CHAPTER 1
Part 3

Conditional Probability and Independence

Definition: Conditional probability

If $P(A) > 0$ then $P(B|A) = \frac{P(A \cap B)}{P(A)}.$

It is mean that the probability that A occurs given that (we know) B occurs.

Example 1:

A coin is flipped twice if assume that all four points in the sample space are equally likely. What is the conditional probability that the both flips results in heads, given that the first flip does?

Solution:

$S = \{HH, TH, HT, TT\}$

$P(HH) = 1/4$

Let $H_1 = \{HH\}$, and $H_2 = \{HT, HH\}$

Now

$$P(H_2|H_1) = \frac{P(H_1 \cap H_2)}{P(H_1)} = \frac{P(HH)}{P(HT, HH)} = \frac{P(HH)}{P(HT) + P(HH)} = \frac{1/4}{1/4 + 1/4} = 1/2.$$

❖ Example 2 :

A family have two kids, what is the probability that they are both boys, given that
i) the older is a boy?
ii) at least one of them is boy?

Solution:

$$S = \{bb, bg, gb, gg\}$$

$$\text{i) } P(\text{both boy} \mid \text{older boy}) = \frac{P(\text{both boy} \cap \text{older boy})}{P(\text{older boy})} = \frac{P(bb)}{P(bg, bb)} = \frac{1/4}{1/4 + 1/4} = 1/2.$$

$$\text{ii) } P(\text{both boy} \mid \text{at least one of them boy}) = \frac{P(\text{both boy} \cap \text{at least one of them boy})}{P(\text{at least one of them boy})}$$

$$= \frac{P(bb)}{P(bg, gb, bb)} = \frac{1/4}{1/4 + 1/4 = 1/4} = 1/3.$$

Definition: The multiplication rule of probability.

$$P(A \cap B) = P(A)P(B \mid A)$$

$$P(B \cap A) = P(B)P(A \mid B)$$

In general

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1A_2)\ldots P(A_n \mid A_1A_2\ldots A_{n-1}).$$

Example 3:

Suppose that un urn contain 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement if we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

Solution:

- using first method

$$P(RR) = \frac{C_2^8}{C_2^{12}} = 0.42.$$

- using probability product (multiplication rule)

$$P(R_1R_2) = P(R_1) \times P(R_2 \mid R_1) = \frac{8}{12} \times \frac{7}{11} = \frac{56}{132} = 0.42.$$

Prove that

$P(A|B)$ is satisfy the axioms of probability

Proof:

$$\text{Since}: P(A|B)=\frac{P(A\cap B)}{P(B)}.$$

$\bullet$ Axiom (1): $0 \le P(A) \le 1$

$$\because P(AB) \ge 0,$$

$$\because P(B) > 0,$$

$$\therefore P(A|B) \ge 0.$$

$\bullet$ Axiom (2): $P(S)=1$

$$P(S|B)=\frac{P(SB)}{P(B)}=\frac{P(B)}{P(B)}=1.$$

$\bullet$ Axiom (3): mutually exclusive if $A_i, (i=1,\dots,n)$ are n mutually excusive then

$$P\left(\bigcup_{i=1}^{n} A_i \,|\, B\right)=\frac{P\left(\bigcup_{i=1}^{n} A_i \cap B\right)}{P(B)}=\frac{P\left(\bigcup_{i=1}^{n} A_i B\right)}{P(B)}$$

$$=\frac{\sum_{i=1}^{n} P(A_i \cap B)}{P(B)}=\frac{\sum_{i=1}^{n} P(B)\times P(A_i|B)}{P(B)}$$

$$=\frac{P(B)\sum_{i=1}^{n} P(A_i|B)}{P(B)}=\sum_{i=1}^{n} P(A_i|B).$$

❖ Example 4:

Urn 1 contains 2 white and 4 red balls, where urn 2 contains 1 red and 1 white ball. A ball is randomly chosen from urn 1 and put into urn 2, and a ball is then randomly selected from urn 2. what is

i) the probability that the ball selected from urn 2 is white

ii) the conditional probability that the transferred ball was white, given that a white ball is selected from urn 2.

