MATH 110

قسم الرياضيات – كلية العلوم

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Chapter 4

Definition 4.1.1: Derivative Function

The function f' defined by the formula

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Is called the *derivative of f with respect to x*. The domain of f' consists of all *x* in the domain of *f* for which the limit exists.

Alternative formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Example

Find the derivative with respect to *x* of $f(x) = x^2 + 4$**.**

Solution

Here we have $f(x) = x^2 + 4$**, so**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^2 + 4] - [x^2 + 4]}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 4 - x^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} (2x + h) = 2x.$$

Definition 4.1.2:

A function *f* is differentiable at *a* if $f'(a)$ exists. It is differentiable on an open interval (*a*, *b*) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$] if it is differentiable at every number in the interval. It is differentiable on a closed interval [*a*, *b*] if it is differentiable on the interior (*a*, *b*) and if the limits

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$ **Right-hand derivative at *a***

$$\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$ **Left-hand derivative at *b***

exist at the endpoints.

Example

Show that the function $f(x)=|x|$ is not differentiable at $x=0$ and find a formula for $f'(x)$.

Solution

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{|h|-|0|}{h}=\lim_{h\to 0}\frac{|h|}{h}.$$

But

$$\frac{|h|}{h}=\begin{cases}1, & h>0\\ -1, & h<0\end{cases}.$$

Then

$$\lim_{h\to 0^-}\frac{|h|}{h}=-1 \quad \text{and} \quad \lim_{h\to 0^+}\frac{|h|}{h}=1.$$

Since these one-sided limits are not equal, the two-sided limit does not exist, and hence f is not differentiable at $x=0$.

Theorem: **Differentiability Implies Continuity**

If a function *f* is differentiable at *c*, then *f* is continuous at *c*.

Defferentiation Rules

In *Section 4.1* we defined the derivative of a function *f* as a limit, and we used that limit to calculate a few simple derivatives. In this section we introduce a few rules that allow us to differentiate a great variety of functions. These rules will enable us to calculate derivatives more efficiently.

Rule 1: **Derivative of a Constant Function**

The derivative of a constant function $f(x)=c$ **is zero; that is**

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}(c)=0.$$

Rule 2: Power Rule for Positive Integers

If *n* is a positive integer, then

$$\frac{\mathrm{d}}{\mathrm{d}\,x}x^n = nx^{n-1}.$$

Rule 3: Constant Multiple Rule

If *f* is a differentiable function of *x*, and *c* is a constant, then

$$\frac{\mathrm{d}}{\mathrm{d}\,x}[cf(x)] = c\frac{\mathrm{d}}{\mathrm{d}\,x}f(x).$$

Rule 4: Sum and Difference Rule

If *f* and *g* are differentiable at *x*, then so are $f+g$ **and** $f-g$ **and**

$$\frac{\mathrm{d}}{\mathrm{d}\,x}[f(x)+g(x)] = \frac{\mathrm{d}}{\mathrm{d}\,x}f(x)+\frac{\mathrm{d}}{\mathrm{d}\,x}g(x),$$
$$\frac{\mathrm{d}}{\mathrm{d}\,x}[f(x)-g(x)] = \frac{\mathrm{d}}{\mathrm{d}\,x}f(x)-\frac{\mathrm{d}}{\mathrm{d}\,x}g(x).$$

Example

If $y = 2x^3 + \frac{3}{2}x^2 - 3x + 7,$ **find** $\dfrac{\mathrm{d}y}{\mathrm{d}x}.$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2x^3 + \tfrac{3}{2}x^2 - 3x + 7)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(2x^3) + \frac{\mathrm{d}}{\mathrm{d}x}(\tfrac{3}{2}x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(-3x) + \frac{\mathrm{d}}{\mathrm{d}x}(7)$$

$$= 2(3x^2) + \tfrac{3}{2}(2x) - 3(1) + 0 = 6x^2 + 3x - 3.$$ ■

Rule 5: **Derivative Product Rule**

If *f* and *g* are differentiable at *x*, then so is their product fg **and**

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) + g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x).$$

Example

If $y = (x^3 + 3)(x^2 - 1)$, **find** $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}[(x^3 + 3)(x^2 - 1)]$$

$$= (x^3 + 3)\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 1) + (x^2 - 1)\frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 3)$$

$$= (x^3 + 3)(2x) + (x^2 - 1)(3x^2)$$

$$= 2x^4 + 6x + 3x^4 - 3x^2$$

$$= 5x^4 - 3x^2 + 6x.$$

Rule 6: Derivative Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then the quotient f/g is differentiable at x, and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$

Example

If $y = \dfrac{x^2 + 3x - 1}{x^3 + 3}$, **find** $\dfrac{dy}{dx}$.

