Stochastic models with memory effects

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Citation: AIP Conf. Proc. **1493**, 639 (2012); doi: 10.1063/1.4765554
View online: http://dx.doi.org/10.1063/1.4765554
View Table of Contents: http://proceedings.aip.org/dbt/dbt.jsp?KEY=APCPCS&Volume=1493&Issue=1
Published by the American Institute of Physics.

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Abstract. We aim in this work to derive non-Markovian stochastic models. We generalize the discrete random walk by using the method of conditional arrival probability and different types of time distribution in order to get the memory effect. Similarly, we apply this method in case of continuous time random walk with different time distributions and different jump distributions to get stochastic models with memory effect. Also, we may know the memory effect from the statistical properties of the model, especially the second moment.

Keywords: transition probability, arrival probability,short memory, long memory, waiting time.

INTRODUCTION

Many phenomena such as volatility in finance, the temperature of the weather have long memory effect. These phenomena need a long memory model to describe them. In order to get stochastic models with memory, we use the discrete random walk (the case when we have discrete states and continuous time) or the continuous random walk CTRW (the case when we have continuous states and continuous time). We need to find their transition probability, which describes the propagator. In case of discrete random walk, it is $P_{ij}(t)$, which describes the probability of being in state j at time t after jump from state i. In case of CTRW it is $P(y,t \mid x)$, which describes the probability of being at state y after state x at time t . To find the transition probability we introduce the conditional arrival probability $J_{ij}(t)$ in case discrete random walk and $J(x,t)$ in case of continuous random walk . This method helps us to derive the stochastic models with memory effect, taking into account the time distribution and jump distribution.

CONDITIONAL ARRIVAL PROBABILITY

Discrete Random Walk

In order to find the master equations of the transition probability for continuous time random walk with discrete state space, we introduce the theory of conditional arrival probability. First we assume the jump process X_t takes N states. We define the conditional transition probability as follows

$$P_{ij}(t) = Pr\{X_t = j \mid X_0 = i\}, \quad i, j = 1, 2, ..., N$$

which is the probability that the process X_t starts from state i at $t = 0$ and it is at state j at time t. We introduce

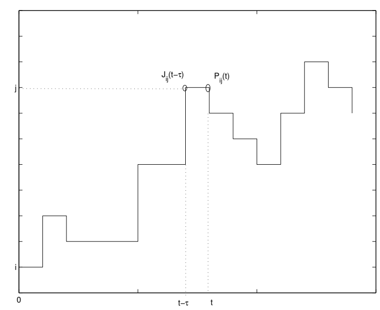

FIGURE 1. Conditional arrival probability $J_{ij}(t-\tau)$ from state i to state j, and conditional transition probability $P_{ij}(t)$.

the conditional arrival probability $J_{ij}(t)$ as the probability that the process X_t starts from state i at time $t = 0$ and arrives at state j at time t; see figure 1. Consider $\phi_j(t)$ the probability density function for the waiting time at state j. The survival function $\Psi_j(t)$ is given by

$$\Psi_j(t) = \int_t^{\infty} \phi_j(t')dt' = 1 - \int_0^t \phi_j(t')dt', \qquad (1)$$

which is the probability that no steps are taken in the time interval $[0,t)$. Moreover, the $n \times n$ transition matrix H with the matrix entries h_{ij} denotes the transition rate from state i to state j and satisfies[1]

$$\sum_{j=1}^{N} h_{ij} = 1. \qquad (2)$$

If $h_{ij} = 1$ only when $j = i + 1$ and zero otherwise, then the discrete random walk process represents a counting

[1] The transition rates must be given constants for each transition.

9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences
AIP Conf. Proc. 1493, 639-648 (2012); doi: 10.1063/1.4765554

process. The balance equation for $J_{ij}(t)$ is

$$J_{ij}(t) = \sum_{k\neq j}^{N} \int_0^t J_{ik}(t-\tau)\phi_k(\tau)h_{kj}d\tau + \phi_i(t)h_{ij}. \quad (3)$$

It follows from the law of total probabilities that $P_{ij}(t)$ obeys the equation

$$P_{ij}(t) = P_{ij}(0)\Psi_j(t) + \int_0^t J_{ij}(t-\tau)\Psi_j(\tau)d\tau, \quad (4)$$

where $P_{ij}(0)$ is the initial condition satisfies

$$P_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad (5)$$

The first term in the right-hand side (RHS) of equation (4) represents the probability of being at the initial state times the probability of no jump up to time t. The second term takes into account the probability of arriving at state j from state i at time $t-\tau$ and the probability of no jump during time τ. We assume that the jump process X_t is homogenous, thus

$$P_{ij}(t) = Pr\{X_{t+h} = j \mid X_t = i\} = Pr\{X_h = j \mid X_0 = i\},$$

In order to find the master equation of the conditional transition probability, we use the Laplace transform of equations (3), (4)

$$\begin{aligned} \tilde{J}_{ij}(s) &= \sum_{k\neq j}^{N} \tilde{J}_{ik}(s)\tilde{\phi}_k(s)h_{kj} + \tilde{\phi}_i(s)h_{ij}, && (6) \\ \tilde{P}_{ij}(s) &= P_{ij}(0)\tilde{\Psi}_j(s) + \tilde{J}_{ij}(s)\tilde{\Psi}_j(s). && (7) \end{aligned}$$

