (E_2 \bigcap E_1{}^c)] = (A \bigcap E_1) \bigcup (A \bigcap E_2 \bigcap E_1{}^c)$

$\Rightarrow m^*(A \bigcap [E_1 \bigcup E_2]) \leq m^*(A \bigcap E_1) + m^*(A \bigcap E_2 \bigcap E_1{}^c)$ (2)

From (1) and (2), we get

$$m^*(A \bigcap [E_1 \bigcup E_2]) + m^*(A \bigcap [E_1 \bigcup E_2]^c) = m^*(A \bigcap [E_1 \bigcup E_2]) + m^*(A \bigcap E_1{}^c \bigcap E_2{}^c)$$

$$\leq m^*(A \bigcap E_1) + m^*(A \bigcap E_2 \bigcap E_1{}^c) + m^*(A \bigcap E_1{}^c \bigcap E_2{}^c) = m^*(A \bigcap E_1) + m^*(A \bigcap E_1{}^c)$$

$= m^*A$ (since E_1 is measurable).

Therefore, $m^*A \geq m^*(A \bigcap [E_1 \bigcup E_2]) + m^*(A \bigcap [E_1 \bigcup E_2]^c)$.

Hence, $E_1 \bigcup E_2$ is measurable.

Corollary 3.5:

The collection M of measurable sets is an algebra of sets.

Lemma 3.6:

Let *A* be any set and $E_1, E_2, \ldots\ldots\ldots, E_n$ be a finite sequence of disjoint measurable sets. Then $m^*(A\cap[\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A\cap E_i)$.

Proof: We will prove the lemma by induction on *n*.

For $n = 1$, $m^*(A\cap[\bigcup_{i=1}^{1} E_i]) = m^*(A\cap E_1)$

and $\sum_{i=1}^{1} m^*(A\cap E_i) = m^*(A\cap E_1)$. So It is true for $n=1$.

Assume it is true for $n-1$ sets, i.e. $m^*(A\cap[\bigcup_{i=1}^{n-1} E_i]) = \sum_{i=1}^{n-1} m^*(A\cap E_i)$, and we will prove that it is true for *n* sets.

Now E_n is measurable, therefore

$$m^*(A\cap[\bigcup_{i=1}^{n} E_i]) = m^*(A\cap[\bigcup_{i=1}^{n} E_i]\cap E_n) + m^*(A\cap[\bigcup_{i=1}^{n} E_i]\cap E_n{}^c).$$

$$A\cap[\bigcup_{i=1}^{n} E_i]\cap E_n = A\cap[\bigcup_{i=1}^{n}(E_i\cap E_n)] = A\cap E_n$$

$$A\cap[\bigcup_{i=1}^{n} E_i]\cap E_n{}^c = A\cap[\bigcup_{i=1}^{n}(E_i\cap E_n{}^c)] = A\cap[\bigcup_{i=1}^{n-1} E_i]$$

Hence, $m^*(A\cap[\bigcup_{i=1}^{n} E_i]) = m^*(A\cap E_n) + m^*(A\cap[\bigcup_{i=1}^{n-1} E_i])$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

So we proved that $m^*(A \cap [\bigcup_{i=1}^{n} E_i]) = \sum_{i=1}^{n} m^*(A \cap E_i)$.

Corollary 3.7:

Let $E_1, E_2, \ldots\ldots\ldots\ldots, E_n$ be a finite sequence of disjoint measurable sets. Then $m^*(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} m^* E_i$.

Proof: Let $A = R$ in lemma 3.6

Theorem 3.8:

The collection M of measurable sets is a σ-algebra; that is the complement of a measurable set is measurable and the union (also intersection) of a countable collection of measurable sets is measurable.

Proof: By corollary 3.5 M is an algebra of sets. So we only have to prove that if $\{E_i\}_{i=1}^{\infty}$ is a countable collection of measurable sets, then $\bigcup_{i=1}^{\infty} E_i$ is measurable.

We can find a sequence $\{F_i\}_{i=1}^{\infty}$ of disjoint measurable sets such that $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$.

Let *A* be any set, by lemma 3.4 $\bigcup_{i=1}^{n} F_i$ is measurable. So we have,

$$m^* A = m^*(A \cap [\bigcup_{i=1}^{n} F_i]) + m^*(A \cap [\bigcup_{i=1}^{n} F_i]^c) \qquad (1)$$

Also, $\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{\infty} F_i \Rightarrow [\bigcup_{i=1}^{n} F_i]^c \supset [\bigcup_{i=1}^{\infty} F_i]^c \Rightarrow A \cap [\bigcup_{i=1}^{n} F_i]^c \supset A \cap [\bigcup_{i=1}^{\infty} F_i]^c$

Therefore, $m^*(A \cap [\bigcup_{i=1}^{n} F_i]^c) \geq m^*(A \cap [\bigcup_{i=1}^{\infty} F_i]^c)$ (2)

By lemma 3.6, $m^*(A \cap [\bigcup_{i=1}^{n} F_i]) = \sum_{i=1}^{n} m^*(A \cap F_i)$ (3)

From (1), (2) and (3), we get $m^* A \geq \sum_{i=1}^{n} m^*(A \cap F_i) + m^*(A \cap [\bigcup_{i=1}^{\infty} F_i]^c)$.