Solution

Applying the Quotient Rule yields

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{x^2 + 3x - 1}{x^3 + 3}\right]$$

$$= \frac{(x^3 + 3)\frac{d}{dx}(x^2 + 3x - 1) - (x^2 + 3x - 1)\frac{d}{dx}(x^3 + 3)}{(x^3 + 3)^2}$$

$$= \frac{(x^3 + 3)(2x + 3) - (x^2 + 3x - 1)(3x^2)}{(x^3 + 3)^2}$$

$$= \frac{(2x^4 + 3x^3 + 6x + 9) - (3x^4 + 9x^3 - 3x^2)}{(x^3 + 3)^2}$$

$$= \frac{-x^4 - 6x^3 + 3x^2 + 6x + 9}{(x^3 + 3)^2}.$$

Example **Finding Higher Derivatives**

The first five derivatives of $y = x^5 - 2x^3 + 3x^2 + x - 1$ are

1st derivative: $y' = 5x^4 - 6x^2 + 6x + 1$

2nd derivative: $y'' = 20x^3 - 12x + 6$

3rd derivative: $y''' = 60x^2 - 12$

4th derivative: $y^{(4)} = 120x$

5th derivative: $y^{(5)} = 120$

The function has derivatives of all order, the 6th and later derivatives all being zero. So,

$$y^{(6)} = 0$$

$$\vdots$$

$$y^{(n)} = 0 \quad (n \geq 6).$$

Derivatives of Trigonometric Functions

Theorem 4.4.1: **Derivative of Trigonometric Functions**

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\csc x) = -\csc x \cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\cot x) = -\csc^2 x$$

Example **Find** $\frac{\mathrm{d}y}{\mathrm{d}x}$ **if,** $y = \sin x - x^3$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x - x^3) = \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) - \frac{\mathrm{d}}{\mathrm{d}x}x^3 = \cos x - 3x^2.$$

Example **Find** $\frac{dy}{dx}$ **if,** $y = \cos x + 2x^2$.

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x + 2x^2) = \frac{d}{dx}(\cos x) + 4x = -\sin x + 4x.$$

Example

Find y'' **if** $y = \csc x$**.**

Solution

$$y' = \frac{d}{dx}(\csc x) = -\csc x \cot x.$$

Then,

$$y'' = \frac{d}{dx}(-\csc x \cot x)$$

$$= -\csc x \frac{d}{dx}(\cot x) - \cot x \frac{d}{dx}(\csc x)$$

$$= -\csc x(-\csc^2 x) - \cot x(-\csc x \cot x)$$

$$= \csc x(\csc^2 x + \cot^2 x).$$

Example **Finding repeated derivatives**

Find the 49th derivative of $\sin x$**.**

Solution

The first few derivatives of $f(x) = \sin x$ **are as follows:**

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(4)}(x) = \sin x$$

$$f^{(5)}(x) = -\cos x$$

Note that the successive derivatives occur in a cycle of length 4 and, in particular, $f^{(n)}(x) = \sin x$ **whenever** ***n*** **is a multiple of 4. Therefore**

$$f^{(48)}(x) = \sin x$$

$$f^{(49)}(x) = \cos x.$$

The Chain Rule and Parametric Equations

Theorem: **The Chain Rule**

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.$$

Example **Applying the Chain Rule**

Find $F'(x)$ if $F(x) = \sqrt{x^3 - 1}$.

Solution

If we let $u = x^3 - 1$ and $y = \sqrt{u}$, then

$$F'(x) = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{u}} \cdot 3x^2 = \frac{1}{2\sqrt{x^3 - 1}} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3 - 1}}.$$

The Chain Rule with Powers of a Function

If *n* is any real number and $u = g(x)$ is differentiable, then

$$\frac{\mathrm{d}}{\mathrm{d}x}(u^n) = nu^{n-1}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Alternatively

$$\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x).$$

More generally, the Chain Rule gives

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^u\right) = \mathrm{e}^u \cdot \frac{\mathrm{d}u}{\mathrm{d}x}.}$$

Example **Differentiate an exponential function with natural base.**

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ **if** $y = \mathrm{e}^{\sin x}$ **.**

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\sin x} = \mathrm{e}^{\sin x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \mathrm{e}^{\sin x} \cdot \cos x = \cos x\, \mathrm{e}^{\sin x}.$$

