Since (x_n) and (y_n) are Cauchy, we can make the right side as small as we please. This implies that the limit in (2) exists because **R** is complete.

We must also show that the limit in (2) is independent of the particular choice of representatives. In fact, if $(x_n) \sim (x_n')$ and $(y_n) \sim (y_n')$, then by (1),

$$|d(x_n, y_n) - d(x_n', y_n')| \leqq d(x_n, x_n') + d(y_n, y_n') \longrightarrow 0$$

as $n \longrightarrow \infty$, which implies the assertion

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n', y_n').$$

We prove that $\hat{d}$ in (2) is a metric on $\hat{X}$. Obviously, $\hat{d}$ satisfies (M1) in Sec. 1.1 as well as $\hat{d}(\hat{x}, \hat{x}) = 0$ and (M3). Furthermore,

$$\hat{d}(\hat{x}, \hat{y}) = 0 \quad \Longrightarrow \quad (x_n) \sim (y_n) \quad \Longrightarrow \quad \hat{x} = \hat{y}$$

gives (M2), and (M4) for $\hat{d}$ follows from

$$d(x_n, y_n) \leqq d(x_n, z_n) + d(z_n, y_n)$$

by letting $n \longmapsto \infty$.

(b) *Construction of an isometry* $T: X \longrightarrow W \subset \hat{X}$. With each $b \in X$ we associate the class $\hat{b} \in \hat{X}$ which contains the constant Cauchy sequence $(b, b, \cdots)$. This defines a mapping $T: X \longrightarrow W$ onto the subspace $W = T(X) \subset \hat{X}$. The mapping T is given by $b \longmapsto \hat{b} = Tb$, where $(b, b, \cdots) \in \hat{b}$. We see that T is an isometry since (2) becomes simply

$$\hat{d}(\hat{b}, \hat{c}) = d(b, c);$$

here $\hat{c}$ is the class of (y_n) where $y_n = c$ for all n. Any isometry is injective, and $T: X \longrightarrow W$ is surjective since $T(X) = W$. Hence W and X are isometric; cf. Def. 1.6-1(*b*).

We show that W is dense in $\hat{X}$. We consider any $\hat{x} \in \hat{X}$. Let $(x_n) \in \hat{x}$. For every $\varepsilon > 0$ there is an N such that