Since the left side of this inequality is independent of *n*, we have

$$m^* A \geq \sum_{i=1}^{\infty} m^*(A \cap F_i) + m^*(A \cap [\bigcup_{i=1}^{\infty} F_i]^c) .$$

Now, $m^*(A \cap [\bigcup_{i=1}^{\infty} F_i]) = m^*(\bigcup_{i=1}^{\infty} [A \cap F_i]) \leq \sum_{i=1}^{\infty} m^*(A \cap F_i)$.

Hence, $m^*A \geq m^*(A \bigcap [\bigcup_{i=1}^{\infty} F_i]) + m^*(A \bigcap [\bigcup_{i=1}^{\infty} F_i]^c)$.

Finally since $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$, we have $m^*A \geq m^*(A \bigcap [\bigcup_{i=1}^{\infty} E_i]) + m^*(A \bigcap [\bigcup_{i=1}^{\infty} E_i]^c)$, and therefore, $\bigcup_{i=1}^{\infty} E_i$ is measurable.

Lemma 3.9:

The interval (a , ∞) is measurable.

Proof: Let *A* be any set. We want to prove that

$$m^*A \geq m^*(A \bigcap (a,\infty)) + m^*(A \bigcap (a,\infty)^c) = m^*(A \bigcap (a,\infty)) + m^*(A \bigcap (-\infty,a]).$$

Let $A_1 = A \bigcap (a,\infty)$ and $A_2 = A \bigcap (-\infty,a]$, then $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \phi$.

We want to show that $m^*A \geq m^*A_1 + m^*A_2$.

Let $\varepsilon > 0$, then there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ which covers *A* (i.e., $A \subset \bigcup_{n=1}^{\infty} I_n$) and for which $\sum_{n=1}^{\infty} l(I_n) < m^*A + \varepsilon$.

Let $I_n^{'} = I_n \bigcap (a,\infty)$ and $I_n^{''} = I_n \bigcap (-\infty, a]$, then $I_n = I_n^{'} \bigcup I_n^{''}$ and $I_n^{'} \bigcap I_n^{''} = \varphi$ and both $I_n^{'}$ and $I_n^{''}$ are intervals (or empty).

So, $l(I_n) = l(I_n^{'} \bigcup I_n^{''}) = l(I_n^{'}) + l(I_n^{''}) = m^* I_n^{'} + m^* I_n^{''}$.

Since $A_1 = A \bigcap (a,\infty) \subset (\bigcup_{n=1}^{\infty} I_n) \bigcap (a,\infty) = \bigcup_{n=1}^{\infty} (I_n \bigcap (a,\infty)) = \bigcup_{n=1}^{\infty} I_n^{'}$, then

$m^* A_1 \le m^* (\bigcup_{n=1}^{\infty} I_n^{'}) \le \sum_{n=1}^{\infty} m^* I_n^{'}$.

Also $A_2 \subset \bigcup_{n=1}^{\infty} I_n^{''} \Rightarrow m^* A_2 \le m^* (\bigcup_{n=1}^{\infty} I_n^{''}) \le \sum_{n=1}^{\infty} m^* I_n^{''}$.

Thus, $m^* A_1 + m^* A_2 \le \sum_{n=1}^{\infty} m^* I_n^{'} + \sum_{n=1}^{\infty} m^* I_n^{''} = \sum_{n=1}^{\infty} (m^* I_n^{'} + m^* I_n^{''}) = \sum_{n=1}^{\infty} l(I_n) \le m^* A + \varepsilon$.

Therefore, $m^* A_1 + m^* A_2 \le m^* A + \varepsilon \quad \forall \varepsilon > 0$, and hence $m^* A_1 + m^* A_2 \le m^* A$.

Definition 3.10:

A *Borel set* is any set that can be formed from open sets and closed sets through the operations of countable union and countable intersection.

Remarks:

- The collection $\mathcal{B}$ of all Borel sets on a set *X* forms a σ-algebra, known as the *Borel algebra*. The Borel algebra on *X* is the smallest σ-algebra containing all open sets and closed sets.

- Borel sets are important in measure theory, since any measure defined on open sets and closed sets must also be defined on all Borel sets.
- Almost every set that you will run into is a Borel set. It takes a certain amount of work to show that there are some sets which are not Borel sets.