From (7) we obtain

$$\tilde{J}_{ij}(s) = \frac{\tilde{P}_{ij}(s)}{\tilde{\Psi}_j(s)} - P_{ij}(0)\frac{\tilde{\Psi}_j(s)}{\tilde{\Psi}_j(s)}, \quad (8)$$

substitution of (8) into (6) and applying the definition of the initial condition (5), we get

$$\frac{\tilde{P}_{ij}(s)}{\tilde{\Psi}_j(s)} - P_{ij}(0) = \sum_{k\neq j}^{N} \tilde{P}_{ik}(s)\frac{\tilde{\phi}_k(s)}{\tilde{\Psi}_k(s)}h_{kj}. \quad (9)$$

To get the conditional transition probability, we rearrange equation (9) and use the definition of kernel function $\tilde{K}_k(s) = \frac{\tilde{\phi}_k(s)}{\tilde{\Psi}_k(s)}$, we obtain

$$\tilde{P}_{ij}(s) = P_{ij}(0)\tilde{\Psi}_j(s) + \sum_{k\neq j}^{N} \tilde{P}_{ik}(s)\tilde{K}_k(s)h_{kj}\tilde{\Psi}_j(s). \quad (10)$$

If the waiting time is independent identically distributed (iid) random variables for all states then (10) becomes

$$\tilde{P}_{ij}(s) = P_{ij}(0)\tilde{\Psi}(s) + \sum_{k\neq j}^{N} \tilde{P}_{ik}(s)\tilde{\phi}(s)h_{kj}. \quad (11)$$

By inverting Laplace transform, it yields

$$P_{ij}(t) = P_{ij}(0)\Psi(t) + \sum_{k\neq j}^{N} \int_0^t P_{ik}(t-\tau)\phi(\tau)h_{kj}d\tau. \quad (12)$$

The first term in RHS represents the process starting from the origin and staying there until time t; the second term includes the contribution from the jump to state j from different states k and waiting up to time t.

To get the master equation of the conditional transition probability $P_{ij}(t)$ for discrete random walk, we move the first term in the left-hand side in equation (9) to the right and add $sP_{ij}(s)$ to both sides, then we use the definition of the survival function in Laplace domain, $\tilde{\Psi}(s) = \frac{1-\tilde{\phi}(s)}{s}$, and the definition of kernel function, imply

$$\begin{aligned} s\tilde{P}_{ij}(s) - P_{ij}(0) &= -\tilde{K}(s)\tilde{P}_{ij}(s) \\ &\quad + \sum_{k\neq j}^{N} \tilde{P}_{ik}(s)\tilde{K}(s)h_{kj}. && (13) \end{aligned}$$

Inverting Laplace transform with convolution theorem gives the master equation in time domain

$$\begin{aligned} \frac{dP_{ij}(t)}{dt} &= -\int_0^t K(\tau)P_{ij}(t-\tau)d\tau \\ &\quad + \sum_{k\neq j}^{N} \int_0^t K(\tau)P_{ik}(t-\tau)h_{kj}d\tau. && (14) \end{aligned}$$

Next we are going to find the master equations for the conditional transition probability corresponding to different waiting time distributions in the Laplace domain by using the previous equations (11) and (13). Then we convert them to time domain to find the memory effect models.

Exponential distribution waiting time:

Let the waiting times t are independent identically distributed and follow an exponential distribution, i.e.

$$\phi(t) = me^{-mt}, \quad t \geq 0, \quad (15)$$

where m is a positive constant called the rate parameter. The expected value of the distribution is $1/m$ and the variance is $1/m^2$. From figure 2, we can see some features of an exponential distribution: the probability density declines monotonically as the value of waiting time increases, and the curve is steeper as the parameter m is larger. Therefore in this particular case, the waiting time is more likely to be very small and long waiting time seldom happens.

The Laplace transform of the distribution's PDF is

$$\tilde{\phi}(s) = \frac{m}{m+s}, \quad (16)$$

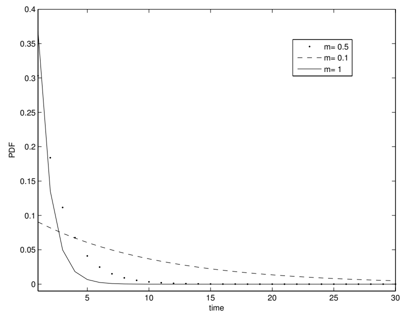

FIGURE 2. The probability density of exponential distribution with various rate parameter m.