We can use the Chain Rule to differentiate an exponential function with any base *a* > 0. Note that

$$a^x = \left(e^{\ln a}\right)^x = e^{(\ln a)x}$$

and the Chain Rule gives

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{(\ln a)x} = e^{(\ln a)x} \cdot \frac{d}{dx}(\ln a)x$$

$$= e^{(\ln a)x} \cdot \ln a = a^x \ln a$$

Definition: **Parametric Curve**

If *x* and *y* are given as functions

$$x = f(t), \quad y = g(t)$$

over an interval of *t*–values, then the set of points $(x, y) = (f(t), g(t))$ **defined by these equations is a parametric curve. The equations are parametric equations for the curve**

Example 10 **Moving counterclockwise on a circle.**

Graph the parametric curve $x = r\cos t,\ y = r\sin t,\ 0 \le t \le 2\pi$.

Solution

Since $x^2 + y^2 = r^2(\cos^2 t + \sin^2 t) = r^2$**, the parametric curves lie along the circle of radius *r*. The parameterization describes a motion that being at the point (*r*, 0) and transverse the circle** $x^2 + y^2 = r^2$ **once counterclockwise, retuning to (*r*, 0) at** $t = 2\pi$.

A parametrized curve $x = f(t)$ **and** $y = g(t)$ **is differentiable at *t* if *f* and *g* are differentiable at *t*. At a point on a differentiable parametrized curve where *y* is also a differentiable function of *x*, the derivative d*y*/d*t*, d*x*/d*t*, and d*y*/d*x* are related by the Chain Rule:**

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Example **Differentiating with a parameter.**

If $x = t^2 - 3$ and $y = 2t + 5$, find the value of $\frac{dy}{dx}$ at $t = 2$.

Solution

Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx./dt} = \frac{2}{2t} = \frac{1}{t}.$$

Then

$$\left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{2}.$$

Example

Find y'' **if** $3x^2 - y^3 = 4$.

Solution

To start, we differentiate both sides of the equation with respect to *x* in order to find $y' = \mathrm{d}y/\mathrm{d}x$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(4)$$

$$6x - 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y^2}, \qquad y \neq 0.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{y^2}\right) = \frac{y^2(2) - 2x(2yy')}{(y^2)^2} = \frac{2y^2 - 4xyy'}{y^4} = \frac{2}{y^2} - \frac{4x}{y^3}y'$$

Finally, we substitute $y' = 2x/y^2$ **to express** y'' **in terms of *x* and *y*.**

$$y'' = \frac{2}{y^2} - \frac{4x}{y^3}\frac{2x}{y^2} = \frac{2}{y^2} - \frac{8x^2}{y^5}.$$

Theorem: **Power Rule for Rational Powers**

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx} x^{p/q} = \frac{p}{q} x^{(p/q)-1}.$$

Derivative of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

Example

Differentiate $y = x^2 \tan^{-1}\sqrt{x}$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 \tan^{-1}\sqrt{x}\right)$$

$$= x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{x}\right) + \tan^{-1}\sqrt{x}\,\frac{\mathrm{d}}{\mathrm{d}x}x^2$$

$$= x^2 \frac{1}{1+(\sqrt{x})^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\right) + \tan^{-1}\sqrt{x}\,(2x)$$

$$= \frac{x^2}{2\sqrt{x}\left(1+x\right)} + 2x\tan^{-1}\sqrt{x}$$

$$= \frac{x^{3/2}}{2\left(1+x\right)} + 2x\tan^{-1}\sqrt{x}.$$

Derivatives of **Logarithmic Function**

Rule 1:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x) = \frac{1}{x \ln a}.$$

Rule 2:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.$$

Rule 3:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln u) = \frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Example

Find $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x)$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x) = \frac{1}{\cos x}\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)$$

$$= \frac{1}{\cos x}(-\sin x) = -\frac{\sin x}{\cos x} = -\tan x.$$

Rule 4: **The Power Rule**

If n is any real number and $f(x) = x^n$ **, then**

$$f'(x) = nx^{n-1}.$$

Example

Differentiate $y = x^{\sqrt{x}}$ **.**

Solution

Using logarithmic differentiation, we have

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x} \ln x \,.$$

Differentiating implicitly with respect to x gives

$$\frac{y'}{y} = \sqrt{x} \cdot \frac{1}{x} + \ln x \cdot \frac{1}{2\sqrt{x}}$$

$$y' = y\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right).$$

Rule 5:

$$e = \lim_{x \to 0} (1 + x)^{1/x}.$$

Rule 6:

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$