Question: Can you find a non-Borel set?

Theorem 3.11:

Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proof: First we are going to prove that each open interval is measurable:

$(-\infty, a] = (a, \infty)^c$, therefore by $(-\infty, a]$ is measurable by proposition 3.2 , lemma 3.9.

$(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$, therefore $(-\infty, b)$ is measurable by theorem 3.8.

Hence, each open interval $(a,b) = (-\infty, b) \bigcap (a, \infty)$ is measurable.

Secondly, we are going to prove that each open set is measurable:

Let O be an open set, then we can write O as the union of a countable number of open intervals and so must be measurable by theorem 3.8.

Finally, we are going to prove that each closed set is measurable:

Let F be a closed set, then F^c is open and so F^c is measurable. This implies that $\left(F^c\right)^c = F$ is measurable.

Thus, the collection M of measurable sets is a σ-algebra that contains all open sets and closed sets and must therefore contain the collection $\mathcal{B}$ of Borel sets since $\mathcal{B}$ is the smallest σ-algebra containing all open sets and closed sets.

Now we are ready to define the Lebesgue measure introduced by Henri Lebesgue in the first decade of the twentieth century.

Definition 3.12:

We define the *Lebesgue measure* m to be the set function obtained by restricting the outer measure m^* to the family M of measurable sets. i.e., if E is measurable, we define the Lebesgue measure mE to be the outer measure of E.

Two important properties of Lebesgue measure are summarized by proposition 3.13 and 3.15:

Proposition 3.13:

Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets. Then

$$m(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} m(E_i)$$

If the sets E_i are pairwise disjoint, then

$$m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

Proof: $m(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} m(E_i)$ by proposition 2.5.

If $\{E_i\}_{i=1}^{n}$ is a finite sequence of disjoint measurable sets, then by corollary 3.7 we have $m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} mE_i$ and so *m* is finitely additive.

Let $\{E_i\}_{i=1}^{\infty}$ be an infinite sequence of pairwise disjoint measurable sets. Then $\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i$, and so $m(\bigcup_{i=1}^{\infty} E_i) \ge m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} mE_i$.

Since the left side of this inequality is independent of *n*, we have $m(\bigcup_{i=1}^{\infty} E_i) \ge \sum_{i=1}^{\infty} mE_i$.

Also $m(\bigcup_{i=1}^{\infty} E_i) \le \sum_{i=1}^{\infty} mE_i$ by proposition 2.5 and hence, $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} mE_i$.

Proposition 3.14:

i) If $A, B \in M$, then $A \setminus B \in M$ and $B \setminus A \in M$.

ii) If $A \subset B$, then $m(B \setminus A) = mB - mA$.

Proof: i) $A \setminus B = A \cap B^c \in M$ since $A \in M$ and $B^c \in M$.

Similarly $B \setminus A = B \cap A^c \in M$

ii) If $A \subset B$, then $B = (B \setminus A) \bigcup A$ and $(B \setminus A) \bigcap A = \phi$.

Then $mB = m(B \setminus A) + mA$ by proposition 3.13, which implies that

$m(B \setminus A) = mB - mA$.

Proposition 3.15:

Let $\{E_i\}_{i=1}^{\infty}$ be an infinite decreasing sequence of measurable sets, i.e., a sequence with $E_{n+1} \subset E_n$. Let $mE_1 < \infty$ then

$$m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n \to \infty} mE_n$$

Proof: Let $E = \bigcap_{i=1}^{\infty} E_i$ and $F_i = E_i \setminus E_{i+1}$.

Then $E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i$ and the sets F_i are pairwise disjoint.

Hence, $m(E_1 \setminus E) = m(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} mF_i = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1})$

$E \subset E_1 \Rightarrow m(E_1 \setminus E) = mE_1 - mE$.

$E_{i+1} \subset E_i \Rightarrow m(E_i \setminus E_{i+1}) = mE_i - mE_{i+1}$.

Therefore, $mE_1 - mE = \sum_{i=1}^{\infty} (mE_i - mE_{i+1}) = \lim_{N \to \infty} \sum_{i=1}^{N} (mE_i - mE_{i+1})$

$$= \lim_{N\to\infty}(mE_1 - mE_2 + mE_2 - mE_3 + ... + mE_{N-1} - mE_N + mE_N - mE_{N+1})$$

$$= \lim_{N\to\infty}(mE_1 - mE_{N+1}) = mE_1 - \lim_{n\to\infty} mE_n$$

So $mE_1 - mE = mE_1 - \lim_{n\to\infty} mE_n$.

Since $mE_1 < \infty$, we have $mE = \lim_{n\to\infty} mE_n$, and therefore $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$.

The following proposition expresses a number of ways in which a measurable set is very nearly a nice set.