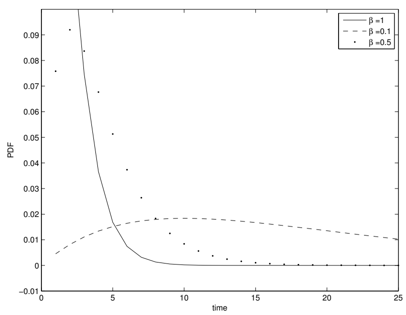

FIGURE 3. The probability density of gamma distribution with shape parameter $\alpha = 2$ and various scale parameter β.

and for survival function

$$\tilde{\Psi}(s) = \frac{1}{m+s}. \tag{17}$$

Therefore, the kernel function will be

$$\tilde{K}(s) = m.$$

Substituting the value of kernel function in (13) gives the master equation in Laplace domain, then by inverting to time domain

$$\frac{dP_{ij}(t)}{dt} = -mP_{ij}(t) + m\sum_{k\neq j}^{N} P_{ik}(t)h_{kj}.$$

This master equation is a classic forward Kolmogorov equation [1], and the conditional transition probability (12) is

$$\begin{aligned} P_{ij}(t) &= P_{ij}(0)e^{-mt} \\ &+ \sum_{k\neq j}^{N} m\int_0^t e^{-m\tau}P_{ik}(t-\tau)h_{kj}d\tau, \end{aligned}$$

The process in this case is Markovian because there is no memory effect. This means the current state of the process at time t does not depend on any previous states at the previous history.

Gamma distribution waiting time:

If the waiting time has gamma distribution then

$$\phi(t) = \frac{\beta^\alpha}{\Gamma(\alpha)}t^{\alpha-1}e^{-\beta t}, \quad t > 0, \tag{18}$$

where α is the shape parameter and β is the scale parameter and both β, α are positive. The expected value of the distribution is $\alpha\beta$ and the variance is $\alpha\beta^2$. Apparently, when $\alpha = 1$ the equation (18) reduces to the form of exponential distribution, hence exponential distribution is a special case of gamma distribution. It is clear from figure 3 that the curve of gamma distribution has only one peak and the peak moves as the parameters α,β vary. Thus gamma distribution can allow for longer waiting times if the proper parameters are chosen.

We will choose the case when $\alpha = 2$ as an example of Gamma distribution to see the memory effect this distribution. The waiting time PDF is

$$\phi(t) = \beta^2 te^{-\beta t}.$$

Using the Laplace transform for the PDF, we obtain the following functions

$$\tilde{\phi}(s) = \frac{\beta^2}{(s+\beta)^2}, \tag{19}$$

$$\tilde{\Psi}(s) = \frac{s+2\beta}{(s+\beta)^2}, \tag{20}$$

$$\tilde{K}(s) = \frac{\beta^2}{s+2\beta}.$$

Inserting them in (13) yields the master equation in the Laplace domain. Hence, in the time domain the master equations will be

$$\begin{aligned} P_{ij}(t) &= \sum_{k\neq j}^{N}\beta^2\int_0^t \tau e^{-\beta\tau}P_{ik}(t-\tau)h_{kj}d\tau \\ &+ P_{ij}(0)e^{-\beta t}(\beta t+1), \end{aligned}$$

$$\begin{aligned} \frac{dP_{ij}(t)}{dt} &= -\beta^2\int_0^t e^{-2\beta\tau}P_{ij}(t-\tau)d\tau \\ &+ \beta^2\sum_{k\neq j}^{N}\int_0^t e^{-2\beta\tau}P_{ik}(t-\tau)h_{kj}d\tau. \end{aligned}$$

Here we can notice the time integral in the right-hand side, which is evidence of memory effects. So the master equation in the case of gamma distributed waiting time is non-Markovian.

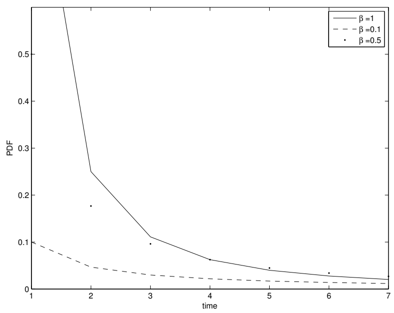

FIGURE 4. The probability density of Pareto distribution with various parameter β and minimum value equals one.

Power-law distribution waiting time:

In the case when the waiting time has a heavy-tailed or power-law distribution, its PDF can be in the following form

$$\phi(t) \sim \frac{\beta}{\Gamma(1-\beta)} t^{-(\beta+1)}, \quad 0 < \beta < 1, \qquad \text{for} \quad t \to \infty. \tag{21}$$

Power-law distribution issued in the description of open system [2]. Power-law correlation is observed in a critical state of an infinite system, but if the system is finite, the finiteness limits the range within which the power-law behavior can be observed. One form of power-law distribution is called **Pareto distribution**, which has the following PDF

$$\phi(t) = \begin{cases} \frac{\beta b^\beta}{t^{\beta+1}} & \text{for} \quad t > b, \\ 0 & \text{for} \quad t < b, \end{cases} \tag{22}$$

where b is the minimum possible value of t and β is a positive parameter. Figure 4 shows Pareto PDF when the minimum value is one and β has various values. This distribution has expected value when $\beta > 1$ equal to $\beta b/(\beta-1)$ or $\beta/(\beta-1)$ for $b = 1$. Also it has variance equal to $(b/(\beta-1))^2(\beta/(\beta-2))$ that exists only for $\beta > 2$. Another example of power-law distribution is the Mittag-Leffler function that is given in [9]

$$\phi(t) = -\frac{d}{dt} E_\beta(-t^\beta), \quad \Psi(t) = E_\beta(-t^\beta).$$