Solution:

i) Let W1 is a white ball from urn 1, W2 is a white ball from urn 2, R1 is a red ball from urn 1, and R2 is a red ball from urn 2.
Urn 2 will contains 3 balls after first chosen

$$\begin{aligned}\mathrm{P(white\ ball\ from\ urn\ 2)} = P(W_2) &= P(W_1W_2) + P(R_1W_2)\\ &= P(W_1)P(W_2 \mid W_1) + P(R_1)P(W_2 \mid R_1)\\ &= \left(\frac{2}{6}\times\frac{2}{3}\right) + \left(\frac{4}{6}\times\frac{1}{3}\right)\\ &= \frac{4}{18} + \frac{4}{18} = \frac{8}{18} = 0.44.\end{aligned}$$

$$\begin{aligned}\mathrm{P(white\ ball\ from\ urn\ 2)} = P(W_2) &= P(W_1W_2) + P(R_1W_2)\\ &= \left(\frac{C_1^2}{C_1^6}\times\frac{C_1^2}{C_1^3}\right) + \left(\frac{C_1^4}{C_1^6}\times\frac{C_1^1}{C_1^3}\right) = \frac{8}{18} = 0.44\end{aligned}$$

ii) The conditional probability that the transferred ball was white, given that a white ball is selected from urn 2.

$$P(W_1 \mid W_2) = \frac{P(W_1 \cap W_2)}{P(W_2)} = \frac{P(W_1 \cap W_2)}{P(W_1W_2, R_1W_2)}$$

$$= \frac{P(W_1W_2)}{P(W_1W_2) + P(R_1W_2)} = \frac{4/18}{8/18} = \frac{4}{8} = \frac{1}{2}.$$

Definition:

If A1, A2, … , An are n mutually exclusive and exhaustive events, that is

$$S = A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i \, .$$

Theorem of total probability

If A1, A2, … , An are n mutually exclusive and exhaustive events then for any

$$B \subset S, P(B) = \sum_{i=1}^{n} P(A_i)P(B|\, A_i).$$

Proof:

We know

$$S = \bigcup_{i=1}^{n} A_i$$

$$\text{and } B = B \cap S = B \cap \left(\bigcup_{i=1}^{n} A_i \right) = \bigcup_{i=1}^{n} (A_i \cap B)$$

$$\therefore B = \bigcup_{i=1}^{n} (A_i \cap B) \Rightarrow P(B) = P\left(\bigcup_{i=1}^{n} (A_i \cap B) \right)$$

Since A_i are MEE

$$B = \sum (A_i \cap B) \Rightarrow P(B) = \sum P(A_i \cap B)$$

Using multiplication rule of probability

$$P(B) = \sum P(A_i) \times P(B|\, A_i).$$

Result:

If $A \subset S$ then for any $B \subset S$ we can calculate $P(B)$ as $P(B) = P(A)P(B \mid A) + P(\overline{A})P(B \mid \overline{A})$
mutually exclusive exhaustive events.

Bay's Theorem

If A1, A2, ... , An are n mutually exclusive and exhaustive events, and B occur if one of Ai's occur

$$P(A_r \mid B) = \frac{P(A_r \cap B)}{P(B)} = \frac{P(A_r)P(B \mid A_r)}{\sum_{i=1}^{n} P(A_i)P(B \mid A_i)}.$$

proof:

$$B = \bigcup_{i=1}^{n} (A_i \cap B)$$

$$\text{and } P(B) = \sum_{i=1}^{n} P(A_i)P(B|A_i)$$

$$\text{then } P(A_r|B) = \frac{P(A_r \cap B)}{P(B)} = \frac{P(A_r)P(B|A_r)}{P(B)} = \frac{P(A_r)P(B|A_r)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Result:

$$\text{If } A \subset S \text{ and } \forall\, B \subset S \text{ then } P(A|B) = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\bar{A})P(B|\bar{A})} \text{ because } A \text{ and } \bar{A} \text{ are MEE.}$$

❖ Example 5:

A bin contains 3 different types of flashlights. The probability that a type 1 flashlight will give over 100 hours of use is 0.7, with the corresponding probabilities for type 2 and type 3 flashlights being 0.4 and 0.3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

1- What is probability that a randomly chosen flashlight will give more than 100 hours of use?