Proposition 3.16:

Let E be a given set. The following statements are equivalent:

i) E is measurable.

ii) Given $\varepsilon > 0$ there exists an open set $O \supset E$ with $m(O \setminus E) < \varepsilon$.

iii) Given $\varepsilon > 0$ there exists a closed set $F \subset E$ with $m(E \setminus F) < \varepsilon$.

Proof: We will prove that (i) $\Leftrightarrow$ (ii). The rest is left as an exercise.

(i) $\Rightarrow$ (ii): Let E be a measurable set.

Case 1: $mE < \infty$:

Given $\varepsilon > 0$ there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ which covers *E* (i.e., $E \subset \bigcup_{n=1}^{\infty} I_n$) and for which $\sum_{n=1}^{\infty} l(I_n) < mE + \varepsilon$.

Let $O = \bigcup_{n=1}^{\infty} I_n$, then *O* is open and $E \subset O$.

Also $mO = m(\bigcup_{n=1}^{\infty} I_n) \le \sum_{n=1}^{\infty} l(I_n) < mE + \varepsilon \Rightarrow mO < mE + \varepsilon \Rightarrow mO - mE < \varepsilon$.

But $E \subset O \Rightarrow m(O \setminus E) = mO - mE < \varepsilon$ and hence (ii) holds.

<u>Case 2:</u> $mE = \infty$:

For every *n* set $B_n = [-n , n]$, and $E_n = E \cap B_n$.

Then $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap B_n) = E \cap (\bigcup_{n=1}^{\infty} B_n) = E \cap R = E$.

E_n is measurable since *E* and B_n are measurable, and $mE_n \le mB_n = 2n < \infty$. So, by applying case 1 on each E_n , there exists an open set $O_n \supset E_n$ with $m(O_n \setminus E_n) < \frac{\varepsilon}{2^n}$.

Let $O = \bigcup_{n=1}^{\infty} O_n$, then *O* is open and $E \subset O$.

$$O \setminus E = (\bigcup_{n=1}^{\infty} O_n) \setminus (\bigcup_{n=1}^{\infty} E_n) = (\bigcup_{n=1}^{\infty} O_n) \cap (\bigcup_{n=1}^{\infty} E_n)^c = (\bigcup_{n=1}^{\infty} O_n) \cap (\bigcap_{n=1}^{\infty} E_n^c)$$

$$= \bigcup_{n=1}^{\infty}[O_n \cap (\bigcap_{n=1}^{\infty} E_n^c)] \subset \bigcup_{n=1}^{\infty}(O_n \cap E_n^c) = \bigcup_{n=1}^{\infty}(O_n \setminus E_n).$$

Therefore, $m(O \setminus E) \leq m\left(\bigcup_{n=1}^{\infty}(O_n \setminus E_n)\right) \leq \sum_{n=1}^{\infty} m(O_n - E_n) < \sum_{n=1}^{\infty}\frac{\varepsilon}{2^n} = \varepsilon$

and hence (ii) holds.

(ii) $\Rightarrow$(i): Assume that (ii) holds.

For each *n* choose an open set O_n such that $O_n \supset E$ and $m(O_n \setminus E) < \frac{1}{n}$.

Let $G = \bigcap_{n=1}^{\infty} O_n$, then $E \subset G$ and *G* is measurable. Also $G \subset O_n \quad \forall n$

$m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n} \ \forall n \Rightarrow m^*(G \setminus E) = 0$ which means that $G \setminus E$ is

measurable by lemma 3.3. But $E = G \setminus (G \setminus E)$, therefore *E* is measurable.

Problem Set 3

1. Prove that every countable set is measurable.

2. Prove that if E_1 , $E_2, \ldots\ldots\ldots E_n$ are measurable then $\bigcup_{i=1}^{n} E_i$ is measurable.

3. Let $\{E_i\}_{i=1}^{\infty}$ be an infinite increasing sequence of measurable sets, i.e., a sequence with $E_{n+1} \supset E_n$. Let $mE_1 < \infty$. Prove that

$$m(\bigcup_{i=1}^{\infty} E_i) = \lim_{n\to\infty} mE_n$$

4. Show that if E_1 and E_2 are measurable then

$$m(E_1 \bigcup E_2) + m(E_1 \bigcap E_2) = mE_1 + mE_2$$

Hint: $(E_1 \bigcup E_2) \setminus E_1 = E_2 \setminus (E_1 \bigcap E_2)$.

5. Show that the condition $mE_1 < \infty$ is necessary in proposition 3.15 by giving a decreasing sequence of measurable sets $\{E_i\}_{i=1}^{\infty}$ with $\bigcap_{i=1}^{\infty} E_i = \phi$ and $mE_i = \infty \ \forall i$.

6. Complete the proof of proposition 3.16

4. Measurable Functions

The measurable functions form one of the most general classes of real functions. They are one of the basic objects of study in analysis.