This function has a Laplace form such as

$$\tilde{\phi}(s) = \frac{1}{s^\beta + 1}, \tag{23}$$

therefore the survival function will be

$$\tilde{\Psi}(s) = \frac{s^{\beta-1}}{1 + s^\beta}, \tag{24}$$

and the kernel function is

$$\tilde{K}(s) = \frac{1}{s^{\beta-1}}.$$

By substituting the kernel function in (13), and multiplying the equation by $s^{\beta-1}$, implies

$$s^\beta \tilde{P}_{ij}(s) - s^{\beta-1} P_{ij}(0) = -\tilde{P}_{ij}(s) + \sum_{k\neq j}^{N} \tilde{P}_{ik}(s) h_{kj}.$$

So, in the time domain we obtain

$$\begin{aligned} P_{ij}(t) &= \sum_{k\neq j}^{N} -\int_0^t \frac{d}{d\tau} E_\beta(-\tau^\beta) P_{ik}(t-\tau) h_{kj} d\tau \\ &\quad + P_{ij}(0) E_\beta(-t^\beta), \qquad (25) \\ \frac{d^\beta P_{ij}(t)}{dt^\beta} &= -P_{ij}(t) + \sum_{k\neq j}^{N} P_{ik}(t) h_{kj}. \qquad (26) \end{aligned}$$

The memory effect appears in equation (25) due to the Mittag-Leffler function and in equation (26) due to the Caputo fractional derivative in the left-hand side, which is defined as [10]

$$\frac{d^\beta}{dt^\beta} P(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t P(\tau)(t-\tau)^{-\beta} d\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)} P(0). \tag{27}$$

Statistical properties

In this section we are going to find the first two moments for continuous time random walk with discrete states (discrete random walk). Back to the master equation (14), and consider the case when

$$h_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

in this case, the solution of the master equation (12) is a counting process, which has the master equation (14)

$$\frac{dP_j(t)}{dt} = \int_0^t K(t-\tau) P_{j-1}(\tau) d\tau - \int_0^t K(t-\tau) P_j(\tau) d\tau,$$

with Laplace form:

$$s\tilde{P}_j(s) - P_0(0) = \tilde{K}(s)\tilde{P}_{j-1}(s) - \tilde{K}(s)\tilde{P}_j(s).$$

We use the induction method to find the probability density $\tilde{P}_j(s)$ in the Laplace form. So for general j we have

$$\tilde{P}_j(s) = \sum_{i=0}^{j} P_i(0) \tilde{\phi}^{j-i}(s) \tilde{\Psi}(s),$$

since $P_i(0) = 0 \forall i \neq 0$, thus

$$\tilde{P}_j(s) = \tilde{\phi}^j(s)\tilde{\Psi}(s). \tag{28}$$

Therefore its expected value is

$$\begin{aligned} <\tilde{X}(s)> &= \sum_{j=0}^{\infty} j\tilde{P}_j(s) = \tilde{\Psi}(s)\frac{\tilde{\phi}(s)}{(1-\tilde{\phi}(s))^2} \\ &= \frac{\tilde{\phi}(s)}{s(1-\tilde{\phi}(s))}, \end{aligned} \tag{29}$$

and the second moment will be

$$\begin{aligned} <\tilde{X}^2(s)> &= \sum_{j=0}^{\infty} j^2\tilde{P}_j(s) = \tilde{\Psi}(s)\frac{\tilde{\phi}(s)(1+\tilde{\phi}(s))}{(1-\tilde{\phi}(s))^3} \\ &= \frac{\tilde{\phi}(s)(1+\tilde{\phi}(s))}{s(1-\tilde{\phi}(s))^2}. \end{aligned} \tag{30}$$

Counting process with Exponential waiting time

If we use the Laplace formula for the exponential distribution (16), (17) in (28), we get

$$\tilde{P}_j(s) = \frac{m^j}{(m+s)^{j+1}}$$

this formula is the Laplace transform of the Poisson process. The expected value of this distribution in the Laplace domain is given by

$$<\tilde{X}(s)> = \sum_{j=0}^{\infty} j\tilde{P}_j(s) = \frac{m}{s^2}.$$

Again this is the Laplace formula of (mt), the expected value of the process in the time domain. In the same way we can find the second moment

$$<\tilde{X}^2(s)> = \sum_{j=0}^{\infty} j^2\tilde{P}_j(s) = \frac{m}{s^2} + \frac{2m^2}{s^3},$$

accordingly,

$$<X_t^2> = mt + m^2t^2.$$

In this case we may notice the process has no memory due to, the second moment is a linear power of time.

Counting process with Gamma distribution waiting time

If we substitute the Laplace form of gamma waiting time PDF (19) when $\alpha = 2$, and its survival function (20) into (28), the counting process will have the following Laplace form distribution

$$\tilde{P}_j(s) = \frac{\beta^{2j}(s+2\beta)}{(s+\beta)^{2j+2}}.$$

From equation (29), the expected value is

$$<\tilde{X}(s)> = \frac{\beta^2}{s^2(s+2\beta)}.$$

Similarly, we obtain the second moment from equation (30)

$$<\tilde{X}^2(s)> = \frac{\beta^2}{s^2(s+2\beta)} + \frac{2\beta^4}{s^3(s+2\beta)^2}.$$

Using the inverse Laplace transform with the convolution theorem, the expected value is

$$<X_t> = \frac{1}{4}[e^{-2\beta t} - 1] + \frac{\beta t}{2},$$

while the second moment will be

$$<X_t^2> = \frac{1}{2} - \frac{\beta t}{2} + \frac{\beta^2 t^2}{2} - \frac{e^{-\beta t}}{2}[\beta t + 1].$$

The expected value and the second moment are the combined linear function of t and the monotonic exponentially decreasing function of t. However, for long time $t \to \infty$ they will be a proportional function of time.

