Ministry of Higher Education, King Abdulaziz University.
Faculty of Science, Department of Mathematics.

Calculators are not allowed. Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

PLEASE SHOW ALL YOUR WORK. No Credit will be given for an answer alone.

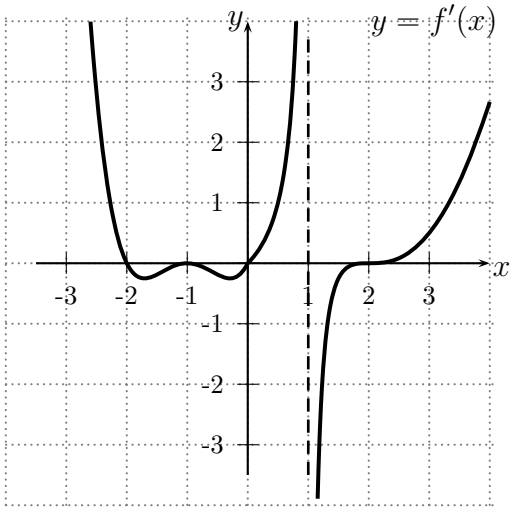
Student Name: _____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

Specialization: Write 1, 2, and 3 in three boxes only according to your desired area of specialization [1 means first choice, 2 means second choice, and 3 means third choice].

Algebra	
Applied Mathematics	
Complex Analysis	
Differential Equations	
Differential Geometry	
Functional Analysis	
Numerical Analysis	
Real Analysis	
Topology	

1. Use the graph of $y = f(x)$ given to the right with the fact that domain of f is the set of all real numbers to answer the following questions



   (a) [3 points] Find all critical numbers of f. Justify your answer.

   (b) [3 points] Find the intervals where the function is increasing and decreasing. Justify your answer.

   (c) [4 points] Classify each critical numbers as local maximum, local minimum, or neither. Justify your answer.

2. Evaluate the following integral

(a) [5 points] $\displaystyle\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \frac{x^8 + x^4 + 2}{\tan x}\, dx$

(b) [5 points] $\displaystyle\int \frac{dx}{\sqrt{e^{2x} - 6}}$

3. (a) [5 points] Prove that for all $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$.

   (b) [5 points] Suppose that $a, b \in \mathbb{R}$ with $a < b$. If f is a continuous function on $[a, b]$ and differentiable on (a, b) such that $f'(x) < 0$, for all $x \in (a, b)$. Show that f is strictly decreasing on $[a, b]$.

4. (a) [5 points] Find a spanning family for the subspace S of $\mathbb{R}^3$ defined by the equation $2x - 5y + 5z = 0$.

   (b) [5 points] Let $A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$. Prove that $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$.

5. (a) [5 points] Find the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$.

   (b) [5 points] Evaluate $\int_C x^2 y \, ds$, where C is the upper half the unit circle.

6. (a) [5 points] Let $\mathbb{Q}$ be the set of all rational numbers. Define an operation $*$ on $\mathbb{Q}$ as follows: $a * b = a + b + ab$ for all $a, b \in \mathbb{Q}$.
   i. Is $(\mathbb{Q}, *)$ a group? Explain your answer.
   ii. Find a solution of the equation $2 * x * 3 = 7$.

   (b) [5 points] i. If G is a group such that $(ab)^2 = a^2b^2$ for all $a, b \in G$. Prove that G is abelian.
   ii. Let H be a subgroup of a group G such that $|G : H| = 2$. Show that H is a normal subgroup in G.

7. (a) [5 points] Discuss the continuity of the given function in the $xy-$plane (on $\mathbb{R}^2$):

$$f(x,y) = \begin{cases} \dfrac{x(y-1)}{x^2+(y-1)^2}, & (x,y) \neq (0,1), \\ 0, & (x,y) = (0,1). \end{cases}$$

(b) [5 points] Write equations for the plane that is parallel to and $\sqrt{6}$ units away from the plane $x - 2y + z = 3$.

8. (a) [5 points] Discuss the existence and uniqueness question for the following value problem $y' = \sqrt{y - x}, \quad y(1) = 2$.

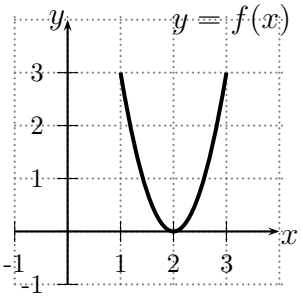
   (b) [5 points] Find a general solution for $y' + x + y + 1 = (x + y)^2 e^{3x}$.

9. (a) [5 points] Prove that $2^n \geq n^2$, for all $n \geq 4$.

   (b) [5 points] If $A; B; C$ are three non-empty sets, prove that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

10. (a) [5 points] Suppose that $b \in \mathbb{R}$ and $f(x) = 3x^2 + bx + 12$ is a polynomial part of its graph shown to the right. Find $f(4)$.



(b) [5 points] If f is a differentiable function on $\mathbb{R}$. Suppose that $f(1) = 3$ and $f'(1) = 5$. Compute $\lim_{x \to 1} \dfrac{(f(x))^2 - 9}{x^2 - 1}$.