If we start with a function *f*, the most important sets that arise from it are those listed in the following proposition:

Proposition 4.1:

Let *E* be a measurable set and let *f* be an extended real-valued function on *E*. Then the following statements are equivalent:

(i) $\forall \alpha \in R$, the set $f^{-1}(\alpha,\infty) = \{x : f(x) > \alpha\}$ is measurable.

(ii) $\forall \alpha \in R$, the set $f^{-1}[\alpha,\infty) = \{x : f(x) \geq \alpha\}$ is measurable.

(iii) $\forall \alpha \in R$, the set $f^{-1}(-\infty, \alpha) = \{x : f(x) < \alpha\}$ is measurable.

(iv) $\forall \alpha \in R$, the set $f^{-1}(-\infty, \alpha] = \{x : f(x) \le \alpha\}$ is measurable.

Proof: (i) $\Leftrightarrow$ (ii)

(i) $\Rightarrow$ (ii): $\{x : f(x) \ge \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \frac{1}{n}\}$.

By (i) $\{x : f(x) > \alpha - \frac{1}{n}\}$ is measurable, and so $\{x : f(x) \ge \alpha\}$ is the intersection of a sequence of measurable sets. Therefore, $\{x : f(x) \ge \alpha\}$ is measurable.

(ii) $\Rightarrow$ (i): $\{x : f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f(x) \ge \alpha + \frac{1}{n}\}$

By (ii) $\{x : f(x) \ge \alpha + \frac{1}{n}\}$ is measurable, and so $\{x : f(x) > \alpha\}$ is the union of a sequence of measurable sets. Therefore, $\{x : f(x) > \alpha\}$ is measurable.

Hence, (i) $\Leftrightarrow$ (ii).

(i) $\Leftrightarrow$ (iv): $\{x : f(x) \le \alpha\} = E \setminus \{x : f(x) > \alpha\} = \{x : f(x) > \alpha\}^c$.

Therefore, $\{x : f(x) \le \alpha\}$ is measurable if and only if $\{x : f(x) > \alpha\}$ is measurable.

(ii) $\Leftrightarrow$ (iii): $\{x : f(x) \ge \alpha\} = E \setminus \{x : f(x) < \alpha\} = \{x : f(x) < \alpha\}^c$.

Therefore, $\{x : f(x) \ge \alpha\}$ is measurable if and only if $\{x : f(x) < \alpha\}$ is measurable.

This shows that the four statements are equivalent.

Definition 4.2:

An extended real-valued function f is said to be (Lebesgue) measurable if its domain is measurable and it satisfies one of the statements in proposition 4.1.

Proposition 4.3:

(i) A continuous function on a measurable set is measurable.

(ii) If f is a measurable function and $E \subset dom\ (f\ )$ is measurable, then the function $f/_E$ is also measurable.

Proof: (i) Let $f : E \to R$ be a continuous function (where E is measurable). We want to prove that f is measurable i.e., $\forall \alpha \in R$, the set $f^{-1}(\alpha, \infty) = \{x : f(x) > \alpha\}$ is measurable.

Since f is continuous, the set $f^{-1}(\alpha, \infty)$ is open $\Rightarrow f^{-1}(\alpha, \infty)$ is measurable and hence f is measurable.

(ii) Let f be a measurable function, and let $E \subset dom\ (f\,)$ be a measurable set. We want to prove that $f/_E : E \to R$ is measurable. Let $\alpha \in R$, then

$\{x \in E : f/_E(x) > \alpha\} = \{x \in E : f(x) > \alpha\} = E \bigcap \{x \in dom(f) : f(x) > \alpha\}$ which is measurable since it is the intersection of two measurable sets (since f is measurable). Therefore, $f/_E$ is measurable.

Remark: While every continuous function is measurable, not every measurable function is continuous.

The following proposition tells us that certain operations performed on measurable functions lead again to measurable functions:

Proposition 4.4:

Let c be a constant and f , g two measurable real-valued functions defined on the same domain. Then the following functions are all measurable: (i) $f + c$ (ii) cf (iii) $f \mp g$ (iv) $|f|$ (v) f^2 (vi) fg.

Proof: (i) The set $\{x: f(x)+c<\alpha\}=\{x: f(x)<\alpha-c\}$ is measurable since f is measurable. Therefore, $f + c$ is measurable.

(ii) The set $\{x: cf(x)<\alpha\}=\{x: f(x)<\frac{\alpha}{c}\}$ if $c > 0$

$$=\{x: f(x)>\frac{\alpha}{c}\} \text{ if } c<0$$

is measurable since f is measurable. Therefore, f is measurable if $c \neq 0$.

If c = 0, then $cf = 0$ which is measurable. (why?)