Counting process with power-law distribution waiting time

In this case, we are going to use the Laplace transforms of the Mittag-Leffler function as an example of power-law waiting time PDF (23) and its survival function (24). Inserting them into (28), the counting process will have the following Laplace form distribution

$$\tilde{P}_j(s) = \frac{s^{\beta-1}}{(1+s^\beta)}\frac{1}{(1+s^\beta)^j}.$$

The expected value is obtained from (29)

$$<\tilde{X}(s)> = \frac{1}{s^{\beta+1}},$$

and the second moment can be found from (30)

$$<\tilde{X}^2(s)> = \frac{1}{s^{\beta+1}} + \frac{2}{s^{2\beta+1}}.$$

Inverting them to the time domain, yields :

$$<X_t> = \frac{t^\beta}{\Gamma(\beta+1)},$$

and

$$<X_t^2> = \frac{t^\beta}{\Gamma(\beta+1)} + \frac{2t^{2\beta}}{\Gamma(2\beta+1)}, \quad 0 < \beta < 1.$$

The expected value and the second moment are nonlinear functions of time. It is decaying very slowly as $t \to \infty$ due to the power-law form formula which is strong evidence of memory effects.

Continuous Random Walk

In the previous section we assumed the jump process X_t has discrete state space, and we derived the master equations for the transition probability corresponds to different distributions of waiting time. These equations depend on the kernel function K and the transition matrix H. The Laplace transform and its inverse play the main role in the derivation. In this section we generalize the continuous time random walk to include the continuous state space, i.e. the state space is the set of real numbers. Also the kernel function is the main factor to derive the master equations besides the jump distribution PDF. Unlike the previous section, here the Fourier-Laplace transform and its inverse will play a fundamental role in the whole processes of derivation and the summation sign will be replaced by the integral sign.
The conditional transition probability P may be defined as

$$\int_a^b P(y,t \mid x)dy = Pr\{a < X_t < b \mid X_0 = x\},$$

$P(y,t \mid x)$ gives the probability of finding the process X_t in the interval $(y, y+dy)$ provided $X_0 = x$. Here x is the backward variable and y is the forward variable. Again we consider the homogenous jump process as in the discrete state space. The initial condition is

$$P(y,0 \mid x) = \delta(y-x).$$

The conditional arrival probability $J(y,t \mid x)$ means the process starts from x at time zero and arrives y at time t. To find it we need the waiting time PDF at point x which is $\phi(x,t)$, the survival function $\Psi(x,t)$ and the probability density for the jump process X_t from point x to point y which is $w(y \mid x)$. Now, let us write the balanc equation for $J(y,t \mid x)$

$$\begin{aligned} J(y,t \mid x) \quad &= \int_{\Re}\int_0^t J(z,t-\tau \mid x)\phi(z,\tau)w(y \mid z)dz \\ &\quad +\phi(x,t)w(y \mid x), \quad x < z < y. \end{aligned} \tag{31}$$

Accordingly, from the law of total probability we have the equation for conditional transition probability $P(y,t \mid x)$ such as

$$\begin{aligned} P(y,t \mid x) &= \delta(y-x)\Psi(x,t) \\ &+ \int_0^t J(y,t-\tau \mid x)\Psi(y,\tau)d\tau. \end{aligned} \tag{32}$$

Let us consider the case when the waiting time distribution is independent identically distributed, i.e. $\phi(x,t) = \phi(t)$. Transferring equations (31) and (32) to the Fourier-Laplace transform give

$$\begin{aligned} \tilde{\hat{J}}(k,s) &= \tilde{\hat{J}}(k,s)\tilde{\phi}(s)\hat{w}(k) \\ &\quad +\tilde{\phi}(s)\hat{w}(k), \end{aligned} \tag{33}$$

$$\tilde{\hat{P}}(k,s) = \tilde{\Psi}(s) + \tilde{\hat{J}}(k,s)\tilde{\Psi}(s). \tag{34}$$

From (34), we have

$$\tilde{\hat{J}}(k,s) = \frac{\tilde{\hat{P}}(k,s)}{\tilde{\Psi}(s)} - 1.$$

Substituting the last formula into (33), yields

$$\frac{\tilde{\hat{P}}(k,s)}{\tilde{\Psi}(s)} - 1 = \frac{\tilde{\hat{P}}(k,s)}{\tilde{\Psi}(s)}\tilde{\phi}(s)\hat{w}(k), \tag{35}$$

hence

$$\tilde{\hat{P}}(k,s) = \tilde{\hat{P}}(k,s)\tilde{\phi}(s)\hat{w}(k) + \tilde{\Psi}(s), \tag{36}$$

$$or$$

$$\begin{aligned} \tilde{\hat{P}}(k,s) &= \frac{\tilde{\Psi}(s)\hat{P}_0(k)}{1-\tilde{\phi}(s)\hat{w}(k)} \\ &= \frac{(1-\tilde{\phi}(s))\hat{P}_0(k)}{s(1-\tilde{\phi}(s)\hat{w}(k))}. \end{aligned} \tag{37}$$

