

**Ministry of Higher Education, K. Abdul-Aziz University.  
Faculty of Science, Department of Mathematics.**

Calculators are not allowed. Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

PLEASE SHOW ALL YOUR WORK. No Credit will be given for an answer alone.

Student Name: \_\_\_\_\_

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	10	10	10	10	10	10	10	10	10	10	100
Score:											

**Specialization:** Write 1, 2, and 3 in three boxes only according to your desired area of specialization [1 means first choice, 2 means second choice, and 3 means third choice].

Algebra	
Applied Mathematics	
Complex Analysis	
Differential Equations	
Differential Geometry	
Functional Analysis	
Numerical Analysis	
Real Analysis	
Topology	

1. (a) 5 points Suppose that  $f$  is differentiable at 0 and

$$\lim_{x \rightarrow 0} \frac{f(x)}{x} = 4, \quad \lim_{x \rightarrow 0} \frac{g(x)}{x} = 2.$$

- (i) Find  $f(0)$ .
- (ii) Find  $f'(0)$ .
- (iii) Find

$$\lim_{x \rightarrow 0} \frac{g(x)}{f(x)}.$$

- (b) (i) Sketch the graph of the function

$$h(x) = |x^2 - 6|x| + 8|.$$

- (ii) At what values of  $x$  is  $h$  not differentiable?

2. (a) 5 points Determine whether the given series converge or diverge.

$$(i) \sum_{n=1}^{\infty} (2^{1/n^2} - 1) \qquad (ii) \sum_{n=1}^{\infty} \sin(1/n^2).$$

- (b) 5 points Let  $p$  be a polynomial function on the set of real numbers. Assume that  $p(0) = p(2) = 3$  and  $p'(0) = p'(2) = -1$ . Evaluate

$$\int_0^2 xp''(x)dx.$$

3. (a) 5 points Prove that the given sequence is convergent and find the limit

$$a_1 = 2 \qquad a_n = \sqrt{3 + a_{n-1}}$$

- (b) 5 points Let  $g$  be the function defined on the set of all real numbers by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ e^x & \text{if } x \text{ is irrational.} \end{cases}$$

What is the set of numbers at which  $g$  is continuous.

4. (a) 5 points Prove that if a matrix is invertible, then the inverse is unique.
- (b) 5 points For what value (or values) of  $m$  is the vector  $(1, 2, m, 5)$  a linear combination of the vectors  $(0, 1, 1, 1)$ ,  $(0, 0, 0, 1)$ , and  $(1, 1, 2, 0)$  ?

5. (a) 5 points Use mathematical induction to prove that

$$1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{1 - 3^n}{-2} \quad n \in \mathbb{Z}, \quad n \geq 1.$$

- (b) 5 points Let  $F(1) = 2$  and  $F(n) = F(n - 1) + \frac{1}{2}$  for all integers  $n > 1$ . Find  $F(101)$ .

6. 10 points Let  $R$  be an equivalence relation on a set  $A$ . Show that

$$[a] = [b] \quad \text{if and only if} \quad (a, b) \in R, \quad \forall a, b \in A.$$

7. (a) 5 points Show that  $\mathbb{Z} \times \mathbb{Z}$  has subgroups not of the form  $n\mathbb{Z} \times m\mathbb{Z}$ .
- (b) 5 points Let  $G$  be a nontrivial group with no proper subgroups except the trivial one. Show that  $G$  is finite and that the order of  $G$  is prime.



8. (a) 5 points An element  $x$  of a ring  $R$  is *nilpotent* if  $x^n = 0$  for some  $n \in \mathbb{Z}^+$ . Show that if  $x$  and  $y$  are nilpotent elements of a commutative ring, then  $x + y$  is also nilpotent.
- (b) 5 points An element  $x$  of a ring  $R$  is *idempotent* if  $x^2 = x$ . If the ring  $R$  has no nonzero nilpotent elements, any idempotent  $e$  of  $R$  commutes with all elements of  $R$ .

9. (a) 5 points Find the Laplace transform of  $f(t) = \cosh t$ .
- (b) 5 points Verify that  $\phi(x) = \frac{\cos x}{\sqrt{x}}$  is a solution of

$$x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0, \quad x > 0.$$

10. (a) 5 points Evaluate

$$\int_0^2 \int_x^2 y^4 \cos(xy^2) dy dx.$$

- (b) 5 points Find the equation of the plane containing the point  $A(1, 2, 1)$  and perpendicular to the planes  $x + y = 2$  and  $2x + y - z = 1$ .