2- Given the flashlight lasted over 100 hours, what is the conditional probability that it was a type 1 flashlight?

Solution:

1.Let A denote the event that the flashlight chosen will give over 100 hours of use,

$$
\begin{aligned}
P(A) &= P(\text{type}1 \cap A) + P(\text{type}2 \cap A) + P(\text{type}3 \cap A) \\
&= P(A \mid \text{type}1)P(\text{type}1) + P(A \mid \text{type}2)P(\text{type}2) + P(A \mid \text{type}3)P(\text{type}1) \\
&= (0.7)(0.2) + (0.4)(0.3) + (0.3)(0.5) = 0.41
\end{aligned}
$$

2- $$P(\text{type}1 \mid A) = \frac{P(\text{type}1 \cap A)}{P(A)} = \frac{P(\text{type}1)P(A \mid \text{type}1)}{P(A)} = \frac{0.2 \times 0.7}{0.41} = 0.34.$$

❖ ***Example 6:***

A class contains 30 students, 12 boys , and 18 girls, 4 boys and 8 girls are superior. If we choose one student randomly chosen, what is probability that :
1- The student is superior.
2-The student is superior if the student is girl.

Solution

$$
\begin{aligned}
1\text{-}\ P(\text{super}) &= P(\text{boy}\ \&\ \text{super}) + P(\text{girl}\ \&\ \text{super}) \\
&= P(\text{boy} \mid \text{super})P(\text{boy}) + P(\text{girl} \mid \text{super})P(\text{girl}) \\
&= \left(\frac{12}{30} \times \frac{4}{12}\right) + \left(\frac{18}{30} \times \frac{8}{18}\right) \\
&= \frac{48}{360} + \frac{144}{540} \\
&= 0.13 + 0.27 \\
&= 0.4.
\end{aligned}
$$

$$2 - P(\text{girl} \mid \text{super}) = \frac{P(\text{girl} \cap \text{super})}{P(\text{super})} = \frac{P(\text{girl})P(\text{super} \mid \text{girl})}{P(\text{super} \cap \text{girl}) + P(\text{super} \cap \text{boy})}$$

$$= \frac{(18/30) \times (8/18)}{(12/30)}$$

$$= \frac{(144/540)}{(12/30)}$$

$$= \frac{0.27}{0.4}$$

$$= 0.67.$$

Definition: Independent Events:

Two events A and B are independent if knowledge that A has occurred does not change the probability that B. $P(A \cap B) = P(A) \times P(B).$

Example 7:

Two coins are flipped and all 4 outcomes are assumed to be equally likely. If E is the event that the first coin lands heads and F is the event that the second coin lands tails, prove that E and F are independent

Solution:

$$E = \{HH, HT\}, \quad P(E) = \frac{2}{4}$$

$$F = \{HT, TT\}, \quad P(F) = \frac{2}{4}$$

$$E \cap F = \{HT\} \Rightarrow P(E \cap F) = \frac{1}{4}$$

$$P(E) \times P(F) = \frac{2}{4} \times \frac{2}{4} = \frac{4}{16} = \frac{1}{4}$$

$$\therefore P(E \cap F) = P(E) \times P(F)$$

$$\therefore E \;\&\; F \text{ are independent.}$$

Note:

If A and B are independent events, then $1)\ \overline{A}, B.\quad 2)\ \overline{A}, \overline{B}.\quad 3)\ A, \overline{B}.$
are also independent events.

Proof:

$$
\begin{aligned}
3)\ P(A \cap \overline{B}) &= P(A) - P(A \cap B) \\
&= P(A) - P(A)P(B) \\
&= P(A)\left[1 - P(B)\right] \\
&= P(A) \times P(B).
\end{aligned}
$$

❖ ***Example 8:***

Let A and B are independent events with P(A)=0.2 and P(B)=0.5 find $P(\overline{A} \cap B)$.