The last equation is equivalent to the Montroll-Weiss equation [1]. By inverting Fourier-Laplace, the equivalent of equation (36) in the time domain is

$$\begin{aligned} P(y,t|x) &= \delta(y-x)\Psi(t) \\ &\quad + \int_{\Re} P(z,t-\tau|x)\phi(\tau)w(z|x)dz. \end{aligned}$$

Recalling equation(35) and adding $s\tilde{\hat{P}}(k,s)$ to both sides then rearranging it to obtain the master equation

$$s\tilde{\hat{P}}(k,s) - \hat{P}_0(k) = -\tilde{K}(s)\tilde{\hat{P}}(k,s) + \tilde{\hat{P}}(k,s)\tilde{K}(s)\hat{w}(k).$$

The inverse Fourier-Laplace transform gives the following master equation

$$\begin{aligned} \frac{\partial}{\partial t}P(y,t) &= -\int_0^t K(\tau)P(y,t-\tau)d\tau \\ &\quad + \int_0^t \int_{\Re} K(\tau)P(z,t-\tau)w(y-z)dzd\tau. \end{aligned} \tag{38}$$

or

$$\begin{aligned} \frac{\partial}{\partial t}P(y,t) &= \int_0^t K(\tau)\Big[-P(y,t-\tau) \\ &\quad + \int_0^t \int_{\Re} P(z,t-\tau)w(y-z)dz\Big]d\tau. \end{aligned}$$

Our following tasks consider the jump process with different continuous PDF, such as Gaussian distribution and Lévy distribution. Then we find the master equation for conditional transition probability in the case when the waiting time has no memory, like exponential distribution, and when it has memory, such as power-law distribution.

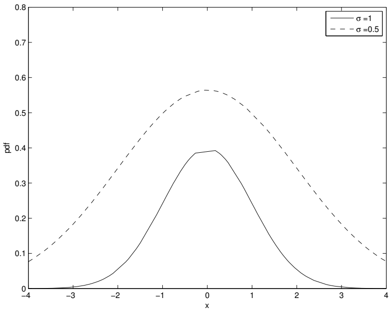

FIGURE 5. The probability density of zero-mean Gaussian distribution with various variance.

Gaussian distribution jump process

If the jump process follows a zero-mean Gaussian distribution, then its PDF has the form

$$w(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-x^2}{2\sigma_x^2}},$$

where σ_x^2 is the variance of the distribution. From the properties of Gaussian distribution, (see figure 5), the process jumps to a state near to where it was at previous time instant and the large jumps seldom occur. In the hydrodynamic limit, we can get the approximation of $\hat{w}(k)$

$$\hat{w}(k) = 1 - \frac{\sigma_x^2 k^2}{2} + o(k^2) \sim 1 - \frac{\sigma_x^2 k^2}{2}, \quad k \to 0. \tag{39}$$

Here, in the hydrodynamic domain the limit of $k \to 0$ is equivalent to $x \to \infty$ in the space domain.

CTRW with Gaussian jump process and Exponential waiting time:

This case in long-time limit corresponds to Brownian motion when the waiting time's mean and jump's variance are finite. To obtain the master equation for the process in Fourier-Laplace domain equations (37), we use the definition of exponential distribution PDF in Laplace transform (16) in the case of invariant waiting time. For the jump process we use the Fourier transform of Gaussian jumps PDF (39). Therefore,

$$\tilde{\hat{P}}(k,s) = \frac{\hat{P}_0(k)}{s + m\frac{\sigma_x^2}{2}k^2}, \tag{40}$$

$$s\tilde{\hat{P}}(k,s) - \hat{P}_0(k) = -\left(\frac{m\sigma_x^2 k^2}{2}\right)\tilde{\hat{P}}(k,s). \tag{41}$$

By inverting the Fourier-Laplace transform, we obtain the diffusion equation when the jump's process follows Gaussian distribution with exponential waiting time

$$\frac{\partial}{\partial t}P(x,t) = \frac{m\sigma_x^2}{2}\frac{\partial^2}{\partial x^2}P(x,t), \tag{42}$$

where the Fourier transform $F\{\frac{\partial^2}{\partial x^2}P(x,t) = -k^2\hat{P}(k,t)\}$. The solution of equation (42) in the time domain is well-known Gaussian, due to the case of finite jump's variance and finite waiting time's mean (for more details see [1])

$$P(x,t) = \frac{1}{\sqrt{2\pi\sigma_x^2 mt}}\exp\Big(-\frac{x^2}{2\sigma_x^2 mt}\Big).$$

The second moment of the CTRW in Laplace form can be obtained from

$$< \tilde{X}^2(s) >= -\frac{\partial^2\tilde{\hat{p}}(k,s)}{\partial k^2}\,|_{k=0}\,. \tag{43}$$

By using (40), this implies

$$\frac{\partial^2\tilde{\hat{p}}(k,s)}{\partial k^2}\,|_{k=0} = -\frac{m\sigma_x^2}{s^2},$$

which in the time domain will be

$$< X_t^2 >= m\sigma_x^2 t.$$

Here we can notice that the process has no memory due to the second moment is a linear power of time.