(iii) First, we are going to prove that if we have two measurable functions h and l on the same domain, then the set $\{x: h(x)<l(x)\}$ is measurable:

$\forall x \in \{x : h(x) < l(x)\}$ there exists a rational number r such that $h(x) < r < l(x)$ and the set $\{x : h(x) < r < l(x)\} = \{x : h(x) < r\} \bigcap \{x : l(x) > r\}$ is measurable since h and l are both measurable. So $\{x : h(x) < l(x)\} = \bigcup_{r \in Q} \{x : h(x) < r < l(x)\}$ is a countable union of measurable sets and hence it is measurable.

Now we will prove that $f + g$ is measurable:

$\{x : f(x) + g(x) < \alpha\} = \{x : f(x) < \alpha - g(x)\}$ is measurable since both the functions f and $\alpha - g$ are measurable. Therefore, $f + g$ is measurable.

$f - g = f + (-g)$ is also measurable.

(iv)The set $\{x : |f(x)| < \alpha\} = \{x : -\alpha < f(x) < \alpha\} = \{x : f(x) < \alpha\} \bigcap \{x : f(x) > -\alpha\}$ is measurable since f is measurable. Therefore, $|f|$ is measurable.

(v)The set $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \bigcup \{x : f(x) < -\sqrt{\alpha}\}$,for $\alpha \geq 0$ is measurable since f is measurable. Therefore, f^2 is measurable.

(vi) $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$ is measurable.

Theorem 4.5:

Let $\{f_n\}$ be a sequence of measurable functions (with the same domain). Then the following functions are all measurable: (i) $\sup\{f_1, f_2, \ldots\ldots f_n\}$ (ii) $\inf\{f_1, f_2, \ldots\ldots f_n\}$

(iii) $\sup_n f_n$ (iv) $\inf_n f_n$ (v) $\overline{\lim} f_n$ (vi) $\underline{\lim} f_n$ (vii) $\lim f_n$.

Proof: (i) Let $h(x)=\sup\{f_1(x), f_2(x),......f_n(x)\}$. We want to prove that the set $\{x: h(x)>\alpha\}$ is measurable for every α. But if $h(x)>\alpha$, then $\exists i$ such that $f_i(x)>\alpha$. So, the set $\{x: h(x)>\alpha\}=\bigcup_{i=1}^{n}\{x: f_i(x)>\alpha\}$ is measurable since it is the finite union of measurable sets. Therefore, *h* is measurable.

(ii) Let $g(x)=\inf\{f_1(x), f_2(x),......f_n(x)\}$. We want to prove that the set $\{x: g(x)<\alpha\}$ is measurable for every α. But if $g(x)<\alpha$, then $\exists i$ such that $f_i(x)<\alpha$. So, the set $\{x: g(x)<\alpha\}=\bigcup_{i=1}^{n}\{x: f_i(x)<\alpha\}$ is measurable since it is the finite union of measurable sets. Therefore, *g* is measurable.

(iii) Let $\tilde{h}(x)=\sup_n f_n(x)$. Then $\{x:\tilde{h}(x)>\alpha\}=\bigcup_{i=1}^{\infty}\{x: f_i(x)>\alpha\}$ is measurabe since it is the countable union of measurable sets. Therefore, $\tilde{h}$ is measurable.

(iv) Let $\tilde{g}(x)=\inf_n f_n(x)$. Then $\{x:\tilde{g}(x)<\alpha\}=\bigcup_{i=1}^{\infty}\{x: f_i(x)<\alpha\}$ is measurabe since it is the countable union of measurable sets. Therefore, $\tilde{g}$ is measurable.

(v) $\overline{\lim} f_n(x)=\inf_n(\sup_{k\geq n} f_k(x))$ is measurable by (iii) and (iv).

(vi) $\underline{\lim} f_n(x)=\sup_n(\inf_{k\geq n} f_k(x))$ is measurable by (iii) and (iv).

(vii) $\lim f_n=\underline{\lim} f_n(x)=\overline{\lim} f_n$ is measurable by (v) and (vi).

Definition 4.6:

A property is said to hold *almost everywhere (a.e.)* if the set of points where it fails to hold is a set of measure zero.

For example, we say $f = g$ a.e. if $m\{x: f(x) \neq g(x)\} = 0$.

Similarly, we say f_n converges to f a.e. if $m\{x: \lim_{n\to\infty} f_n(x) \neq f(x)\} = 0$.

Proposition 4.7:

If f is a measurable function and $f = g$ *a.e.* then g is measurable.

Proof: Let $E = \{x: f(x) \neq g(x)\}$, then $m\,E = 0$ since $f = g$ *a.e.*

Now, $\{x: g(x) > \alpha\} = [\{x: f(x) > \alpha\} \bigcup \{x \in E: g(x) > \alpha\}] \setminus \{x \in E: g(x) \leq \alpha\}$

$\{x: f(x) > \alpha\}$ is measurable since f is measurable.