Solution:

$$P(\overline{A} \cap B) = P(A) \times P(B) = 0.2 \times 0.5 = 0.1.$$

Definition:

The three events E,F,G are said to be independent events if EF, EG, FG, EFG are independent events.

$$P(E \cap F) = P(E) \times P(F)$$
$$P(E \cap G) = P(E) \times P(G)$$
$$P(G \cap F) = P(G) \times P(F)$$
$$P(E \cap F \cap G) = P(E) \times P(F) \times P(G).$$

Prove

that $(A \cup B), C$ *is independent if A, B, C independent events.*

Proof:

$$\begin{aligned} P[(A \cup B)C] &= P[AC \cup BC] \\ &= P(AC) + P(BC) - P(ABC) \\ &= (P(A) \times P(C)) + (P(B) \times P(C)) - (P(A) \times P(B) \times P(C)) \\ &= [P(A) + P(B) - (P(A) \times P(B))] \times P(C) \\ &= P(A \cup B) \times P(C). \end{aligned}$$

Example 9:

If S={1,2,3,4} and A={1,2}, B={1,3}, C={1,4}, prove that three events are independent.

Solution:

$$P(A \cap B) = P(A) \times P(B)$$

$$1/4 = 2/4 \times 2/4, \qquad \text{then } A, B \text{ are indep.}$$

$$P(A \cap C) = P(A) \times P(C)$$

$$1/4 = 2/4 \times 2/4, \qquad \text{then } A, C \text{ are indep.}$$

$$P(B \cap C) = P(B) \times P(C)$$

$$1/4 = 2/4 \times 2/4, \qquad \text{then } B, C \text{ are indep.}$$

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

$$1/4 \neq 2/4 \times 2/4 \times 2/4, \qquad \text{then } A, B, C \text{ are not indep.(dependent).}$$

Note:

When $P(A)>0$ and $P(B)>0$ *then mutually exclusive events* $\Rightarrow$ *independent events*

When $P(A)=0$ or $P(B)=0$ *independent events* $\Rightarrow$ *not mutually exclusive events*

Example 10:

Suppose that we toss 2 fair dice. Let E1 denote the event that the sum of the dice is 6 and E2 denote the event that the first die equal 4. Prove that E1 and E2 are independent.

Solution:

$$E_1 = \{(1,5),(2,4),(3,3),(4,2),(5,1)\} \Rightarrow P(E_1) = \frac{5}{36}$$

$$E_2 = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\} \Rightarrow P(E_2) = \frac{6}{36}$$

$$E_1 \cap E_2 = \{(4,2)\} \Rightarrow P(E_1 \cap E_2) = \frac{1}{36}$$

$$P(E_1) \times P(E_2) \neq P(E_1 \cap E_2)$$

$$\therefore E_1 \text{ and } E_2 \text{ are dependent.}$$

❖ Example 11:

Two fair dice are thrown. Let E denote the event that the sum of the dice is 7. Let F denote the event that the first die equals 4, and let G be the event that the second die equal 3. Prove that the three events are independent.

Solution:

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \Rightarrow P(E) = \frac{6}{36}$$

$$F = \{(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)\} \Rightarrow P(F) = \frac{6}{36}$$

$$G = \{(1,3),(2,3),(3,3),(4,3),(5,3),(6,3)\} \Rightarrow P(G) = \frac{6}{36}$$

$$P(E \cap F) = P(E) \times P(F) \Rightarrow \frac{1}{36} = \frac{6}{36} \times \frac{6}{36} \Rightarrow \frac{1}{36} = \frac{36}{1296}$$

$\therefore$ E, F are independent.

$$P(E \cap G) = P(E) \times P(G) \Rightarrow \frac{1}{36} = \frac{6}{36} \times \frac{6}{36} \Rightarrow \frac{1}{36} = \frac{36}{1296}$$

$\therefore$ E, G are independent.

$$P(F \cap G) = P(F) \times P(G) \Rightarrow \frac{1}{36} = \frac{6}{36} \times \frac{6}{36} \Rightarrow \frac{1}{36} = \frac{36}{1296}$$

$\therefore$ F, G are independent.

$$P(E \cap F \cap G) = P(E) \times P(F) \times P(G) \Rightarrow \frac{1}{36} \neq \frac{6}{36} \times \frac{6}{36} \times \frac{6}{36} \Rightarrow \frac{1}{36} = \frac{36}{1296}$$

$\therefore$ E, F, G are dependent.

Thank You !