CTRW with Gaussian jump process and Power-law waiting time:

Here the waiting time PDF is heavy-tailed, so that the mean waiting time is infinite while the jump's variance is still kept finite. The asymptotic behavior of a heavy-tailed waiting time PDF is given by

$$\phi(t) \sim (t)^{-(1+\beta)} \quad \text{as} \quad t \to \infty. \tag{44}$$

Consequently, the long time limit corresponds to

$$\tilde{\phi}(s) \sim 1 - (\lambda s)^\beta \quad \text{as} \quad s \to 0, \tag{45}$$

where λ is a parameter with units of time. Similarly, inserting these Laplace transforms of power-law PDF and the Fourier transform of jump distribution PDF (39) into (37) obtains the master equations as follows

$$\tilde{\hat{P}}(k,s) = \frac{s^{\beta-1}\hat{P}_0(k)}{s^\beta + K_\beta k^2} \tag{46}$$

$$s^\beta\tilde{\hat{P}}(k,s) - s^{\beta-1}\hat{P}_0(k) = -\Big(K_\beta k^2\Big)\tilde{\hat{P}}(k,s),$$

where $K_\beta = \sigma_x^2/(2\lambda^\beta)$. Hence, the master equation is a time-fractional equation (see [8])

$$\frac{\partial^\beta}{\partial t^\beta}P(x,t) = K_\beta \frac{\partial^2}{\partial x^2}P(x,t), \qquad (47)$$

where $\frac{\partial^\beta}{\partial t^\beta}P(x,t)$ is the Caputo fractional derivative defined by eqrefcaputo.The time-fractional equation (47) is equivalent to the Fractional Fokker Planck equation given by Barkai *et al.*[3], with solution equivalent to (46) in the Laplace-Fourier domain. A closed-form solution for (47) can be found in terms of the Wright function (see [4]) and in terms of the Fox function such as(see [1])

$$P(x,t) = \frac{1}{\sqrt{4\pi K_\beta t^\beta}} H^{2,0}_{1,2}\Big[\frac{x^2}{4K_\beta t^\beta} \;\Big|\; \begin{matrix} (1-\beta/2,\beta) \\ (0,1),(1/2,1) \end{matrix} \Big]$$

The second moment of CTRW can be found by substituting (46) into (43), hence

$$\frac{\partial^2 \tilde{\hat{p}}(k,s)}{\partial k^2}\,|_{k=0} = -\frac{2K_\beta}{s^{\beta+1}}.$$

Accordingly, in the time domain it will be

$$< X_t^2 > = \frac{2K_\beta t^\beta}{\Gamma(\beta+1)} = \frac{\sigma_x^2}{\lambda^\beta}\frac{t^\beta}{\Gamma(\beta+1)}.$$

In this result the second moment is a fractional power, so the process has long memory.

Lévy distribution jump process

Lévy distribution looks similar to normal distribution in the center, but the tails are much flatter than those of Gaussian distribution. The variance of this PDF is infinite. There is no general explicit form for $w(x)$, but the Lévy distribution may be written as a power-law distribution for a large value of stochastic variable x [2]

$$w(x) \sim \sigma_x^{-\alpha}|x|^{-(\alpha+1)}, \quad \text{for} \quad |x| > \sigma_x, \quad 0 < \alpha < 2.$$

In a hydrodynamic limit of jump it will have the form,

$$\hat{w}(k) = exp(-\sigma_x^\alpha |k|^\alpha) \sim 1 - \sigma_x|k|^\alpha, \quad \text{for} \quad k \to 0. \qquad (48)$$

The fact that power-law distribution may lack a typical scale is reflected in Lévy processes, by the property that the variance of Lévy processes is infinite for $\alpha < 2$. Stochastic processes with infinite variance are extremely difficult to use and raise fundamental questions when applied to a real system. For example, in the finance system, an infinite variance would complicate the important task of risk estimation.

CTRW with Lévy jump process and Exponential waiting time:

Although the jump length variance is infinite, the process is of Markovian nature due to the finiteness of the waiting time's mean [5]. We apply the same procedure when the waiting time is exponentially distributed and has the Laplace forms (16) for its PDF and the jump process has Lévy distribution corresponding to (48) in a Fourier domain. Consequently, the master equations (37) will be

$$\begin{aligned} \tilde{\hat{P}}(k,s) &= \frac{\hat{P}_0(k)}{s + m\sigma_x^\alpha |k|^\alpha}, \\ s\tilde{\hat{P}}(k,s) - \hat{P}_0(k) &= -K_\alpha |k|^\alpha \tilde{\hat{P}}(k,s), \end{aligned}$$

where $K_\alpha = m\sigma_x^\alpha$. Inverting the Fourier-Laplace transform to get the master equation in time-space domain

$$\frac{\partial}{\partial t}P(x,t) = K_\alpha \frac{\partial^\alpha}{\partial |x|^\alpha}P(x,t), \qquad (49)$$

where $\frac{\partial^\alpha}{\partial |x|^\alpha}$ is space-fractional derivative known as **Riesz** fractional derivative of order $\alpha, 0 < \alpha < 2$ defined by [8]

$$\frac{\partial^\alpha}{\partial |x|^\alpha}P(x) = \Gamma(1+\alpha)\frac{sin(\alpha\pi/2)}{\pi} * \int_0^\infty \frac{P(x+\xi) - 2P(x) + P(x-\xi)}{\xi^{1+\alpha}}d\xi.$$