$\{x \in E: g(x) > \alpha\} \subset E \Rightarrow m\{x \in E: g(x) > \alpha\} \leq mE = 0 \Rightarrow m\{x \in E: g(x) > \alpha\} = 0$

So by lemma 3.3, $\{x \in E: g(x) > \alpha\}$ is measurable. Similarly $\{x \in E: g(x) \leq \alpha\}$ is measurable.

Therefore, $\{x: g(x) > \alpha\}$ is measurable for each α and hence g is measurable.

Definition 4.8:

If A is any set we define the *characteristic function* χ_A of the set A to be

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Proposition 4.9:

χ_A is measurable if and only if A is measurable.

Proof: Let χ_A be a measurable function. By definition, $\chi_A(x) = 1$ if and only if $x \in A$. So $A = \{x: \chi_A(x) = 1\} = \{x: \chi_A(x) \geq 1\} \bigcap \{x: \chi_A(x) \leq 1\}$.

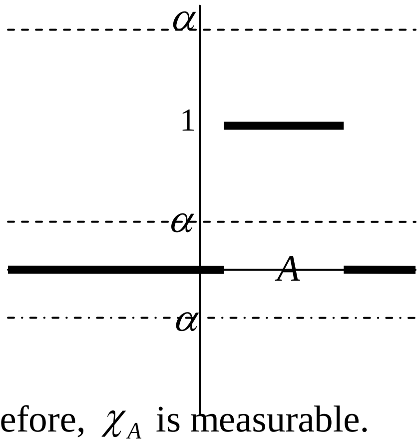
The two sets on the right are measurable since χ_A is measurable, and hence A is measurable.

Conversely, let A be a measurable set. Then A^c is also measurable. Consider the set $\{x : \chi_A(x) < \alpha\}$. We have three cases:

(i) $\alpha \leq 0$: $\{x : \chi_A(x) < \alpha\} = \varphi$ which is measurable.

(ii) $0 < \alpha \leq 1$: $\{x : \chi_A(x) < \alpha\} = A^c$ which is measurable.

(iii) $\alpha > 1$: $\{x : \chi_A(x) < \alpha\} = R$ which is measurable.



Hence, $\forall \alpha \in R$ the set $\{x : \chi_A(x) < \alpha\}$ is measurable. Therefore, χ_A is measurable.

<u>Proposition 4.10:</u>

i) $\chi_{A\cap B} = \chi_A \chi_B$

ii) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \chi_B$

iii) $\chi_{A^c} = 1 - \chi_A$

<u>Proof:</u> i) $\chi_{A\cap B}(x) = \begin{cases} 1 & x \in A \bigcap B \\ 0 & x \notin A \bigcap B \end{cases}$

Then if $x \in A \bigcap B \Rightarrow x \in A \wedge x \in B \Rightarrow \chi_A(x) = 1 \wedge \chi_B(x) = 1 \Rightarrow \chi_A \chi_B(x) = 1$.

If $x \notin A \bigcap B \Rightarrow x \notin A \vee x \notin B \Rightarrow \chi_A(x) = 0 \vee \chi_B(x) = 0 \Rightarrow \chi_A \chi_B(x) = 0$.

Therefore, $\chi_{A\cap B} = \chi_A \chi_B$.

iii) $\chi_{A^c}(x) = \begin{cases} 1 & x \in A^c \\ 0 & x \notin A^c \end{cases} = \begin{cases} 1 & x \notin A \\ 0 & x \in A \end{cases}$

On the other hand, 1-$\chi_A(x) = 1 - \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} = \begin{cases} 0 & x \in A \\ 1 & x \notin A \end{cases}$

Therefore, $\chi_{A^c} = 1 - \chi_A$.

ii) $\chi_{A\cup B} = 1 - \chi_{(A\cup B)^c} = 1 - \chi_{A^c \cap B^c} = 1 - \chi_{A^c}\chi_{B^c} = 1 - (1-\chi_A)(1-\chi_B)$

$$= 1 - (1 - \chi_A - \chi_B + \chi_A\chi_B) = \chi_A + \chi_B - \chi_A\chi_B .$$

Definition 4.11:

A real-valued function φ is called *simple* if it is measurable and assumes only a finite number of values, (i. e. the range of φ is a finite set).

If φ is a simple function and has the values α_1, α_2,…….., α_n, then $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where $A_i = \{x : \varphi(x) = \alpha_i\}$.

Proposition 4.12:

The sum of two simple functions is simple.

Proof: Let φ and ψ be two simple functions and let $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $\psi = \sum_{j=1}^{m} \alpha_j \chi_{B_j}$ where $A_i = \{x : \varphi(x) = \alpha_i\}$ and $B_j = \{x : \psi(x) = \beta_j\}$.