The space-fractional derivative are obtained from the assumption that the random jump has a Lévy distribution, with power-law tails. The solution of the fractional space differential equation (49) can be obtained by using the Fox function, the result being [1]

$$P(x,t) = \frac{1}{\alpha|x|} H^{2,2}_{1,1}\Big[\frac{|x|}{(K_\alpha t)^{1/\alpha} t^\beta} \;\Big|\; \begin{matrix} (1,1/\alpha),(1,1/2) \\ (1,1),(1,1/2) \end{matrix} \Big]$$

For limit $\alpha \to 2$, (49) goes to the diffusion equation and the classical Gaussian solution is recovered. When $m = \sigma_x = 1$, the solution of the space-fractional derivative equation $\frac{\partial}{\partial t}P(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha}P(x,t)$ as given by Scalas [7]

$$P(x,t) = t^{-1/\alpha}L_\alpha(xt^{-1/\alpha}),$$

where L_α is the Lévy standardized probability density function:

$$L_\alpha(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} exp\Big(-iqx - |q|^\alpha\Big)dq.$$

The second moment of CTRW in the case of exponential waiting time and Lévy jump process $< X_t^2 > \to \infty$ when $0 < \alpha < 2$, for $\alpha = 2$ corresponds to the case of the Gaussian jump process.

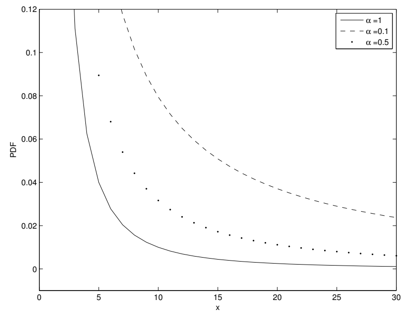

FIGURE 6. The probability density of Lévy distribution

CTRW with Lévy jump process and Power-law waiting time:

In this case CTRW has heavy-tailed distribution for both waiting time and jump process. Accordingly, the waiting time's mean and the jump's variance are both infinite. Substituting the Laplace forms of power-law waiting time (45), the Fourier form of Lévy jump process (48) into (37) gets the master equations of CTRW, thus

$$\begin{aligned}\tilde{\hat{P}}(k,s) &= \frac{\lambda^\beta s^\beta \hat{P}_0(k)}{s[\lambda^\beta s^\beta + \sigma_x^\alpha |k|^\alpha]},\\ &= \frac{s^{\beta-1}\hat{P}_0(k)}{s^\beta + K_{\alpha,\beta}|k|^\alpha},\\ s^\beta \tilde{\hat{P}}(k,s) - s^{\beta-1}\hat{P}_0(k) &= -K_{\alpha,\beta}|k|^\alpha \tilde{\hat{P}}(k,s),\end{aligned}$$

where $K_{\alpha,\beta} = \sigma_x^\alpha/\lambda^\beta$. Hence, the master equation is a space-time fractional derivative equation [8]

$$\frac{\partial^\beta}{\partial t^\beta}P(x,t) = K_{\alpha,\beta}\frac{\partial^\alpha}{\partial |x|^\alpha}P(x,t). \quad (50)$$

The solution of this equation, when $\lambda = \sigma_x = 1$, is defined by Scalas [6]

$$P(x,t) = t^{-\beta/\alpha} W_{\alpha,\beta}(x t^{-\beta/\alpha}),$$

where $W_{\alpha,\beta}(u)$ is given by

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-iku} E_\beta(-|k|^\alpha),$$

that is the inverse Fourier transform of a Mittag-Leffler function. In the case $\beta = 1$ and $\alpha = 2$, the fractional equation reduces to the ordinary diffusion equation and the solution $P(x,t)$ becomes the Gaussian probability density function as demonstrated in (42) and its solution. In the general case $0 < \beta < 1$ and $0 < \alpha < 2$, the function $W_{\alpha,\beta}(u)$ is still a probability density function evolving in time and it belongs to the class of Fox function. Finally, the second moment of CTRW also diverges. In [1] they found another value for the second moment called imaginary mean squared displacement.

CONCLUSION

In this work we demonstrated our method of conditional arrival probability, which was used to derive the conditional transition probability for the random walk with continuous time and discrete states or continuous states. In addition to the arrival probability we used different time of waiting time distribution in the case of random walk with discrete states, and different waiting time distributions and jump distributions in the case of random walk with continuous states. We got many stochastic models with memory effect depending on the waiting time distribution. Also we explained the memory effect of the process from its second moment. Our future work is finding a numerical solution for these models.

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