Note that $\varphi + \psi$ is measurable since both φ and ψ are measurable.

Now if $x \in A_i$ then $\varphi(x) = \alpha_i$, if $x \in B_j$ then $\psi(x) = \beta_j$ and if $x \in A_i \cap B_j$ then $(\varphi + \psi)(x) = \alpha_i + \beta_j$.

So we can write $\varphi+\psi$ as $\varphi+\psi=\sum_{i,j=1}^{nm}(\alpha_i+\beta_j)\chi_{A_i\cap B_j}$ (Note that some of the sets $A_i\cap B_j$ might be empty). Hence, $\varphi+\psi$ is a simple function.

Proposition 4.13:

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions defined on a measurable set *E* such that $f_n\to f$ a.e. on *E,* then *f* is a measurable function.

Proof: Let $E_o=\{x\in E:\lim_{n\to\infty} f_n(x)\neq f(x)\}$. Then mE_o = 0.

Define a sequence $\{\tilde{f}_n\}_{n=1}^{\infty}$ on *E* by $\tilde{f}_n(x)=\begin{cases} f_n(x) & x\in E\setminus E_o \\ 0 & x\in E_o \end{cases}$

Then $\tilde{f}_n=f_n\ a.e.\Rightarrow\tilde{f}_n$ is measurable by proposition 4.7.

Therefore, $\{\tilde{f}_n\}_{n=1}^{\infty}$ is a sequence of measurable functions.

Define a function $\tilde{f}$ on *E* by $\tilde{f}(x)=\begin{cases} f(x) & x\in E\setminus E_o \\ 0 & x\in E_o \end{cases}$

Then $\lim_{n\to\infty}\tilde{f}_n=\tilde{f}\ \Rightarrow\ \tilde{f}$ is measurable by theorem 4.5

But $\tilde{f}=f\ a.e.$, and hence *f* is measurable.

Proposition 4.14:

Let *E* be a measurable set of finite measure, and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions defined on *E*. Let *f* be a real-valued function such that $f_n(x)\to f(x)$ pointwise. Then given $\varepsilon>0$ and $\delta>0$ there is a measurable set $A\subset E$ with $mA<\delta$ and a natural number *N* such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall x \notin A \text{ and } \forall n \geq N$$

Proof: Let $\varepsilon > 0$ and let $G_n = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}$, and set $E_N = \bigcup_{n=N}^{\infty} G_n$.

So $E_N = \{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\}$.

If $x \in E_{N+1} \Rightarrow |f_n(x) - f(x)| \geq \varepsilon$ for some $n \geq N+1$

$\Rightarrow |f_n(x) - f(x)| \geq \varepsilon$ for some $n > N \Rightarrow x \in E_N$

Therefore $E_{N+1} \subset E_N$. So $\{E_N\}_{N=1}^{\infty}$ is a decreasing sequence of measurable sets(why?).

Also $\forall x \in E : f_n(x) \to f(x)$, therefore there exists a natural number N such that

$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N \Rightarrow x \notin E_N$.

This means that $\forall x \in E$ there exists some E_N such that $x \notin E_N$. So $\bigcap_{N=1}^{\infty} E_N = \varphi$.

By proposition 3.15 we have: $m(\bigcap_{N=1}^{\infty} E_N) = \lim_{N \to \infty} mE_N \Rightarrow \lim_{N \to \infty} mE_N = 0$.

Hence, given $\delta > 0$ there exists a natural number N such that $mE_N < \delta$, i.e.,

$m\{x \in E : |f_n(x) - f(x)| \geq \varepsilon \text{ for some } n \geq N\} < \delta$.

Let $A = E_N$, then $mA < \delta$ and if $x \notin A$, then $|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq N$.

Remark:

Proposition 4.14 states that if $\{f_n\}_{n=1}^{\infty}$ converges to f pointwise, then $\{f_n\}_{n=1}^{\infty}$ is nearly uniformly convergent to f.

Problem Set 4:

1. Let *f* be a measurable function defined on a measurable set *E*, and let *O* be an open set. Prove that the set $\{x \in E : f(x) \in O\}$ is measurable.

2. Let $f:[0,1] \to R$ be defined by $f(x) = \begin{cases} \sqrt{x} & x \notin Q \\ 0 & x \in Q \end{cases}$.

   Prove that *f* is measurable.

3. Let *f* be a real-valued function defined on a measurable set *E* and let E_1 , E_2 be measurable sets such that $E_1 \bigcup E_2 = E, E_1 \bigcap E_2 = \varphi$. Assume that $f /_{E_1}$ and $f /_{E_2}$ are measurable. Prove that *f* is measurable.

4. Prove that the product of two simple functions is simple.